3.THE ALGEBRA OF CANONICAL COMMUTATION RELATIONS

3.1 Fock spaces

In classical mechanics a point system is characterized by its coordinates $Q_i(t)$ and impulsion $P_i(t)$, $i = 1 \dots n$. In the Hamiltonian description of motion equations there exists a fundamental function $H(P, Q)$ of motion, which describes the system and satisfies Euler-Lagrange equations:

$$\frac{\partial H}{\partial P_i} = \dot{Q}_i \ , \quad \frac{\partial H}{\partial Q_i} = -\dot{P}_i \ .$$

If $f(P, Q)$ is a functional of the trajectory, we then have the evolution equation

$$\frac{df}{dt} = \frac{\partial f}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial f}{\partial Q}\frac{\partial Q}{\partial t}$$

or else

$$\frac{df}{dt} = \{f, H\}$$

where $\{g, f\}$ denote the *Poisson bracket* of f by h:

$$\{g, h\} = \frac{\partial g}{\partial P}\frac{\partial h}{\partial Q} - \frac{\partial g}{\partial Q}\frac{\partial h}{\partial P} \ .$$

In particular we have

$$\begin{aligned}\{P_i, P_j\} &= \{Q_i, Q_j\} = 0 \\ \{P_i, Q_j\} &= \delta_{ij} \ .\end{aligned}$$

It happens that it is not exactly the definitions of the P_i and Q_i which is important, but the relations above. Indeed, a change of coordinates $P'(P, Q)$, $Q'(P, Q)$ will give rise to the same motion equations if and only if P' and Q' satisfy the relations above.

In quantum mechanics it is essentially the same situation. We have a self-adjoint operator H (the Hamiltonian) which describes all the evolution of the system via the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = H \ \psi(t) \ .$$

There are also self-adjoint operators Q_i, P_i which represent the position and the impulsion of the system and which evolve following

$$\begin{aligned}Q_i(t) &= e^{itH} Q_i e^{itH} \\ P_i(t) &= e^{-itH} P_i e^{itH} \ .\end{aligned}$$

Thus any observable A defined from P and Q satisfies the evolution equation

$$\frac{d}{dt}A(t) = -\frac{i}{\hbar}[A(t), H]$$

where $[\cdot, \cdot]$ denotes the commutator.

But the operators P_i, Q_i satisfy the relation

$$[P_i, P_j] = [Q_i, Q_j] = 0$$
$$[Q_i, P_j] = i\hbar\delta_{ij} I \ .$$

Once again, it is not the choice of the representations of P_i and Q_i which is important, it is the relations above. It is called *commutation relation.*

In quantum field theory we have an infinite number of degrees of freedom. The operators position and impulsion are indexed by $\mathbb{R}^3$ (for example): we have a field of operators and the relations

$$[P(x), P(y)] = [Q(x), Q(y)] = 0$$
$$[Q(x), P(y)] = i\hbar\delta(x - y) I \ .$$

If one puts $a(x) = \frac{1}{\sqrt{2}}(Q(x) + iP(x))$ and $a^*(x) = \frac{1}{\sqrt{2}}(Q(x) - iP(x))$ then $a(x)$ and $a^*(x)$ are mutually adjoint and satisfy the *canonical commutation relations* (CCR)

$$[a(x), a(y)] = [a^*(x), a^*(y)] = 0$$
$$[a(x), a^*(y)] = \hbar\delta(x - y) I \ .$$

Actually it happens that these equations are valid only for a particular family of particles: the bosons (photons, mesons, gravitons,...). There is another family of particles: the fermions (electrons, muons, neutrinos, protons, neutrons, baryons,...) for which the correct relations are the *canonical anticommutation relations* (CAR)

$$\{b(x), b(y)\} = \{b^*(x), b^*(y)\} = 0$$
$$\{b(x), b^*(y)\} = \hbar\delta(x - y) I$$

where $\{A, B\} = AB + BC$ is the *anticommutator* of operators.

A natural problem, which has given rise to a huge literature, is to find concrete realisations of these relations. Let us see the simplest example: find two self-adjoint operators P and Q such that

$$QP - PQ = i\hbar I \ .$$

In a certain sense there is only one solution. This solution is realized on $L^2(\mathbb{R})$ by $Q = x$ (multiplication by x) and $P = i\hbar\frac{d}{dx}$. It is the *Schrödinger representation* of the CCR. But in full generality this problem is not well-posed. We need to be able to define the operators PQ and QP on good common domains. One can construct pathological counter-examples (Reed-Simon).

The problem is well-posed if we transform it in terms of bounded operators. Let $W_{x,y} = e^{-i(xP - yQ)}$ and $W_z = W_{x,y}$ when $z = x + iy \in \mathbb{C}$. We then have the *Weyl commutation relations*

$$W_z W_{z'} = e^{-i\Im\langle z, z'\rangle} W_{z+z'} \ .$$

Posed in these terms the problem has only one solution: the symmetric Fock space (Stone-von Neumann theorem).

The anticommutation relations as they are written with $b(x)$ and $b^*(x)$ have a more direct solution for $b(x)$ and $b^*(x)$ have to be bounded. We will come back to that later.

The importance of Fock space comes from the fact they give an easy realization of the CCR and CAR. They are also a natural tool for quantum field theory, second quantization... (all sorts of physical important notions that we will not develop here). The physical ideal around Fock spaces is the following. If $\mathcal{H}$ is the Hilbert space describing a system of one particle, then $\mathcal{H}\otimes\mathcal{H}$ describes a system consisting of two particles of the same type. The space $\mathcal{H}^{\otimes n}=\mathcal{H}\otimes\cdots\otimes\mathcal{H}$, n-fold, describes n such particles. Finally the space $\oplus_{n\in\mathbb{N}}\mathcal{H}^{\otimes n}$ describes a system where there can be any number of such particles which can disappear (annihilate) or be created. But depending on the type of particles (bosons or fermions) we deal with, there are some symmetries which force to look at certain subspaces of $\oplus_n\mathcal{H}^{\otimes n}$. We did not aim to describe the physics behind Fock spaces (we are not able to), but we just wanted to motivate them. Let us come back to mathematics.

Let $\mathcal{H}$ be a complex Hilbert space. For any integer $n\geq 1$ put

$$\mathcal{H}^{\otimes n}=\mathcal{H}\otimes\cdots\otimes\mathcal{H}$$

the *n-fold tensor product* of $\mathcal{H}$. That is, the Hilbert space obtained after completion of the pre-Hilbert space of finite linear combinations of elements of the form $u_1\otimes\cdots\otimes u_n$, with the scalar product

$$\langle u_1\otimes\cdots\otimes u_n, v_1\otimes\cdots\otimes v_n\rangle=\langle u_1,v_1\rangle\cdots\langle u_n,v_n\rangle\ .$$

For $u_1,\ldots,u_n\in\mathcal{H}$ we define the *symmetric tensor product*

$$u_1\circ\cdots\circ u_n=\frac{1}{n!}\sum_{\sigma\in S_n}u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}\ ,$$

where S_n is the group of permutations of $\{1,2\ldots n\}$, and the *antisymmetric tensor product*

$$u_1\wedge\cdots\wedge u_n=\frac{1}{n!}\sum_{\sigma\in S_n}\varepsilon_\sigma u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}\ ,$$

where ε_σ is the signature of the permutation σ.

