Lecture 3
OPERATOR SEMIGROUPS

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Abstract This lecture is an introduction to the theory of Operator Semigroups and its main ingredients: different types of continuity, associated generator, dual and predual semigroups, Stone's Theorem. The lecture also starts with a complete introduction to the Bochner integral.

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This lecture makes use of basic elements of bounded and unbounded operator theory. The elements we use are rather elementary in most of the lecture: domains, closed operators, spectrum, etc; only when dealing with Stone's Theorem we make use of the functional calculus for general self-adjoint operators. All these ingredients may be found in Lecture 1.

3.1 Bochner Integral

Before entering into the theory of operator semigroups we shall make clear basic constructions and properties of the Bochner integrals, that is, the integration theory for functions taking values in Banach spaces. These notions and results are not only useful when considering operator semigroups but they are also be used over and ever in many subject of interest for us, in particular when dealing with integration on Fock spaces or with Quantum Stochastic Calculus.

3.1.1 Dual Topologies on Banach Spaces

We shall make use of many well-known properties associated to the different topologies associated to Banach spaces. We recall without proof the main results that we use. They are all very classical theorems, proofs may be easily found in the usual literature (see the notes at the end of the chapter).

Definition 3.1. Let $\mathcal{B}$ be a Banach space with a norm $\|\cdot\|$. The dual of $\mathcal{B}$ is the space $\mathcal{B}^*$ of continuous linear forms on $\mathcal{B}$.

Recall that we denote the action on $\mathcal{B}$ of a linear form ϕ by $B \mapsto \langle \phi\,,\, B\rangle$ instead of $B \mapsto \phi(B)$.

Theorem 3.2. *When equipped with the norm*

$$\|\phi\| = \sup\left\{|\langle \phi\,,\, B\rangle|\ ;\ B \in \mathcal{B}\,,\ \|B\| = 1\right\}$$

the space $\mathcal{B}^*$ is a Banach space.

Definition 3.3. The dual space $\mathcal{B}^*$ of $\mathcal{B}$ allows to define a *weak topology* on $\mathcal{B}$. It is the topology on $\mathcal{B}$ induced by the seminorms $B \mapsto |\langle \phi\,,\, B\rangle|$, for all $\phi \in \mathcal{B}^*$.

Definition 3.4. Note that every element $B \in \mathcal{B}$ defines a continuous linear form $\phi \mapsto \langle \phi\,,\, B\rangle$ on $\mathcal{B}^*$, hence $\mathcal{B}$ is a subset of $\mathcal{B}^{**}$, the dual of $\mathcal{B}^*$. In general $\mathcal{B}^{**}$ is not equal to $\mathcal{B}$. When this is the case, the space $\mathcal{B}$ is called *reflexive*. Every Hilbert space is a reflexive Banach space, but this is not the only case. For example, the spaces $L^p(X, \mathcal{F}, \mu)$ are reflexive if $1 < p < \infty$. The space

$L^1(X, \mathcal{F}, \mu)$ is a well-known example of Banach space which is not reflexive in general.

Definition 3.5. On the Banach space $\mathcal{B}^*$ one can define also a weak topology: the $*$-*weak topology*, also often called σ-*weak topology*. It is the smallest topology on $\mathcal{B}^*$ which makes the linear forms $\phi \mapsto \langle \phi \,, B \rangle$ being continuous, for all $B \in \mathcal{B}$. It is generated by the seminorms $\phi \mapsto |\langle \phi \,, B \rangle|$ for all $B \in \mathcal{B}$.

Concerning these two weak topologies, here are the main theorems which will be of much use for us. First of all, here are two very basic results concerning the weak and the norm topologies on $\mathcal{B}$.

Proposition 3.6.

1) The weak topology is weaker than the norm topology.

2) The weak and the norm closure of convex sets coincide.

The first basic results concerning the $*$-weak topology on $\mathcal{B}^*$ are the following.

Theorem 3.7.

1) On $\mathcal{B}^*$ the weak topology is stronger than the $*$-weak topology.

2) The space $\mathcal{B}^*$ is closed for the $*$-weak topology.

3) A linear form λ on $\mathcal{B}^*$ is $*$-weakly continuous if and only if there exists an element $B \in \mathcal{B}$ such that

$$\lambda(\phi) = \langle \phi \,, B \rangle$$

for all $\phi \in \mathcal{B}^*$.

The main interest of the $*$-weak topology is the following result: this topology makes the bounded balls of $\mathcal{B}^*$ being compact, which is not the case of the other topologies in infinite dimension.

Theorem 3.8 (Banach-Alaoglu Theorem). *The unit ball of $\mathcal{B}^*$ is compact for the $*$-weak topology.*

Finally, a theorem which is very useful in order to prove that a linear form is $*$-weakly continuous.

