STOPPING TIMES IN QUANTUM MECHANICS

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QUANTUM MECHANICS

(First version)

1. States

• State space= a complex Hilbert space $\mathcal{H}$.

• Possible states = norm 1 vectors $\Psi \in \mathcal{H}$: *wave functions.*

2. Observables

• Observables of the system $\mathcal{H}$ = self-adjoint operators on $\mathcal{H}$.

• Possible numerical outcomes for the measure of the observable H = the (real) spectrum $\sigma(H)$ of H.

• If $\mathcal{H}$ is in a state Ψ, if the observable H admits a spectral measure $A \mapsto \mathbb{1}_A(H)$, then the probability to measure H with values in the set A is

$$< \Psi\,,\, \mathbb{1}_A(H)\,\Psi > .$$

Also equal to

$$||\mathbb{1}_A(H)\,\Psi||^2 = \mathrm{Tr}\big(|\Psi\rangle\langle\Psi|\,\mathbb{1}_A(H)\big)\,.$$

3. Time evolution

- Energy observable H = *Hamiltonian* of the system.
- Consider the unitary group

$$U_t = e^{-itH}\,.$$

If Ψ_0 is the state at time 0, then it becomes

$$\Psi_t = U_t \Psi_0$$

at time t.

OPEN QUANTUM SYSTEMS

• Quantum system $\mathcal{H}$ in interaction with another $\mathcal{K}$:

$$\mathcal{H} \otimes \mathcal{K}\,.$$

• The typical situation: we have access to $\mathcal{H}$ only.

– System $\mathcal{K}$ is too complicated or unknown (environment, heat bath, noisy channel ...)

– System $\mathcal{K}$ not accessible (shared EPR pair, ...)

• State Ψ on $\mathcal{H} \otimes \mathcal{K}$, observable X on $\mathcal{H}$, what does a measurement of X give?

$$\mathrm{Prob}(X \in A) = \mathrm{Tr}\big(\rho\, \mathbb{1}_A(X)\big)$$

with $\rho = \mathrm{Tr}_{\mathcal{K}}(|\Psi\rangle\langle\Psi|)$.

• The observer of $\mathcal{H}$ does not see Ψ but only

$$\rho = \mathrm{Tr}_{\mathcal{K}}(|\Psi\rangle\langle\Psi|)\,.$$

QUANTUM MECHANICS

(Second version)

1. States

• States = positive, trace-class operators ρ on $\mathcal{H}$ with trace 1, the *density matrices* of $\mathcal{H}$.

$$\rho = \sum_n \lambda_n \, |\Psi_n\rangle\langle\Psi_n|$$

($\lambda_n \geq 0,\ \sum_n \lambda_n = 1$).

2. Observables

• If $\mathcal{H}$ is in a state ρ, if the observable H admits the spectral measure $A \mapsto \mathbb{1}_A(H)$, then the probability that the measurement of H lies in A is

$$\mathrm{Tr}(\rho \, \mathbb{1}_A(H)) \, .$$

3. Time evolution

• If ρ_0 is the state at time 0, it becomes

$$\rho_t = U_t \, \rho_0 \, U_t^*$$

at time t.

3'. Noisy channels

What is the most general transformation for a state?

$$\rho \mapsto \rho \otimes \omega \mapsto U(\rho \otimes \omega)U^* \mapsto \mathrm{Tr}_{\mathcal{K}} \left(U(\rho \otimes \omega)U^* \right) .$$

It is a *completely positive map* on $\mathcal{L}_1(\mathcal{H})$

$$\mathcal{L}(\rho) = \sum_i L_i \, \rho \, L_i^*$$

with

$$\sum_i L_i^* L_i = I$$

(Stinespring, Kraus).

3". Time-dependant case

General time evolution of an open quantum system = $(P_t)_{t \geq 0}$ semigroup of completely positive maps

$$P_t = e^{tL}$$

with

$$L(\rho) = -i[H,\rho] - \frac{1}{2}\sum_n (L_n L_n^* \rho + \rho L_n L_n^* - 2L_n^* \rho L_n)$$

(Lindblad).

QUANTUM PROBABILITY

1. Setup

• A *quantum probability space* is $(\mathcal{H}, \rho)$

$\big[\neq (\Omega, \mathcal{F}, P)\big]$.

• A *quantum random variable* is a self-adjoint operator X on $\mathcal{H}$

$\big[\neq$ a measurable function $X \ : \ \Omega \to \mathbb{R}\big]$.

• The distribution of X in the state ρ is the probability measure μ on $\mathbb{R}$ given by

$$\mu(A) = \mathrm{Tr}(\rho\, \mathbb{1}_A(X))\,.$$

Or else

$$\int f(x)\, d\mu(x) = \mathrm{Tr}(\rho f(X))$$

$$\widehat{\mu}(t) = \mathrm{Tr}(\rho\, e^{itX})\,.$$

$\big[\neq \mu = X \circ P\big]$.

2. Connecting to classical theory

• When one is given *a single* observable X it is the same situation as classical theory

– if X is an observable, one can represent it as a multiplication operator on some $(\Omega, \mathcal{F}, P)$,

– if X is a classical random variable then take $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$ and $\mathcal{M}_X$.

• It holds the same for any family $(X_i)_{i \in I}$ of commuting observables.