The closed subspace of $\mathcal{H}^{\otimes n}$ generated by the $u_1\circ\cdots\circ u_n$ (*resp.* $u_1\wedge\cdots\wedge u_n$) is denoted $\mathcal{H}^{\circ n}$ (*resp.* $\mathcal{H}^{\wedge n}$). It is called the n-fold symmetric (*resp.* antisymmetric) tensor product of $\mathcal{H}$.

Sometimes, when the notation is clear, one denotes by $\mathcal{H}_n$ the space $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\circ n}$ or $\mathcal{H}^{\wedge n}$, and one calls it the *n-th chaos* of $\mathcal{H}$. In any of the three cases we put

$$\mathcal{H}_0=\mathbb{C}\ .$$

The element $1\in\mathbb{C}=\mathcal{H}_0$ plays an important role. One denotes it by $\mathbb{1}$ (usually by Ω in the literature) and one calls it the *vacuum vector.*

If one computes

$$\langle u_1\wedge\cdots\wedge u_n, v_1\wedge\cdots\wedge v_n\rangle=\frac{1}{(n!)^2}\sum_{\sigma,\tau\in S_n}\varepsilon_\sigma\varepsilon_\tau\langle u_{\sigma(1)},v_{\tau(1)}\rangle\cdots\langle u_{\sigma(n)},v_{\tau(n)}\rangle$$

one finds

$$\frac{1}{n!}\det(\langle u_i,v_j\rangle)_{ij}.$$

In order to remove the $n!$ factor we put a scalar product on $\mathcal{H}^{\wedge n}$ which is different from the one induced by $\mathcal{H}^{\otimes n}$, namely:

$$\langle u_1 \wedge \cdots \wedge u_n, v_1 \wedge \cdots \wedge v_n \rangle_\wedge = \det(\langle u_i, v_j \rangle)_{ij} \ .$$

This way, we have

$$\|u_1 \wedge \cdots \wedge u_n\|_\wedge^2 = n! \|u_1 \wedge \cdots \wedge u_n\|_\otimes^2 \ .$$

In the same way, on $\mathcal{H}^{\circ n}$ we put

$$\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle_\circ = \mathrm{per}(\langle u_i, v_j \rangle)_{ij} \ ,$$

where per denotes the *permanent* of the matrix (that is, the determinant without the minus signs). This way we get

$$\|u_1 \circ \cdots \circ u_n\|_\circ^2 = n! \|u_1 \circ \cdots \circ u_n\|_\otimes^2 \ .$$

We call *free (or full) Fock space* over $\mathcal{H}$ the space

$$\Gamma_f(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} \ .$$

We call *symmetric (or bosonic) Fock space* over $\mathcal{H}$ the space

$$\Gamma_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circ n} \ .$$

We call *antisymmetric (or fermionic) Fock space* over $\mathcal{H}$ the space

$$\Gamma_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\wedge n}.$$

It is understood that in the definition of $\Gamma_f(\mathcal{H})$, $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$ each of the spaces $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\circ n}$ or $\mathcal{H}^{\wedge n}$ is equipped with its own scalar product $\langle \cdot, \cdot \rangle_\otimes$, $\langle \cdot, \cdot \rangle_\circ$ or $\langle \cdot, \cdot \rangle_\wedge$. In other words, the elements of $\Gamma_f(\mathcal{H})$ (*resp.* $\Gamma_s(\mathcal{H})$, $\Gamma_a(\mathcal{H})$) are those series $f = \sum_{n \in \mathbb{N}} f_n$ such that $f_n \in \mathcal{H}^{\otimes n}$ (*resp.* $\mathcal{H}^{\circ n}$, $\mathcal{H}^{\wedge n}$) for all n and

$$\|f\|^2 = \sum_{n \in \mathbb{N}} \|f_n\|_\varepsilon^2 < \infty$$

for $\varepsilon = \otimes$ (*resp.* $\circ$, $\wedge$).

If one want to write everything in terms of the usual tensor norm, we simply have that an element $f = \sum_{n \in \mathbb{N}} f_n$ is in $\Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$) if $f_n \in \mathcal{H}^{\circ n}$ (*resp.* $\mathcal{H}^{\wedge n}$) for all n and

$$\|f\|^2 = \sum_{n \in \mathbb{N}} n! \|f_n\|_\otimes^2 < \infty \ .$$

The simplest case is obtained by taking $\mathcal{H} = \mathbb{C}$, this gives $\Gamma_s(\mathbb{C}) = \ell^2(\mathbb{N})$. If $\mathcal{H}$ is of finite dimension n then $\mathcal{H}^{\wedge m} = 0$ for $m > n$ and thus $\Gamma_a(\mathcal{H})$ is of finite dimension 2^n; this is never the case for $\Gamma_s(\mathcal{H})$.

In physics, one usually consider bosonic or fermionic Fock spaces over $\mathcal{H} = L^2(\mathbb{R}^3)$.

In quantum probability it is the space $\Gamma_s(L^2(\mathbb{R}^+))$ which is important for quantum stochastic calculus (we will meet this space during the second semester).

We now only consider symmetric Fock spaces $\Gamma_s(\mathcal{H})$.

For a $u \in \mathcal{H}$ one notes that $u \circ \cdots \circ u = u \otimes \cdots \otimes u$. The *coherent vector* (or *exponential vector*) associated to u is

$$\varepsilon(u) = \sum_{n \in \mathbb{N}} \frac{u^{\otimes n}}{n!}$$

so that

$$\langle \varepsilon(u), \varepsilon(v) \rangle = e^{\langle u,v \rangle}$$

in $\Gamma_s(\mathcal{H})$.

Proposition 3.1 – *The vector space $\mathcal{E}$ of finite linear combinations of coherent vectors, is dense in $\Gamma_s(\mathcal{H})$.*

Every finite family of coherent vectors is linearly independent.

Proof

Let us prove the independence. Let $u_1 \ldots u_n \in \mathcal{H}$. The set

$$E_{i,j} = \big\{ u \in \mathcal{H}; \langle u, u_i \rangle \neq \langle u, u_j \rangle \big\},$$

for $i \neq j$, is open and dense in $\mathcal{H}$. Thus the set $\bigcap_{i,j} E_{i,j}$ is non empty. Thus there exists a $v \in \mathcal{H}$ such that the $\theta_j = \langle v, u_j \rangle$ are two by two different. Now, if $\sum_{i=1}^n \alpha_i \, \varepsilon(u_i) = 0$ this implies that

$$0 = \Big\langle \varepsilon(zv), \sum_{i=1}^n \alpha_i \varepsilon(u_i) \Big\rangle = \sum_{i=1}^n \alpha_i e^{z\theta_i}$$

for all $z \in \mathbb{C}$. Thus the α_i all vanish and the family $\big\{\varepsilon(u_1) \ldots \varepsilon(u_n)\big\}$ is free.