Theorem 3.9 (Krein-Smulian Theorem). *A convex subset $\mathcal{C}$ in $\mathcal{B}^*$ is $*$-weakly closed if and only if its intersection with the balls $\{\phi \in \mathcal{B}^* \,;\ \|\phi\| \leq r\}$ of $\mathcal{B}^*$ are $*$-weakly closed.*

3.1.2 Bochner Measurability

Before entering into the construction of the Bochner integral we shall establish a few technical lemmas concerning the notion of measurability we shall use for defining these integrals.

The general setup in which we shall be interested is the following. We consider a measured space $(E, \mathcal{E}, \mu)$, with the measure μ being σ-finite. We consider a Banach space $\mathcal{B}$ and a function f from E to $\mathcal{B}$.

Definition 3.10. We say that the function f is *weakly measurable* if, for all $\phi \in \mathcal{B}^*$, the mapping $x \mapsto \langle \phi\,,\, f(x)\rangle$ is measurable from $(E, \mathcal{E})$ to $\mathbb{C}$.

We say that the function f has a *separable range* if its range $f(E)$ is a separable subset of $\mathcal{B}$.

There are two cases for which this second property is satisfied and which will be very useful for us.

Proposition 3.11.

1) If $\mathcal{B}$ is a separable Banach space, then any function $f : E \mapsto \mathcal{B}$ has a separable range.

2) Whatever is the Banach space $\mathcal{B}$, if E is a separable topological space and if the function $f : E \mapsto \mathcal{B}$ is continuous, then f has a separable range.

Proof. The proof of 1) is immediate for any subset of a separable metric space is separable (exercise). The proof of 2) is also immediate for the continuous image of a separable space is always separable (exercise). □

Remark: Actually all the discussion that we shall have here does not exactly needs f to have a separable range, but an *almost surely separable range*, that is, there exists $N \in \mathcal{E}$ such that $\mu(N) = 0$ and such that the set $f(E \setminus N) \subset \mathcal{B}$ is separable. If the function f from E to $\mathcal{B}$ has an almost surely separable range, then we can change f on the null set N into a function $\widetilde{f}$ which has full range being separable. But it is easy to check that the constructions and the theorems which follow concerning the Bochner integral of f with respect to μ are not affected by this change on a null set.

Definition 3.12. From now on, a function f from E to $\mathcal{B}$ which is weakly measurable and has a separable range is called a *Bochner measurable* function.

We now prove a kind of separability property of $\mathcal{B}^*$, when $\mathcal{B}$ is separable. Put

$$\mathcal{C}_1 = \{\phi \in \mathcal{B}^*\,;\ \|\phi\| = 1\}$$

Lemma 3.13. *If $\mathcal{B}$ is a separable Banach space, then there exists a countable set $\mathcal{Y}$ included in $\mathcal{C}_1$ such that, for all $\phi \in \mathcal{C}_1$, there exists a sequence (ϕ_n) in $\mathcal{Y}$ satisfying*

$$\lim_{n\to+\infty} \langle \phi_n \, , \, y\rangle = \langle \phi \, , \, y\rangle$$

for all $y \in \mathcal{B}$*.*

Proof. Let $\mathcal{X} = \{y_n \, ; \; n \in \mathbb{N}\}$ be a countable dense subset of $\mathcal{B}$. For every $n \in \mathbb{N}^*$ consider the mapping Φ_n from $\mathcal{C}_1$ to $\mathbb{R}^n$ defined by

$$\Phi_n(\phi) = (\langle \phi \, , \, y_1\rangle \, , \ldots , \langle \phi \, , \, y_n\rangle) \; .$$

As $\mathbb{R}^n$ is a separable metric space then $\Phi_n(\mathcal{C}_1) \subset \mathbb{R}^n$ is also separable. Hence, for all $n \in \mathbb{N}^*$, there exists $\{\phi_{n,k} \in \mathcal{C}_1 \, ; \; k \in \mathbb{N}\}$ such that the set $\{\Phi_n(\phi_{n,k}) \, ; \; k \in \mathbb{N}\}$ is dense in $\Phi_n(\mathcal{C}_1)$. In particular, for every $\phi \in \mathcal{C}_1$ there exists a sequence $\{\phi_{n,k_n} \, ; \; n \in \mathbb{N}\}$ such that

$$|\langle \phi_{n,k_n} \, , \, y_i\rangle - \langle \phi \, , \, y_i\rangle| \leq \frac{1}{n}$$

for all $i = 1, \ldots , n$. In particular, $\lim_n \langle \phi_{n,k_n} \, , \, y_i\rangle = \langle \phi \, , \, y_i\rangle$ for all $i \in \mathbb{N}$. This implies easily that $\lim_n \langle \phi_{n,k_n} \, , \, y\rangle = \langle \phi \, , \, y\rangle$ for all $y \in \mathcal{B}$. □

This lemma has the following useful consequence.