• The difference lies when considering *non-commuting* observables on $\mathcal{H}$. Each one is like a classical random variable, but on its own space.

STOPPING TIMES

1. Setup

• A Hilbert space $\mathcal{H}$, a filtration of sub-Hilbert spaces $(\mathcal{H}_t)_{t\in\mathbb{R}^+}$, with associated projectors $\mathbb{E}_t$.

• A (quantum) stopping time T is an increasing family of orthogonal projectors $\mathbb{1}_{T\leq t}$ such that

$$\mathbb{E}_u \mathbb{1}_{T\leq t} = \mathbb{1}_{T\leq t}\, \mathbb{E}_u$$

for all $u \geq t$.

• Set $\mathbb{1}_{T=\infty} = I - \lim_{t\to\infty} \mathbb{1}_{T\leq t}$.

• Equivalently T can be seen as generalized observable with spectrum $\subset \mathbb{R}^+ \cup \{+\infty\}$.

• Physically, T is not an observable. It cannot be measured directly. Only $\mathbb{1}_{T\leq t}$ can be.

RESULTS

• Most definitions and properties can be extended.

$$\mathcal{H}_T = \{f \in \mathcal{H};\ \mathbb{1}_{T\le t} f \in \mathcal{H}_t, \forall t\}\,.$$

$$\mathcal{H}_{T-} = \overline{\mathrm{span}}\{\mathbb{1}_{T>t} f;\ f \in \mathcal{H}_t, t \in \mathbb{R}^+\} \cup \mathcal{H}_0\,.$$

• $S \le T$ if $\mathbb{1}_{T\le t} \ge \mathbb{1}_{S\le t}$ for all t.

• One can also define $S < T$ (more tricky), predictable stopping times.

Theorem – *On the Fock space Φ, for every predictable quantum stopping time T we have $\Phi_T = \Phi_{T-}$.*

• This generalizes the following remark:

Every normal martingale ($< X, X >_t = t$) with the predictable representation property has a quasi left-continuous natural filtration.

It shows that this property is not really probabilistic, but more intrinsic to the Fock space structure (continuous tensor product of Hilbert spaces).

• Results on strong Markov property for quantum processes, quantum Dirichlet problem,... have been obtained.

STOPPING QUANTUM PROCESSES

• One of the main problems: given a family of operators $(X_t)_{t\in\mathbb{R}^+}$ (for example an observable evolving with time), and a (finite) stopping time T: how to define X_T?

• In the classical case (for T discrete)

$$X_T = \sum_i X_{t_i} 1\!\!1_{T=t_i}$$

For a general stopping time, pass to the limit on discrete approximation of T, that is pass to the limit on

$$\sum_i X_{t_{i+1}} 1\!\!1_{T\in[t_i,t_{i+1}[}\,.$$

• In the quantum case, what shall we consider?

$$\sum_i X_{t_{i+1}} 1\!\!1_{T\in[t_i,t_{i+1}[}\,.$$

$$\sum_i 1\!\!1_{T\in[t_i,t_{i+1}[} X_{t_{i+1}}\,.$$

$$\sum_i 1\!\!1_{T\in[t_i,t_{i+1}[} X_{t_{i+1}} 1\!\!1_{T\in[t_i,t_{i+1}[}\,.$$

something else?

• Condition for the convergence of the two first forms are workable. But the resulting object is not satisfactory (not preserving self-adjointness, not adapted,...).

• The third one has some advantages: it preserves self-adjointness, it is the only one to have adaptedness properties (w.r.t. $\mathcal{H}_T$), ... But finding a general condition for the convergence is a completely open problem.

• It is maybe a too strong projection (like conditionning w.r.t. $\sigma(T)$ instead of $\mathcal{F}_T$ in classical probability).

• A surprising computation:

$$W_{T_n} = -2T_n \, .$$

PHYSICAL EXAMPLES

• Yet no true applications in quantum physics.

• Most examples are as follows: under some quantum evolution a particular observable evolves: $(N_t)_{t\in\mathbb{R}^+}$ and gives rise to a *commutative* family of self-adjoint operators. Namely, a classical stochastic process (often a Markov process). The natural stopping times of this process can be considered in the general non-commutative setup. They give rise to quantum stopping times.

• But purely non-commutative examples are lacking. The naive definitions fail.

• For example, take the free particle. It undergoes the Schrödinger evolution driven by the Laplacian. Let Q_t be the position observable at time t. Let us try to define the *entrance time T in $\mathbb{R}^+$* for the particle.

We must have

$$(T > t) \subset \cap_{s\leq t}(Q_s \subset \mathbb{R}^-)\,.$$

But

Theorem – *For all $s < t$ we have*

$$(Q_s \subset \mathbb{R}^-) \cap (Q_t \subset \mathbb{R}^-) = \{0\}\,.$$

Idea: By unitary transforms $s = 0$. If ϕ is a wave function with support in $\mathbb{R}^-$ then its Fourier transform $\widehat{\phi}$ has support on all $\mathbb{R}$. But ϕ is the wave function for the position of the particle and $\widehat{\phi}$ is the wave function for the speed of the particle. As a consequence, immediately after 0, ϕ_t is spread all over $\mathbb{R}$, and it stays so. Hence $\phi \notin (Q_t \subset \mathbb{R}^-)$.