In order to show the density, we first notice that the set $\{u \circ \cdots \circ u, u \in \mathcal{H}\}$ is total in $\Gamma_s(\mathcal{H})$ for

$$u_1 \circ \cdots \circ u_n = \sum_{\varepsilon_i = \pm 1} (\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n)^{\circ n} \; .$$

But $u^{\circ n} = \frac{d^n}{dt^n} \varepsilon(tu)_{|t=0}$. This gives the result. ■

Corollary 3.2 – *If $S \subset \mathcal{H}$ is dense subset, then the space $\mathcal{E}(S)$ generated by the $\varepsilon(u)$, $u \in S$, is dense in $\Gamma_s(\mathcal{H})$.*

Proof

We have

$$\|\varepsilon(u) - \varepsilon(v)\|^2 = e^{\|u\|^2} + e^{e\|v\|^2} - 2\Re e^{\langle u,v \rangle} \; .$$

Thus the mapping $u \mapsto \varepsilon(u)$ is continuous. We now conclude easily from Proposition 3.1. ■

Theorem 3.3 – *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two Hilbert spaces. Then there exists a unique unitary isomorphism*

$$\begin{aligned} U : \Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) &\longrightarrow \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2) \\ \varepsilon(u \oplus v) &\longmapsto \varepsilon(u) \otimes \varepsilon(v). \end{aligned}$$

Proof

The space $\mathcal{E}(\mathcal{H}_i)$ is dense in $\Gamma_s(\mathcal{H}_i)$, $i = 1, 2$, and $\{\varepsilon(u) \otimes \varepsilon(v)\ ;\ u \in \mathcal{H}_1,\ v \in \mathcal{H}_2\}$ is total in $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$. Furthermore, we have

$$\begin{aligned}\langle \varepsilon(u \oplus v), \varepsilon(u' \oplus v')\rangle &= e^{\langle u \oplus v, u' \oplus v'\rangle} \\ &= e^{\langle u, u'\rangle + \langle v, v'\rangle} \\ &= e^{\langle u, u'\rangle} e^{\langle v, v'\rangle} \\ &= \langle \varepsilon(u), \varepsilon(u')\rangle \langle \varepsilon(v), \varepsilon(v')\rangle \\ &= \langle \varepsilon(u) \otimes \varepsilon(v), \varepsilon(u') \otimes \varepsilon(v')\rangle\ .\end{aligned}$$

Thus the mapping U is isometric. One concludes easily. ■

An example we will follow all along this chapter: the space $\Gamma_s(\mathbb{C})$. It is equal to $\ell^2(\mathbb{N})$ but it can be advantageously interpreted as $L^2(\mathbb{R})$. Indeed, let U be the mapping from $\Gamma_s(\mathbb{C})$ to $L^2(\mathbb{R})$ which maps $\varepsilon(z)$ to the function $f_z(x) = (2\pi)^{-1/4} e^{zx - s^2/2 - x^2/4}$. It is easy to see that U extends to a unitary isomorphism. We will come back to this example later.

There is an interesting characterization of the space $\Gamma_s(\mathcal{H})$ which says roughly that $\Gamma_s(\mathcal{H})$ is the exponential of $\mathcal{H}$. Idea which is already confirmed by Theorem 3.3.

Theorem 3.4 – *Let $\mathcal{H}$ be a separable Hilbert space. If K is a Hilbert space such that there exists a mapping*

$$\begin{aligned}\lambda : \mathcal{H} &\longrightarrow K \\ u &\longmapsto \lambda(u)\end{aligned}$$

satisfying

i) $\langle \lambda(u), \lambda(v)\rangle = e^{\langle u, v\rangle}$ *for all* $u, v \in \mathcal{H}$

ii) $\{\lambda(u)\ ;\ u \in \mathcal{H}\}$ *is total in* K.

Then there exists a unique unitary isomorphism

$$\begin{aligned}U : K &\longrightarrow \Gamma_s(\mathcal{H}) \\ \lambda(u) &\longmapsto \varepsilon(u).\end{aligned}$$

Proof

Clearly U is isometric and maps a dense subspace onto a dense subspace. ■

It is useful to stop a moment in order to describe $\Gamma_s(\mathcal{H})$ when $\mathcal{H}$ is of the form $L^2(E, \mathcal{E}, m)$. We are going to see that if $(E, \mathcal{E}, m)$ is a measured, non atomic, σ-finite, separable measured space then $\Gamma(L^2(E, \mathcal{E}, m))$ can be written as $L^2(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$ for an explicit measured space $(\mathcal{P}, \mathcal{E}_{\mathcal{P}}, \mu)$.

If $\mathcal{H} = L^2(E, \mathcal{E}, m)$, then $\mathcal{H}^{\otimes n}$ interprets naturally as $L^2(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ and $\mathcal{H}^{\circ n}$ interprets as $L^2_{sym}(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$ the space of symmetric, square integrable functions on E^n.

If $f(x_1 \dots x_n)$ is a n-variable *symmetric* function on E then

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for all $\sigma \in S_n$. If the x_i are two by two different we thus can see f as a function on the *set* $\{x_1, \dots, x_n\}$. But as m is non atomic, almost all the $(x_1 \dots x_n) \in E^n$ satisfy $x_i \neq x_j$ once $i \neq j$. An element of $\Gamma_s(\mathcal{H})$ is of the form $f = \sum_{n \in I\!N} f_n$ where each f_n is a function on the n-element subsets of E. Thus f can be seen as a function on the finite subsets of E.

More rigorously, let $\mathcal{P}$ be the set of finite subsets of E. Then $\mathcal{P} = \cup_{n \in I\!N} \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of n-elements subsets of E. Let f_n be an element of $L^2_{sym}(E^n, \mathcal{E}^{\otimes n}, m^{\otimes n})$, we define f on $\mathcal{P}$ by

$$\begin{cases} f(\sigma) = 0 & \text{if } \sigma \in \mathcal{P} \text{ and } |\sigma| = n \ , \\ f(\{x_1, \dots, x_n\}) = f_n(x_1, \dots, x_n) & \text{otherwise.} \end{cases}$$

Let $\mathcal{E}_\mathcal{P}$ be the smallest σ-field on $\mathcal{P}$ which makes all these functions measurable on $\mathcal{P}$.

Let $\Delta_n \subset E^n$ be the set of $(x_1 \dots x_n)$ such that $x_i \neq x_j$ once $i \neq j$. By the non-atomicity of m, we have $m(E^n \backslash \Delta^n) = 0$. For $F \in \mathcal{E}_\mathcal{P}$ we put

$$\mu(F) = 1\!\!1_\emptyset(F) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Delta_n} 1\!\!1_{F \cap \mathcal{P}_n}(x_1, \dots, x_n) dm(x_1) \cdots dm(x_n) \ .$$

For example, if $E = I\!R$ with the Lebesgue structure, then $\mathcal{P}_n$ can be identified to the increasing simplex $\Sigma_n = \{x_1 < \cdots < x_n \in I\!R\} \subset I\!R^n$. Thus $\mathcal{P}^n$ inherits the Lebesgue measure from $I\!R^n$.