Lemma 3.14. *If* f *is a Bochner measurable function from* E *to* $\mathcal{B}$ *then the function* $x \mapsto \|f(x)\|$ *is measurable from* E *to* $\mathbb{R}$*.*

Proof. For $a > 0$ and $\phi \in \mathcal{B}^*$ consider the sets

$$A = \{x \in E \, ; \; \|f(x)\| \leq a\}$$

and

$$A_\phi = \{x \in E \, ; \; |\langle \phi \, , \, f(x)\rangle| \leq a\} \; .$$

We obviously have

$$A \subset \bigcap_{\phi \in \mathcal{C}_1} A_\phi \; .$$

By the Hahn-Banach Theorem, for every $x \in E$ there exists $\phi_0 \in \mathcal{C}_1$ such that $\langle \phi_0 \, , \, f(x)\rangle = \|f(x)\|$; this implies that

$$\bigcap_{\phi \in \mathcal{C}_1} A_\phi \subset A$$

and hence the two sets are equal.

Consider the closure $\mathcal{X}$ of the linear span of the range of f. It is a separable Banach space by hypothesis. Let $\mathcal{Y}$ be the countable set associated to $\mathcal{X}$ by Lemma 3.13. We claim that we have

$$A = \bigcap_{\phi \in \mathcal{C}_1} A_\phi = \bigcap_{\phi \in \mathcal{Y}} A_\phi \; .$$

Indeed, one inclusion is obvious and, for the converse inclusion, consider $x \in E$ such that $|\langle \psi\,,\, f(x)\rangle| \leq a$ for all $\psi \in \mathcal{Y}$. By Lemma 3.13 every $\langle \phi\,,\, f(x)\rangle$, with $\phi \in \mathcal{C}_1$, is a limit of a sequence $\langle \psi_n\,,\, f(x)\rangle$, with $(\psi_n) \subset \mathcal{Y}$. Hence, we have $|\langle \phi\,,\, f(x)\rangle| \leq a$ for all $\phi \in \mathcal{C}_1$. This proves the announced equality of sets.

Finally, as f is weakly measurable, the sets A_ϕ are measurable, hence so is A. This proves the measurability of $x \mapsto \|f(x)\|$. □

3.1.3 Construction of the Bochner Integral

We can now pass to the construction of the Bochner integral. Let f be Bochner measurable function from E to $\mathcal{B}$. By Lemma 3.14 the function $x \mapsto \|f(x)\|$ is measurable from E to $\mathbb{R}$.

Definition 3.15. We say that f is *Bochner integrable* if furthermore

$$\int_E \|f(x)\|\ \mathrm{d}\mu(x) < \infty\,.$$

We wish to show that f being Bochner integrable is a sufficient condition for the integral

$$\int_E f(x)\,\mathrm{d}\mu(x)$$

to be a well-defined and to be an element of $\mathcal{B}$.

The construction actually follows the usual construction of real-valued integrals. First of all, assume that f is a *simple* function, that is, f is of the form

$$f(x) = \sum_{n\in\mathbb{N}} f_n\, \mathbb{1}_{A_n}(x)$$

for some measurable partition $(A_n) \subset \mathcal{E}$ of E and some $f_n \in \mathcal{B}$. Clearly such a function f is Bochner measurable. If f is furthermore Bochner integrable, that is, if

$$\sum_{n\in\mathbb{N}} \|f_n\|\ \mu(A_n) < \infty\,,$$

we put

$$\int_E f(x)\,\mathrm{d}\mu(x) = \sum_{n\in\mathbb{N}} \mu(A_n)\, f_n\,.$$

This way we have defined an element of $\mathcal{B}$ and clearly we have

$$\left\|\int_E f(x)\,\mathrm{d}\mu(x)\right\| \leq \int_E \|f(x)\|\ \mathrm{d}\mu(x)\,.$$

The difference with the usual integration theory for real-valued functions is that, in the general Banach context, approximating measurable functions by simple functions is not obvious.

Lemma 3.16. *Let f be a Bochner measurable function from E to $\mathcal{B}$, then there exists a sequence (f_n) of simple functions from E to $\mathcal{B}$ such that*

$$\lim_{n\to+\infty} \sup_{x\in E} \|f(x) - f_n(x)\| = 0\,.$$

Proof. Let $\mathcal{X}$ be the closure of the range of f. As $\mathcal{X}$ is separable by hypothesis there exists a countable subset $\mathcal{Y} = \{y_n\,;\ n \in \mathbb{N}\} \subset \mathcal{X}$ which is dense in $\mathcal{X}$. For all $n \in \mathbb{N}$ define the function g_n from E to $\mathbb{R}$ by

$$g_n(x) = \|f(x) - y_n\|\ .$$

As f is weakly measurable all the functions $x \mapsto f(x) - y_n$ are weakly measurable. They also obviously have separable range. Hence, by Lemma 3.14, the functions g_n are measurable.

