On the Lenstra constant associated to continued fractions

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Suppose that a class of continued fraction expansions of real numbers are defined by a fixed algorithm. Then convergents p_n and q_n can be defined similar to the simple case. A positive number C_1 is said to be the Legendre constant associated to this class if the the following holds:
(i) for any irrational number x, if $|x - \frac{p}{q}| < C_1 \frac{1}{q^2}$ then $\frac{p}{q} = \frac{p_n}{q_n}$ for some $n \geq 0$. On the other hand, for any $\varepsilon > 0$, there exist x and $\frac{p}{q}$ such that $|x - \frac{p}{q}| < (C_1 + \varepsilon)\frac{1}{q^2}$ and $\frac{p}{q} \neq \frac{p_n}{q_n}$ for any $n \geq 0$.

A positive number C_2 is said to be the Lenstra constant associated to this class if the the following holds:
(ii) for almost every irrational number x,

$$\lim_{N \to \infty} \frac{\sharp\{1 \leq n \leq N \,:\, q_n^2 |x - \frac{p_n}{q_n}| < s\}}{N} \begin{cases} = & \frac{s}{K} \quad \text{if } 0 \leq s \leq C_2 \\ < & \frac{s}{K} \quad \text{if } C_2 \leq s \leq 1 \end{cases}$$

where K is a positive constant depending on the class of continued fractions.

In the case of simple continued fractions, it is known that $C_1 = C_2 = \frac{1}{2}$. In this talk, we will see that $C_1 = C_2$ if C_1 exists. As an application, we determine the Legendre constants explicitly for Rosen's continued fractions associated to Hecke groups.