The measure μ we have defined is σ-finite, it possesses only one atom: $\{\emptyset\}$ which has mass 1. We call $(\mathcal{P}, \mathcal{E}_\mathcal{P}, \mu)$ the *symmetric measure space* over $(E, \mathcal{E}, m)$. This construction is due to Guichardet.

For all $u \in L^2(E, \mathcal{E}, m)$ one defines by π_u the element of $L^2(\mathcal{P}, \mathcal{E}_\mathcal{P}, \mu)$ which satisfies

$$\pi_u(\sigma) = \begin{cases} 1 & \text{if } \quad \sigma = \emptyset \\ \prod_{s \in \sigma} u(s) & \text{otherwise} \end{cases}$$

for all $\sigma \in \mathcal{P}$.

Theorem 3.4 – *The mapping $\pi_u \longmapsto \varepsilon(u)$ extends to a unitary isomorphism from $L^2(\mathcal{P}, \mathcal{E}_\mathcal{P}, \mu)$ onto $\Gamma_s(L^2(E, \mathcal{E}, m))$.*

Proof

Clearly $\langle \pi_u, \pi_v \rangle = e^{\langle u, v \rangle} = \langle \varepsilon(u), \varepsilon(v) \rangle$. The set of functions π_u is total in $L^2(\mathcal{P}, \mathcal{E}_\mathcal{P}, \mu)$. One concludes easily. ■

3.2 Creation and annihilation operators

We now come back to general symmetric and antisymmetric Fock spaces $\Gamma_s(\mathcal{H})$ and $\Gamma_a(\mathcal{H})$.

For $u \in \mathcal{H}$ we define the following operators:

$$\begin{array}{rccc}
a^*(u): & \mathcal{H}^{\circ n} & \longrightarrow & \mathcal{H}^{\circ(n+1)} \\
& u_1 \circ \cdots \circ u_n & \longmapsto & u \circ u_1 \circ \cdots \circ u_n \\[2ex]
b^*(u): & \mathcal{H}^{\wedge n} & \longrightarrow & \mathcal{H}^{\wedge(n+1)} \\
& u_1 \wedge \cdots \wedge u_n & \longmapsto & u \wedge u_1 \wedge \cdots \wedge u_n \\[2ex]
a(u): & \mathcal{H}^{\circ n} & \longrightarrow & \mathcal{H}^{\circ(n-1)} \\
& u_1 \circ \cdots \circ u_n & \longmapsto & \sum_{i=1}^n \langle u, u_i\rangle u_1 \circ \cdots \circ \widehat{u_i} \circ \cdots \circ u_n \\[2ex]
b(u): & \mathcal{H}^{\circ n} & \longrightarrow & \mathcal{H}^{\circ(n-1)} \\
& u_1 \wedge \cdots \wedge u_n & \longmapsto & \sum_{i=1}^n (-1)^i \langle u, u_i\rangle u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_n \ .
\end{array}$$

These operators are respectively called *bosonic creation operator*, *fermionic creation operator*, *bosonic annihilation operator* and *fermionic annihilation operator*.

Notice that $a^*(u)$ and $b^*(u)$ depend linearly on u, whereas $a(u)$ and $b(u)$ depend antilinearly on u. Actually, one often finds in the literature notations with "bras" and kets": $a^*_{|u\rangle}$, $b^*_{|u\rangle}$, $a_{\langle u|}$, $b_{\langle u|}$.

Note that

$$\begin{aligned}
a^*(u)\mathbb{1} &= b^*(u)\mathbb{1} = u \\
a(u)\mathbb{1} &= b(u)\mathbb{1} = 0 \ .
\end{aligned}$$

All the operators above extend to the space $\Gamma_s^f(\mathcal{H})$(*resp.* $\Gamma_a^f(\mathcal{H})$) of finite sums of chaos that is those $f = \sum_{n\in\mathbb{N}} f_n \in \Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$) such that only a finite number of f_n do not vanish. This subspace is dense in the corresponding Fock space. It is included in the domain of the operators $a^*(u)$, $b^*(u)$, $a(u)$, $b(u)$ (defined as operators on $\Gamma_s(\mathcal{H})$ (*resp.* $\Gamma_a(\mathcal{H})$)), and it is stable under their action. On this space we have the following relations:

$$\begin{aligned}
\langle a^*(u)f, g\rangle &= \langle f, a(u)g\rangle \\
[a(u), a(v)] &= [a^*(u), a^*(v)] = 0 \\
[a(u), a^*(v)] &= \langle u, v\rangle I \\
\langle b^*(u)f, g\rangle &= \langle f, b(u)g\rangle \\
\{b(u), b(v)\} &= \{b^*(u), b^*(v)\} = 0 \\
\{b(u), b^*(v)\} &= \langle u, v\rangle I
\end{aligned}$$

In other words, when restricted to $\Gamma_s^f(\mathcal{H})$ (*resp.* $\Gamma_a^f(\mathcal{H})$) the operators $a(u)$ and $a^*(u)$ (*resp.* $b(u)$ and $b^*(u)$) are mutually adjoint and they satisfy the *CCR* (*resp.* *CAR*).

Proposition 3.6 – *For all $u \in \mathcal{H}$ we have*

i) $b^*(u)^2 = 0$,

ii) $\|b(u)\| = \|b^*(u)\| = \|u\|$.

Proof

The anticommutation relation $\{b^*(u), b^*(u)\} = 0$ means $2b^*(u)b^*(u) = 0$, this gives *i)*.

We have

$$\begin{aligned} b^*(u)b(u)b^*(u)b(u) &= b^*(u)\{b(u), b^*(u)\}b(u) \\ &= \|u\|^2 b^*(u)b(u) \ . \end{aligned}$$

Thus

$$\begin{aligned} \|b(u)\|^4 = \|b^*(u)b(u)b^*(u)b(u)\| &= \|u\|^2 \|b^*(u)b(u)\| \\ &= \|u\|^2 \|b(u)\|^2 \ . \end{aligned}$$

As the operator $b(u)$ is null if and only if $u = 0$ we easily deduce that $\|b(u)\| = \|u\|$. ■

The identity *i)* expresses the so-called *Pauli exclusion principle*: "One cannot have together two fermionic particles in the same state".

The bosonic case is less simple for the operators $a^*(u)$ and $a(u)$ are never bounded. Indeed, we have $a(u)v^{\circ n} = n\langle u, v\rangle v^{\circ(n-1)}$, thus the coherent vectors are in the domain of $a(u)$ and

$$a(u)\varepsilon(v) = \langle u, v\rangle \varepsilon(v) \ .$$

In particular

$$\begin{aligned} \sup_{\|h\|=1} \|a(u)h\| &\geq \sup_{v \in \mathcal{H}} \|a(u)e^{-\|v\|^2/2}\varepsilon(v)\| \\ &= \sup_{v \in \mathcal{H}} |\langle u, v\rangle| = +\infty \ . \end{aligned}$$

Thus $a(u)$ is not bounded.