For all $n \in \mathbb{N}$ and all $k \in \mathbb{N}^*$ define the sets $A_k^n = \{x \in E\,;\ g_n(x) \leq 1/k\}$. They are measurable subsets of E. As $\mathcal{Y}$ is dense in $\mathcal{X}$, for every $x \in E$, every $k \in \mathbb{N}^*$ there exists at least a $y_n \in \mathcal{Y}$ such that $\|f(x) - y_n\| \leq 1/k$. Hence we have $E = \cup_n A_k^n$, for all $k \in \mathbb{N}^*$.

For all fixed $k \in \mathbb{N}^*$ define for all $n \geq 1$

$$B_k^0 = A_k^0 \qquad \text{and} \qquad B_k^n = A_k^n \setminus \bigcup_{j<n} A_k^j\,.$$

They are measurable subsets of E and, for each fixed $k \in \mathbb{N}^*$, the family $(B_k^n)_{n\in\mathbb{N}}$ forms a partition of E. Define the functions

$$f_k(x) = \sum_{n\in\mathbb{N}} y_n \mathbb{1}_{B_k^n}(x)\,.$$

We clearly have

$$\|f(x) - f_k(x)\| \leq \frac{1}{k}\,,$$

for all $x \in E$. We have proved the uniform approximation property. □

Corollary 3.17. *If f is a Bochner integrable function from E to $\mathcal{B}$ then there exists a sequence (f_n) of simple and Bochner integrable functions from E to $\mathcal{B}$ such that*

$$\lim_{n\to+\infty} \int_E \|f(x) - f_n(x)\|\, \mathrm{d}\mu(x) = 0\,.$$

Proof. As μ is σ-finite and $\|f(\cdot)\|$ is integrable, for every $\varepsilon > 0$ there exists a measurable set $K \in \mathcal{E}$ such that $\mu(K) < \infty$ and

$$\int_{E\setminus K} \|f(x)\| \, \mathrm{d}\mu(x) \leq \varepsilon\,.$$

Consider a sequence (f_k) such as given by Lemma 3.16. We have

$$\begin{aligned}\int_E \|f(x) - f_k \mathbb{1}_K(x)\| \, \mathrm{d}\mu(x) &\leq \int_E \|f(x) - f(x)\mathbb{1}_K(x)\| \, \mathrm{d}\mu(x) + \\ &\qquad + \int_E \|f(x) - f_k(x)\| \, \mathbb{1}_K(x) \, \mathrm{d}\mu(x) \\ &\leq \varepsilon + \frac{1}{k}\mu(K)\,.\end{aligned}$$

It is easy to conclude now. □

In particular, the sequence $(\int_E f_n(x) \, \mathrm{d}\mu(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{B}$ for

$$\left\| \int_E \big(f_n(x) - f_m(x)\big) \, \mathrm{d}\mu(x) \right\| \leq \int_E \|f_n(x) - f_m(x)\| \, \mathrm{d}\mu(x)$$

which tends to 0 when n and m tend to $+\infty$. Hence this sequence converges in $\mathcal{B}$.

Definition 3.18. We denote by

$$\int_E f(x) \, \mathrm{d}\mu(x)$$

its limit in $\mathcal{B}$. This integral is called the *Bochner integral of f with respect to the measure μ*.

Note that this limit does not depend on the choice of the sequence (f_n) (exercise). This limit obviously satisfies the inequality

$$\left\| \int_E f(x) \, \mathrm{d}\mu(x) \right\| \leq \int_E \|f(x)\| \, \mathrm{d}\mu(x)\,. \tag{3.1}$$

We now give a useful characterization of the Bochner integral in terms of the action of elements of $\mathcal{B}^*$

Theorem 3.19. *Let f be a Bochner measurable and Bochner integrable function from $(E, \mathcal{E}, \mu)$ to $\mathcal{B}$. Then its Bochner integral $\int_E f(x) \, \mathrm{d}\mu(x)$ is the unique element of $\mathcal{B}$ which satisfies*

$$\boxed{\left\langle \phi\,,\, \int_E f(x) \, \mathrm{d}\mu(x) \right\rangle = \int_E \langle \phi\,,\, f(x) \rangle \, \mathrm{d}\mu(x)\,,} \tag{3.2}$$

for all $\phi \in \mathcal{B}^*$.

Proof. Let us first check that the integrals

$$\int_E \langle \phi\,,\, f(x) \rangle \, \mathrm{d}\mu(x)$$

are well-defined. As f is Bochner measurable, the function $x \mapsto \langle \phi\,,\, f(x) \rangle$ is measurable from E to $\mathbb{C}$. It is also absolutely integrable for

$$\int_E |\langle \phi\,,\, f(x) \rangle| \, \mathrm{d}\mu(x) \leq \int_E \|\phi\| \, \|f(x)\| \, \mathrm{d}\mu(x)\,.$$

Hence the integral above is well-defined.

We claim that the mapping $\lambda : \phi \mapsto \int_E \langle \phi\,,\, f(x) \rangle \, \mathrm{d}\mu(x)$ is *-weakly-continuous. Indeed, if (ϕ_n) converges *-weakly to $\phi \in \mathcal{B}^*$ then $\langle \phi_n\,,\, f(x) \rangle$ converges to $\langle \phi\,,\, f(x) \rangle$ for all x (by hypothesis) and the sequence (ϕ_n) is bounded (by the Uniform Boundedness Principle). By Lebesgue's Theorem this shows that $\lambda(\phi_n)$ converges to $\lambda(\phi)$. This proves that λ is *-weakly-continuous.

By Theorem 3.7 this proves that there exists a unique element $g \in \mathcal{B}$ such that

$$\lambda(\phi) = \langle \phi\,,\, g \rangle \ .$$

The fact the the Bochner integral $g = \int_E f(x)\, dx$ satisfies the relation (3.2) is obtained easily for simple functions. One then passes to the limit for general Bochner integrable functions. □

In the case of separable Hilbert spaces the definition of the Bochner integral becomes much simpler. This result is an easy consequence of the results established previously, we leave the proof to the reader.

Theorem 3.20. *Let $\mathcal{H}$ be a separable Hilbert space and let f be a function from E to $\mathcal{H}$.*

1) The function f is Bochner measurable if and only if, for all $\phi \in \mathcal{H}$, the function $x \mapsto \langle \phi\,,\, f(x) \rangle$ is measurable from E to $\mathbb{C}$.

2) If f is Bochner integrable then, for all $\phi \in \mathcal{H}$, we have

$$\int_E |\langle f(x)\,,\, \phi \rangle| \, \mathrm{d}\mu(x) < \infty$$

and the Bochner integral $\int_E f(x)\, \mathrm{d}\mu(x)$ is the unique element of $\mathcal{H}$ which satisfies

$$\left\langle \int_E f(x)\, \mathrm{d}\mu(x)\,,\, \phi \right\rangle = \int_E \langle f(x)\,,\, \phi \rangle \, \mathrm{d}\mu(x) \tag{3.3}$$

for all $\phi \in \mathcal{H}$.

3.1.4 Actions of Operators

The usual properties of Integration Theory, such as additivity, linearity, Change of Variable Formula (when $E=\mathbb{R}^n$), Fundamental Theorem of Calculus (when $E=\mathbb{R}$), Lebesgue's Dominated Convergence Theorem, ... remain valid in the context of Banach-valued functions. We shall not develop these results here. In this subsection we concentrate only on results which are specific to Bochner integrals and which admit no interesting equivalent result for usual integrals in finite dimension: the behavior of Bochner integrals with respect to the action of linear operators.

Proposition 3.21. *If f is a Bochner integrable function from E to $\mathcal{B}$ and if T is a bounded operator on $\mathcal{B}$, then $x\mapsto \mathsf{T}f(x)$ is Bochner integrable too. Furthermore we have*

$$\mathsf{T}\int_E f(x)\,\mathrm{d}\mu(x)=\int_E \mathsf{T}f(x)\,\mathrm{d}\mu(x)\,. \tag{3.4}$$

Proof. If ϕ is a continuous linear form on $\mathcal{B}$ then $\phi\circ\mathsf{T}$ is also a continuous linear form on $\mathcal{B}$. This shows that $\mathsf{T}f$ is weakly measurable. The range of $\mathsf{T}f$ is the continuous image by T of the range of f, hence it is separable too. The norm-integrability of $\mathsf{T}f$ is obvious. This proves that $\mathsf{T}f$ is Bochner integrable.

The identity (3.4) is immediate when f is simple. We then conclude easily in the general case, using an approximation by simple functions (Corollary 3.17) and using the continuity of T. □

In the case of Hilbert spaces we have an extension of Proposition 3.21, but first of all we need a little lemma on measurability.