The action of $a^*(u)$ can be also be made explicit. Indeed, we have

$$a^*(u)v^{\circ n} = u \circ v \circ \cdots \circ v = \frac{d}{d\varepsilon}(u + \varepsilon v)^{\circ n}_{|\varepsilon=0} \ .$$

Thus $\varepsilon(v)$ is in the domain of $a^*(u)$ and

$$a^*(u)\varepsilon(v) = \frac{d}{d\varepsilon}\varepsilon(u + \varepsilon v)_{|\varepsilon=0} \ .$$

The operators $a(u)$ and $a^*(u)$ are thus closable (they have a densely defined adjoint). We extend them by closure, while keeping the same notations $a(u)$, $a^*(u)$.

Proposition 3.7 – *We have $a^*(u) = a(u)^*$.*

Proof

On $\Gamma_s^f(\mathcal{H})$ we have $\langle f, a(u)g\rangle = \langle a^*(u)f, g\rangle$. We extend this relation to $f \in \operatorname{Dom} a^*(u)$. The mapping $g \mapsto \langle f, a(u)g\rangle$ is thus continuous and $f \in \operatorname{Dom} a(u)^*$. We have proved that $a^*(u) \subset a(u)^*$.

Conversely, if $f \in \operatorname{Dom} a(u)^*$ and if $h = a(u)^* f$. We decompose f and h in chaoses: $f = \sum_n f_n$ and $h = \sum_n h_n$. We have $\langle f, a(u)g\rangle = \langle h, g\rangle$ for all $g \in \Gamma_s^f(\mathcal{H})$.

Thus, taking $g \in \mathcal{H}^{\circ n}$ we get $\langle f_{n-1}, a(u)g\rangle = \langle h_n, g\rangle$ that is, $\langle a^*(u)f_{n-1}, g\rangle = \langle h_n, g\rangle$. This shows that $h_n = a^*(u)f_{n-1}$ This way $\sum_n \|a^*(u)f_n\|^2$ is finite, f belongs to $\mathrm{Dom}\, a^*(u)$ and $a^*(u)f = a(u)^*f$. ■

In physics, the space $\mathcal{H}$ is often $L^2(\mathbb{R}^3)$. An element h_n of $\mathcal{H}^{\circ n}$ is thus a symmetric function of n variables on $\mathbb{R}^3$. With our definitions we have

$$\big(a(f)h_n\big)(x_1 \dots x_{n-1}) = \int h_n(x_1 \dots x_{n-1}, x)\overline{f}(x)\; dx$$

and

$$\big(a^*(f)h_n\big)(x_1 \dots x_{n+1}) = \sum_{i=1}^{n+1} h_n(x_1 \dots \hat{x}_i \dots x_{n+1})f(x_i)\ .$$

But in the physic literature one often use creation and annihilation operators indexed by the points of $\mathbb{R}^3$, instead of the elements of $L^2(\mathbb{R}^3)$. One can find there $a(x)$ and $a^*(x)$ formally defined by

$$a(f) = \int \overline{f}(x)a(x)\; dx$$

$$a^*(f) = \int f(x)a^*(x)\; dx$$

with

$$\big(a(x)h_n\big)(x_1 \dots x_{n-1}) = h_n(x_1 \dots x_{n-1}, x)$$

$$\big(a^*(x)h_n\big)(x_1 \dots x_{n+1}) = \sum_{i=1}^{n+1} \delta(x - x_i)h_n(x_1 \dots \hat{x}_i \dots x_{n+1})\ .$$

If one comes back to our example $\Gamma_s(\mathbb{C}) \simeq L^2(\mathbb{R})$, we have the creation and annihilation operators $a^*(z)$, $a(z)$, $z \in \mathbb{C}$. They are actually determined by two operators $a^* = a^*(1)$ and $a = a(1)$. They operate on coherent vectors by

$$a \in \varepsilon(z) = z(\varepsilon(z), \quad a^*\varepsilon(z) = \frac{d}{d\varepsilon}\varepsilon(z+\varepsilon)_{|\varepsilon=0}\ .$$

On $L^2(\mathbb{R})$ this gives

$$af_z(x) = zf_z(x) = \Big(\frac{d}{dx} + \frac{x}{2}\Big)f_z(x)$$

$$a^*f_s(x) = \frac{d}{d\varepsilon}f_{z+\varepsilon}(x)_{|\varepsilon=0} = (x-z)f_z(x) = \Big(\frac{x}{2} - \frac{d}{dx}\Big)f_z(x)\ .$$

The operators $Q = a + a^*$ and $P = i(a - a^*)$ are thus respectively represented by the operator x and $2i\frac{d}{dx}$ on $L^2(\mathbb{R})$. That is, the Schrödinger representation of the CCR (with $\hbar = 2$).

To conclude in this section, note that the operator $Q = a+a^*$ is an observable, in the physical sens. If we are given a state on $L^2(\mathbb{R})$, for example the vaccum state $\mathbb{1}$,then the observable Q has a natural probability law. This law isthe one which describes the probabilistic behaviour of the observable Q if one tries to measure it in the state $1\!\!1$. One can also see this law in the following way: the mapping $t \mapsto \,<1\!\!1\,,\, e^{itQ}1\!\!1>$ satisfies the Bochner criterion and is thus the Fourier transform of some probability measure μ.

From the postulats of quantm mechanics, the n-th moment of this law are given by

$$<1\!\!1\,,\, Q^n 1\!\!1>.$$

Passing by the $L^2(I\!\!R)$ interpretation, this quantity equals

$$<f_0\,,\, x^n f_0> = \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2}\, dx$$

that is the n-th moment of the standard normal law $\mathcal{N}(0,1)$. The observable Q, in the vaccum state, follows the $\mathcal{N}(0,1)$ law.

3.3 Second quantization

If one is given an operator A from an Hilbert space $\mathcal{H}$ to another $\mathcal{K}$, it is possible to rise, in a natural way, this operator into an operator $\Gamma(A)$ from $\Gamma_s(\mathcal{H})$ to $\Gamma_s(K)$ (and in a similar way from $\Gamma_a(\mathcal{H})$ to $\Gamma_a(K)$) by putting

$$\Gamma(A)(u_1 \circ \cdots \circ u_n) = Au_1 \circ \cdots \circ Au_n \ .$$

One easily sees that

$$\Gamma(A)\varepsilon(u) = \varepsilon(Au) \ .$$

This operator $\Gamma(A)$ is called the *second quantization* of A.