Lemma 3.22. *Let T be a closable operator from $\mathrm{Dom}\,\mathsf{T}\subset\mathcal{H}$ to $\mathcal{H}$ and let f be a Bochner measurable function from E to $\mathcal{H}$ such that $f(x)\in\mathrm{Dom}\,\mathsf{T}$ for all $x\in E$. Then the function $x\mapsto\mathsf{T}f(x)$ is Bochner measurable.*

Proof. As T is closable it admits a densely defined adjoint T^*. For all $\psi\in\mathrm{Dom}\,\mathsf{T}^*$ and all $x\in E$ we have

$$\langle\psi\,,\,\mathsf{T}f(x)\rangle=\langle\mathsf{T}^*\psi\,,\,f(x)\rangle\ .$$

In particular, for all $\psi\in\mathrm{Dom}\,\mathsf{T}^*$, the function $x\mapsto\langle\psi\,,\,\mathsf{T}f(x)\rangle$ is measurable (by Proposition **??**). As $\mathrm{Dom}\,\mathsf{T}^*$ is dense in $\mathcal{H}$, it is easy to see that the mapping $x\mapsto\langle\psi\,,\,\mathsf{T}f(x)\rangle$ is measurable from E to $\mathbb{C}$, for all $\psi\in\mathcal{H}$. This gives the Bochner measurability of $\mathsf{T}f$, by Proposition **??** again. □

Proposition 3.23. *Let f be a Bochner integrable function from E to a separable Hilbert space $\mathcal{H}$. Let T be a closable operator on $\mathcal{H}$, with closure $\overline{\mathsf{T}}$, such that $f(x)$ belongs to $\mathrm{Dom}\,\mathsf{T}$ for all $x\in E$ and such that*

$$\int_E \|\mathsf{T} f(x)\| \, \mathrm{d}\mu(x) < \infty \,.$$

Then $\int_E f(x)\, \mathrm{d}\mu(x)$ *belongs to* $\mathrm{Dom}\, \overline{\mathsf{T}}$ *and*

$$\overline{\mathsf{T}} \int_E f(x)\, \mathrm{d}\mu(x) = \int_E \mathsf{T} f(x)\, \mathrm{d}\mu(x) \,. \tag{3.5}$$

Proof. By Lemma 3.22 the function $\mathsf{T} f$ is Bochner measurable. By hypothesis it is also norm-integrable, hence the Bochner integral $\int_E \mathsf{T} f(x)\, \mathrm{d}\mu(x)$ is well-defined.

As T is closable it admits a densely adjoint T^*. Let $\psi \in \mathrm{Dom}\, \mathsf{T}^*$, then

$$\begin{aligned}
\left\langle \mathsf{T}^*\psi \,,\, \int_E f(x)\, \mathrm{d}\mu(x) \right\rangle &= \int_E \langle \mathsf{T}^*\psi \,,\, f(x) \rangle \, \mathrm{d}\mu(x) \qquad \text{(by Theorem 3.20)} \\
&= \int_E \langle \psi \,,\, \mathsf{T} f(x) \rangle \, \mathrm{d}\mu(x) \\
&= \left\langle \psi \,,\, \int_E \mathsf{T} f(x)\, \mathrm{d}\mu(x) \right\rangle \quad \text{(by Theorem 3.20)} \,.
\end{aligned}$$

This proves that $\int_E f(x)\, \mathrm{d}\mu(x)$ belongs to the domain of $\mathsf{T}^{**} = \overline{\mathsf{T}}$ and that

$$\overline{\mathsf{T}} \int_E f(x)\, \mathrm{d}\mu(x) = \int_E \mathsf{T} f(x)\, \mathrm{d}\mu(x) \,.$$

This proves the announced relation (3.5). □

3.2 Operator Semigroups

We now discuss the basic properties of operator semigroups and in particular their different notions of continuity.

3.2.1 Definitions

Definition 3.24. Let $\mathcal{B}$ be a Banach space. An *operator semigroup*, or simply a *semigroup*, on $\mathcal{B}$ is a family (T_t) of bounded linear operators on $\mathcal{B}$, indexed by $t \in \mathbb{R}^+$, such that

i) $\mathsf{T}_0 = \mathrm{I}$,

ii) $\mathsf{T}_s\, \mathsf{T}_t = \mathsf{T}_{s+t}$ for all $s, t \in \mathbb{R}^+$.

In general these two conditions are far too general and a continuity assumption has to be added. There are several types of continuity conditions that are usually considered.

Definition 3.25. A semigroup (T_t) on $\mathcal{B}$ is

– *uniformly continuous* if $t \mapsto \mathsf{T}_t$ is continuous for the operator-norm,

– *strongly continuous* if, for all $f \in \mathcal{B}$, the mapping $t \mapsto \mathsf{T}_t\, f$ is continuous from $\mathbb{R}^+$ to $\mathcal{B}$,

– *weakly continuous* if, for all $f \in \mathcal{B}$, all $\phi \in \mathcal{B}^*$, the mapping $t \mapsto \langle \phi\,,\, \mathsf{T}_t\, f\rangle$ is continuous from $\mathbb{R}^+$ to $\mathbb{C}$,

– *continuous* if the mapping $(t, f) \mapsto \mathsf{T}_t\, f$ is continuous from $\mathbb{R}^+ \times \mathcal{B}$ to $\mathcal{B}$.