One must be careful that even if A is bounded operator, $\Gamma(A)$ is not bounded in general. Indeed, if $\|A\| > 1$ then $\Gamma(A)$ is not bounded. But on easily sees that

$$\Gamma(AB) = \Gamma(A)\Gamma(B)$$

and

$$\Gamma(A^*) = \Gamma(A)^*.$$

In particular if A is unitary, then so is $\Gamma(A)$. Even more, if $(U_t)_{t\in I\!\!R}$ is a strongly continuous one parameter group of unitary operators then so is $(\Gamma(U_t)_{t\in I\!\!R})$.∎

In other words, if $U_t = e^{itH}$ for some self-adjoint operator H, then $\Gamma(U_t) = e^{itH'}$ for some self-adjoint operator H'. The operator H' is denoted $\Lambda(H)$ (or sometimes $d\Gamma(H)$ in the litterature) and called the *differential second quantization of* H.

One easily checks that

$$\Lambda(H)\, u_1 \circ \cdots \circ u_n = \sum_{i=1}^{n} u_1 \circ \cdots \circ Hu_i \circ \cdots \circ u_n$$

and $\Lambda(H)1\!\!1 = 0$.

In particular, if $H = I$ we have

$$\Lambda(I)\, u_1 \circ \cdots \circ u_n = n\, u_1 \circ \cdots \circ u_n \ .$$

This operator is called *number operator*.

Proposition 3.7 – *We have*

$$\Lambda(H)\varepsilon(u) = a^*(Hu)\varepsilon(u)\ .$$

Proof

We have $\Lambda(H)u^{\circ n} = n(Hu)\circ u \circ \cdots \circ u$. Thus

$$\Lambda(H)\frac{u^{\circ n}}{n!} = (Hu)\circ \frac{u^{\circ(n-1)}}{(n-1)!} = a^*(Hu)\frac{u^{\circ(n-1)}}{(n-1)!}\ .$$

■

Proposition 3.8 – *For all $u \in \mathcal{H}$, we have*

$$\Lambda(|u\rangle\langle u|) = a^*_{|u\rangle}a_{\langle u|}\ .$$

Proof

Indeed, we have

$$\begin{aligned}\Lambda(|u\rangle\langle u|)\varepsilon(v) &= a^*(\langle u,v\rangle u)\varepsilon(v)\\ &= \langle u,v\rangle a^*(u)\varepsilon(v)\\ &= a^*_{|u\rangle}a_{\langle u|}\varepsilon(v)\ .\end{aligned}$$

■

Coming back to $L^2(I\!\!R)$, there is only one differential second quantization:

$$\Lambda(I) = \Lambda = a^*a\ .$$

We obtain

$$\begin{aligned}\Lambda &= \Big(\frac{x}{2} - \frac{d}{dx}\Big)\Big(\frac{x}{2} + \frac{d}{dx}\Big)\\ &= \frac{x^2}{4} - \frac{d^2}{dx^2} - \frac{1}{2}\end{aligned}$$

that is

$$\Lambda + \frac{1}{2} = \frac{x^2}{4} - \frac{d^2}{dx^2}$$

the Hamiltonian of the one dimensional harmonic oscillator.

Note that Λ is self-adjoint and its law in the vacuum state is just the Dirac mass in 0, for $\Lambda\mathbb{1} = 0$.

3.4 Weyl operators

Let $\mathcal{H}$ be a Hilbert space. Let G be the group of displacements of $\mathcal{H}$ that is,

$$G = \big\{(U,u)\ ;\ \ U \in \mathcal{U}(\mathcal{H}),\ \ u \in \mathcal{H}\big\}\ ,$$

where $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$. This group acts on $\mathcal{H}$ by

$$(U,u)h = Uh + u\ .$$

The composition law of G is thus

$$(U,u)(V,v) = (UV, Uv+u)$$

and in particular

$$(U,u)^{-1} = (U^*, -U^*u) \ .$$

For every $\alpha = (U,u) \in G$ one defines the *Weyl operator* W_α on $\Gamma_s(\mathcal{H})$ by

$$W_\alpha\, \varepsilon(v) = e^{-\|u\|^2/2 - \langle u, Uv\rangle} \varepsilon(Uv+u) \ .$$

In particular

$$W_\alpha W_\beta = e^{-i\Im\langle u, Uv\rangle} W_{\alpha\beta}$$

for all $\alpha = (U,u)$, $\beta = (V,v)$ in G. These are called the *Weyl commutation relations.*

Proposition 3.8 – *The Weyl operators W_α are unitary.*

Proof

We have

$$\begin{aligned}
\langle W_\alpha \varepsilon(k), W_\alpha \varepsilon(\ell)\rangle &= e^{-\|u\|^2 - \langle Uk, u\rangle - \langle u, U\ell\rangle} \langle \varepsilon(Uk+u), \varepsilon(U\ell+u)\rangle \\
&= e^{-\|u\|^2 - \langle Uk, u\rangle - \langle u, U\ell\rangle} e^{\langle Uk+u, U\ell+u\rangle} \\
&= e^{\langle Uk, U\ell\rangle} = e^{\langle k,\ell\rangle} = \langle \varepsilon(k), \varepsilon(\ell)\rangle \ .
\end{aligned}$$

Thus W_α extends to an isometry. But we furthermore have

$$\begin{aligned}
W_\alpha W_{\alpha^{-1}} &= e^{-i\Im\langle u, -UU^*u\rangle} W_{\alpha\alpha^{-1}} \\
&= e^{-i\Im(-\|u\|^2)} W_{(I,0)} \\
&= I \ .
\end{aligned}$$

Thus W_α is invertible. ■

The mapping :

$$\begin{aligned}
G &\longrightarrow \mathcal{U}(\Gamma(\mathcal{H})) \\
\alpha &\longmapsto W_\alpha
\end{aligned}$$

is a unitary projective representation of G.

If one considers the group $\widetilde{G}$ of (U,u,t), $U \in \mathcal{U}(\mathcal{H})$, $u \in \mathcal{H}$ and $t \in \mathbb{R}$ with

$$(U,u,t)(V,v,s) = (UV, Uv+u, t+s+\Im\langle u, Uv\rangle) \ ;$$

we obtain the so-called *Heisenberg group* of $\mathcal{H}$. The mapping $(U,u,t) \longmapsto W_{(U,u)}e^{it}$ is thus a unitary representation of $\widetilde{G}$.

Conversely, if $W_{(U,u,t)}$ is a unitary representation of the Heisenberg group of $\mathcal{H}$ we then have

$$W_{(U,u,t)} = W_{(U,u,0)} W_{(I,0,t)}$$

and

$$W_{(I,0,t)} = W_{(I,0,s)} W_{(I,0,t+s)} \ .$$

This means that

$$W_{(U,u,t)} = W_{(U,u,0)}e^{itH}$$

for some self-adjoint operator H, and the $W_{(U,u,0)}$ satisfy the Weyl commutation relations.

If we come back to our Weyl operators $W_{(U,u)}$ one easily sees that

$$W_{(U,u)} = W_{(I,u)}W_{(U,0)} \ .$$

By definition $W_{(U,0)}\varepsilon(k) = \varepsilon(Uk)$ and thus

$$W_{(U,0)} = \Gamma(U) \ .$$

Finally, write W_u for $W_{(I,u)}$. Then

$$W_u W_v = e^{-i\Im\langle u,v\rangle} W_{u+v} \ .$$

These relations are often also called Weyl commutation relations. As a consequence $(W_{(I+tu)})_{t\in\mathbb{R}}$ is a unitary group; it is strongly continuous (*exercise*).