Clearly the uniform continuity implies the strong continuity, which itself implies the weak continuity. The continuity implies the strong continuity. We shall see later on more relations between these different continuity notions for semigroups.

3.2.2 Uniform Continuity

The case of uniformly continuous semigroups is the simplest one and is easy to characterize. But, first of all, let us make clear some properties of the Bochner integral in this context.

Proposition 3.26. *Let (T_t) be a uniformly continuous semigroup of operators on $\mathcal{B}$. Then the mapping $t \mapsto \mathsf{T}_t$, from $\mathbb{R}^+$ to the space of bounded operators on $\mathcal{B}$, is Bochner integrable on any compact interval. For all $f \in \mathcal{B}$, the mapping $t \mapsto \mathsf{T}_t\, f$, from $\mathbb{R}^+$ to $\mathcal{B}$, is also Bochner integrable on any compact interval. The Bochner integral $\int_0^t \mathsf{T}_s\, \mathrm{d}s$ defines a bounded operator on $\mathcal{B}$ which satisfies*

$$\left(\int_0^t \mathsf{T}_s\, \mathrm{d}s\right) f = \int_0^t \mathsf{T}_s\, f\, \mathrm{d}s \tag{3.6}$$

for all $f \in \mathcal{B}$.

Furthermore, if H is any bounded operator on $\mathcal{B}$ then $(\mathsf{H}\,\mathsf{T}_t)_{t\in\mathbb{R}^+}$ and $(\mathsf{T}_t\,\mathsf{H})_{t\in\mathbb{R}^+}$ are Bochner integrable on any compact interval and we have

$$\mathsf{H}\left(\int_0^t \mathsf{T}_s\, \mathrm{d}s\right) = \int_0^t \mathsf{H}\,\mathsf{T}_s\, \mathrm{d}s\,, \tag{3.7}$$

$$\left(\int_0^t \mathsf{T}_s\, \mathrm{d}s\right) \mathsf{H} = \int_0^t \mathsf{T}_s\,\mathsf{H}\, \mathrm{d}s\,. \tag{3.8}$$

Proof. The mapping $t \mapsto \mathsf{T}_t$ is continuous by hypothesis, hence it is weakly-measurable and by Proposition 3.11 it has separable range. This means that it is Bochner measurable. By the continuity of $t \mapsto \|\mathsf{T}_t\|$, this mapping is also Bochner integrable on all compact interval.

Exactly the same arguments show that $t \mapsto \mathsf{T}_t\, f$ is Bochner integrable on any compact interval.

The identity (3.6) is obvious when $t \mapsto \mathsf{T}_t$ is a simple function, it is then rather easy to see that it remains true in the general case, by an approximation argument.

The fact that $(\mathsf{H}\,\mathsf{T}_t)_{t\in\mathbb{R}^+}$ and $(\mathsf{T}_t\,\mathsf{H})_{t\in\mathbb{R}^+}$ are Bochner integrable on any compact interval is immediate. Now, we have, by (3.6)

$$\left(\int_0^t \mathsf{T}_s\,\mathrm{d}s\right)\mathsf{H}\,f = \int_0^t \mathsf{T}_s\,\mathsf{H}\,f\,\mathrm{d}s = \left(\int_0^t \mathsf{T}_s\,\mathsf{H}\,\mathrm{d}s\right) f\,,$$

for all $f \in \mathcal{B}$. This gives (3.8).

On the other hand, by Proposition 3.21 and Equation (3.6), we have

$$\mathsf{H}\left(\int_0^t \mathsf{T}_s\,\mathrm{d}s\right) f = \mathsf{H}\left(\int_0^t \mathsf{T}_s\,f\,\mathrm{d}s\right) = \left(\int_0^t \mathsf{H}\,\mathsf{T}_s\,f\,\mathrm{d}s\right) = \left(\int_0^t \mathsf{H}\,\mathsf{T}_s\,\mathrm{d}s\right) f\,,$$

for all $f \in \mathcal{B}$. This proves (3.7). □

We can now pass to the main characterization theorem concerning uniformly continuous semigroups.

Theorem 3.27. *Let (T_t) be a semigroup on $\mathcal{B}$. Then the following assertions are equivalent.*

i) The semigroup (T_t) is uniformly continuous.

ii) There exists a bounded operator Z on $\mathcal{B}$ such that

$$\mathsf{T}_t = \mathrm{e}^{t\,\mathsf{Z}}$$

for all $t \in \mathbb{R}^+$.

In this holds, then the operator Z is given by

$$\mathsf{Z} = \lim_{h\to 0} \frac{1}{h}\left(\mathsf{T}_h - \mathrm{I}\right),$$

on all $\mathcal{B}$.