Proposition 3.9 – *We have*

$$W_{(I,tu)} = e^{it\frac{1}{i}(a(u)-a^*(u))} \ .$$

Proof

$$\begin{aligned}
\frac{1}{i}\frac{d}{dt}\Big|_{t=0} W_{(I,tu)}\varepsilon(k) &= \frac{1}{i}\frac{d}{dt}\Big|_{t=0} e^{-\frac{t^2}{2}\|u\|^2 - t\langle u,k\rangle}\varepsilon(k+tu)\\
&= -\frac{1}{i}\langle u,k\rangle\varepsilon(k) + \frac{1}{i}\frac{d}{dt}\Big|_{t=0}\varepsilon(k+tu)\\
&= \frac{1}{i}(-a(u)+a^*(u))\varepsilon(k) \ .
\end{aligned}$$

■

Coming back to our example on $L^2(\mathbb{R})$, the Weyl operators are defined by

$$W_z\varepsilon(z') = e^{-\frac{|z|^2}{2} - \overline{z}z'}\varepsilon(z+z') \ .$$

These operators are very helpfull for computing the law of some observables.

Proposition 3.10 – *The observable* $1/i\,(za^* - \overline{z}a)$ *follows a law* $\mathcal{N}(0,|z|^2)$ *in the vaccum state.*

Proof

We have

$$\begin{aligned}
<\mathbb{1}\,,\, e^{t(za^*-\overline{z}a)}\mathbb{1}> &= <\mathbb{1}\,,\, W_{itz}\mathbb{1}>\\
&= <\mathcal{E}(0)\,,\, W_{itz}\mathcal{E}(0)>\\
&= <\mathcal{E}(0)\,,\, \mathcal{E}(itz)>e^{-t^2|z|^2/2}\\
&= e^{-t^2|z|^2/2}.
\end{aligned}$$

$\blacksquare$

Proposition 3.11 – *The observable $\Lambda + \alpha I$ follows the law δ_α in the vaccum state.*

Proof

Indeed,

$$< \mathbb{1}\,,\, (\Lambda + \alpha I)^n \mathbb{1} > = \alpha^n < \mathbb{1}\,,\, \mathbb{1} > = \alpha^n.$$

$\blacksquare$

Let us now compute the law, in the vaccum state, of the observable $\Lambda + za^* - \overline{z}a + |z|^2 I$, sum of the two previous observables.

Lemma 3.12 –

$$W_{-z} e^{it\Lambda} W_z = e^{it(\Lambda + za^* - \overline{z}a + |z|^2 I)}.$$

Proof

It suffices to show that $W_{-z}\Lambda W_z = \Lambda + za^* - \overline{z}a + |z|^2 I$. We have

$$\begin{aligned}
< \mathcal{E}(z_1)\,,\, W_{-z}\Lambda W_z\, \mathcal{E}(z_2) > &= < aW_z\mathcal{E}(z_1)\,,\, aW_z\mathcal{E}(z_2) > \\
&= < (z_1 + z)\mathcal{E}(z_1 + z)\,,\, (z_2 + z)\mathcal{E}(z_2 + z) > \times \\
&\qquad \times e^{-|z|^2 - \overline{z}z_2 - \overline{z_1}z} \\
&= (\overline{z_1}z_2 + \overline{z_1}z + z_2\overline{z} + |z|^2)e^{\overline{z_1}z_2}.
\end{aligned}$$

$\blacksquare$

Proposition 3.13 – *The law of the observable $\Lambda + za^* - \overline{z}a + |z|^2 I$ in the vaccum state is the Poisson law $\mathcal{P}(|z|^2)$.*

Proof

We have

$$\begin{aligned}
< \mathbb{1}\,,\, e^{it(\Lambda + za^* - \overline{z}a + |z|^2 I)}\mathbb{1} > &= < W_z\mathbb{1}\,,\, e^{it\Lambda}W_z\mathbb{1} > \\
&= e^{-|z|^2} < \mathcal{E}(z)\,,\, e^{it\Lambda}\mathcal{E}(z) > \\
&= e^{-|z|^2} e^{|z|^2 e^{it}} \\
&= e^{|z|^2(e^{it} - 1)}.
\end{aligned}$$

$\blacksquare$

Comments: This result may seem very surprising: the sum of a Gaussian variable and a deterministic one, gives a Poisson distribution! Of course such a phenomena cannot be realised with usual random variables. What does this mean?

Actually it is one of the manifestation of the fact that **the random phenomena attached to quantum mechanics cannot be modelized by usual probability theory**.

There has been many attemps to give a probabilistic model to the stochastic phenomena of quantum mechanics, such as the "hidden variable theory" which tried to give a model of the randomness in measurement by saying that many parameters of the systems are unknown to us (hidden variables) since the begining and that their effects appear at the measurement and give this uncertainty, this randomness.

But it has been proved by Bell, that if one tries to attach random variables behind each observable and try to find (complicated) rules that explain the principles of quantum mechanics, then this reaches an impossibility. Indeed, taking the spin of a particle in three well-choosen directions, one obtains three Bernoulli variables, but their correlations cannot be obtained by any triplet of classical Bernoulli variables.

What then can we do to express the probabilistic effects of measurement in a probabilistic language? Actually, one does need to look very far away. Quantum mechanics in itself, in its axioms, contains the germ of a new probability theory. Indeed, now accept to consider a probability space to be a couple $(\mathcal{H}, \Psi)$, where $\mathcal{H}$ is a Hilbert space and Ψ is a normalized vector of $\mathcal{H}$; instead of the usual $(\Omega, \mathcal{F}, P)$. Accept to consider a random variable to be a self-adjoint operator A on $\mathcal{H}$, instead of a measurable function $X : \Omega \to I\!R$. Accept that the probability distribution of A under the state Ψ is the one described above:

$$E \longmapsto < \Psi \, , \, 1\!\!1_E(A)\Psi >.$$

Then what do we obtain ?

Actually this probability theory, as stated here, is equivalent to the usual one when considering a single random variable. Indeed, a classical probabilistic situation $(\Omega, \mathcal{F}, P, X)$ is easily seen to be also a quantum one $(\mathcal{H}, \Psi, A)$ by putting $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$, $\Psi = 1\!\!1$ and $A = \mathcal{M}_X$ the operator of multiplication by X. The (quantum) distribution of A is then the same as the (classical) distribution of X.

Conversely, given a quantum triplet $(\mathcal{H}, \Psi, A)$, then by the spectral theorem the operator A can be represented as a multiplication operator on some measured space (by diagonalization).

Where does the difference lie? When considering two non commuting observables on $\mathcal{H}$ (for example P, Q on $L^2(I\!R)$), then each of them is a classical random variable, but on its own probability space (they cannot be diagonalized simultaneously). We have put together two classical random variables which have nothing to do together, in a same context. We have not stick the together by declaring them independant, there is a dependency ($[Q, P] = iI$) which has some consequences (uncertainty principle for example) which cannot be expressed in classical terms.

This is to say that what quantum mechanics teaches us is that the observables, under measurement, behave as true random variables, but each one with its own random, and that their interdependency cannot be expressed in a simpler way than the axioms of quantum mechanics.

It is then not a surprise that adding two observables which do not commute we obtain a distribution which has nothing to do with the convolution of their respective distributions. This is the example above. One also obtains a surprising one with $P^2 + Q^2$ (exercise).

3.5 The CCR algebra

We now denote by $W(f)$ the Weyl operator $W_{(I,f)}$, $f \in \mathcal{H}$.

Theorem 3.14 – *Let $\mathcal{K}$ be any (algebraic) subspace of a Hilbert space $\mathcal{H}$. There exists a C^*-algebra, denoted $CCR(\mathcal{K})$ of operators on $\Gamma(\mathcal{H})$, unique up to isomorphism, generated by nonzero elements $W(f)$, $f \in \mathcal{K}$, such that*

$$W(f)^* = W(-f) \text{ for all } f \in \mathcal{K}$$
$$W(f)W(g) = W(f+g)e^{-i\Im < f\,,\, g >} \text{ for all } f, g \in \mathcal{K}.$$

Proof

The existence of a C^*-algebra satisfying the two conditions is obvious. It suffices to consider the C^*-algebra generated by the Weyl operators $W(f), f \in \mathcal{K}$ of $\Gamma(\mathcal{H})$.

We now give the proof of uniqueness but it can omited by the reader as it makes use of tools that are not pertinent for us.

Put $b(f,g) = \exp(-i\Im < f\,,\, g >/2)$ for all $f, g \in \mathcal{K}$. For every $F \in \ell^2(\mathcal{K})$ put

$$(R_b(g)F)(f) = b(f,g)F(f+g)$$
$$(R(g)F)(f) = F(f+g).$$

then R is a unitary representation of the additive abelian group $\mathcal{K}$ in $\ell^2(\mathcal{K})$ and R_b is also a unitary representation, but up to a multiplier b.

Assume we have $\mathcal{U}_1$ and $\mathcal{U}_2$, two CCR algebras on $\mathcal{K}$, with associated Weyl elements W_i, $i = 1, 2$. Assume they are faithfully represented in $\mathcal{H}_1$ and $\mathcal{H}_2$. On the space $\ell^2(\mathcal{K}; \mathcal{H}_i) = \ell^2(\mathcal{K}) \otimes \mathcal{H}_i$ we put

$$((W_i \times R)(g)\Psi)(f) = W_i(g)\Psi(f+g).$$

Finally, define U_i, unitary operator on $\ell^2(\mathcal{K}; \mathcal{H}_i)$ by

$$(U_i\Psi)(f) = W_i(f)\Psi(f).$$

Then a simple computation proves that

$$U_i(W_i \times R)(g)U_i^* = I_i \otimes R_b(g).$$

If $\mathcal{B}_i$ denotes the C^*-algebra generated by $\{(W_i \times R)(g); g \in \mathcal{K}\}$, then there exists a $*$-isomorphism τ from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that

$$\tau((W_1 \times R)(g)) = (W_2 \times R)(g).$$

Thus it would be sufficient now to find $*$-isomorphisms τ_i from $\mathcal{U}_i$ to $\mathcal{B}_i$ such that

$$\tau_i(W_i(g)) = (W_i \times R)(g).$$

Now set $W = W_i$. It suffices to show that

$$\left|\left|\sum_{i=1}^{n} \lambda_i (W \times R)(f_i)\right|\right| = \left|\left|\sum_{i=1}^{n} \lambda_i W(f_i)\right|\right|$$

for all $\lambda_i \in \mathbb{C}$, $f_i \in \mathcal{K}$.

The representation $W \times R$ is, via Fourier transform on $\ell^2(\mathcal{K})$, unitary equivalent to the representation $W \times \widehat{R}$ on $\ell^2(\widehat{\mathcal{K}}, \mathcal{H})$ defined by

$$((W \times \widehat{R})(g)\Psi)(\chi) = W(g)\chi(g)\Psi(\chi), \chi \in \widehat{\mathcal{K}}$$

and hence

$$\left|\left|\sum_{i=1}^{n} \lambda_i (W \times R)(f_i)\right|\right| = \sup_{\chi \in \widehat{\mathcal{K}}} \left|\left|\sum_{i=1}^{n} \lambda_i \chi(f_i) W(f_i)\right|\right|.$$

The set of characters $\{\chi_g\}$ on $\mathcal{K}$ of the form $\chi_g(f) = b(f,g)^2$ is dense in $\widehat{\mathcal{K}}$ for it is a subgroup with annihilator zero.

Note that $\chi_g(f)W(f) = W(g)W(f)W(g)^*$, thus

$$\begin{aligned}\left|\left|\sum_{i=1}^{n} \lambda_i (W \times R)(f_i)\right|\right| &= \sup_{g \in \mathcal{K}} \left|\left|\sum_{i=1}^{n} \lambda_i \chi_g(f_i) W(f_i)\right|\right| \\ &= \sup_{g \in \mathcal{K}} \left|\left| W(g) \sum_{i=1}^{n} \lambda_i W(f_i) W(g)^* \right|\right| \\ &= \left|\left|\sum_{i=1}^{n} \lambda_i W(f_i)\right|\right|.\end{aligned}$$

■

Proposition 3.15– *Let $\mathcal{K} \subset \mathcal{H}$ be a subspace of $\mathcal{H}$. It follows that $CCR(\mathcal{K}) = CCR(\mathcal{H})$ if and only if $\mathcal{K} = \mathcal{H}$.*

Proof

If $\mathcal{K} \neq \mathcal{H}$, then consider the representation of $CCR(\mathcal{H})$ on $\ell^2(\mathcal{H})$ defined by

$$(W(g)F)(f) = b(f,g)F(f+g)$$

with the same notation as in the proof of uniqueness in the above theorem. If $g \in \mathcal{H} \setminus \mathcal{K}$ then

$$\left(\left(W(g) - \sum_{i=1}^{n} \lambda_i W(g_i)\right) F\right)(f) = b(f,g)\left(F(f+g) + \right.$$
$$\left. - \sum_{i=1}^{n} \lambda_i b(f, g_i - g) F(f + g_i)\right).$$

If F is supported by $\mathcal{K}$ then

$$\left|\left|\left(W(g) - \sum_{i=1}^{n} \lambda_i W(g_i)\right) F\right|\right| \geq ||F||$$

for the vector $f \mapsto F(f+g)$ is orthogonal to each of the vectors $f \mapsto b(f, g_i - g)F(f + g - i)$.

Therefore

$$\inf_{A \in CCR(\mathcal{K})} ||W(g) - A|| \geq 1$$

and hence $W(g) \notin CCR(\mathcal{K})$. ■