Proof. If Z is a bounded operator on $\mathcal{B}$ then the series

$$\mathrm{e}^{t\,\mathsf{Z}} = \sum_{n\in\mathbb{N}} \frac{t^n \mathsf{Z}^n}{n!}$$

is convergent in operator norm. This defines a bounded operator $\mathrm{e}^{t\,\mathsf{Z}}$ on $\mathcal{B}$ and obviously we have

$$\left\|\mathrm{e}^{t\,\mathsf{Z}}\right\| \leq \mathrm{e}^{t\|\mathsf{Z}\|}\,.$$

It is also easy to check that $\mathsf{T}_t = \mathrm{e}^{t\,\mathsf{Z}}$, $t \in \mathbb{R}^+$, defines a uniformly continuous semigroup (proof left to the reader). We have proved that ii) implies i).

Assume that i) is satisfied. The mapping $t \mapsto \mathsf{T}_t$ is Bochner integrable on any compact interval and by the Fundamental Theorem of Calculus, one can choose h small enough such that

$$\left\| h^{-1} \int_0^h \mathsf{T}_s \,\mathrm{d}s - \mathrm{I} \right\| < 1\,.$$

This means that the operator $\mathsf{X} = \int_0^h \mathsf{T}_s \,\mathrm{d}s$ is invertible, for the inverse series

$$\frac{1}{h} \sum_{n \in \mathbb{N}} \left(\mathrm{I} - \frac{1}{h} \mathsf{X} \right)^n$$

converges in operator norm. In particular X has a bounded inverse X^{-1}. Put

$$\mathsf{Z} = (\mathsf{T}_h - \mathrm{I})\, \mathsf{X}^{-1}\,.$$

We have, using Proposition 3.21 and the Change of Variable Formula,

$$\begin{aligned}
(\mathsf{T}_t - \mathrm{I})\, \mathsf{X} &= \int_t^{t+h} \mathsf{T}_s \,\mathrm{d}s - \int_0^h \mathsf{T}_s \,\mathrm{d}s \\
&= \int_h^{t+h} \mathsf{T}_s \,\mathrm{d}s - \int_0^t \mathsf{T}_s \,\mathrm{d}s \\
&= (\mathsf{T}_h - \mathrm{I}) \int_0^t \mathsf{T}_s \,\mathrm{d}s\,.
\end{aligned}$$

As all the operators above commute, we get by Proposition 3.26

$$\mathsf{T}_t - \mathrm{I} = \int_0^t \mathsf{T}_s\, \mathsf{Z} \,\mathrm{d}s.$$

In particular $\frac{\mathrm{d}}{\mathrm{d}t}\, \mathsf{T}_t$ exists and is equal to $\mathsf{T}_t\, \mathsf{Z}$. This means that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathsf{T}_t\, \mathrm{e}^{-t\,\mathsf{Z}} \right) = 0$$

and hence $\mathsf{T}_t = \mathrm{e}^{t\,\mathsf{Z}}$. □

Definition 3.28. The operator Z given by the theorem above is called the *generator* of (T_t).

3.2.3 Strong and Weak Continuity

We have seen that the uniform continuity for the semigroup is the signature of a bounded generator. Outside of this case, the strong continuity assumption is a very reasonable assumption and it has many consequences.

Before hand, we need to recall (without proof) a famous theorem of Functional Analysis (proofs may be found in any textbook in Functional Analysis, such as [RS80]).

Theorem 3.29 (Uniform Boundedness Principle). *Let* $(\mathsf{T}_i)_{i\in I}$ *be any family of bounded operators from a Banach space* $\mathcal{B}$ *to a normed linear space* $\mathcal{A}$*. If for each* $x \in \mathcal{B}$ *the set* $\{\|\mathsf{T}_i\, x\|\ ;\ i \in I\}$ *is bounded, then the set* $\{\|\mathsf{T}_i\|\ ;\ i \in I\}$ *is bounded.*

We can now state our theorem on strong continuity.

Theorem 3.30. *Let* (T_t) *be a semigroup on* $\mathcal{B}$*. Then the following assertions are equivalent.*

i) The semigroup (T_t) *is strongly continuous.*

ii) The semigroup (T_t) *is strongly continuous at* 0^+*.*

iii) The semigroup (T_t) *is continuous.*

In that case there exist positive constants M *and* β *such that*

$$\|\mathsf{T}_t\| \le M\, \mathrm{e}^{\beta t} \tag{3.9}$$

for all $t \in \mathbb{R}^+$*.*

Proof. It is obvious that i) implies ii) and that iii) implies i). We have only have to prove that ii) implies iii).

Assume that (T_t) is strongly continuous at 0^+, that is, $\lim_{t\to 0} \mathsf{T}_t\, f = f$, for all $f \in \mathcal{H}$. We claim that there exists a $n \in \mathbb{N}^*$ such that

$$C_n = \sup\{\|\mathsf{T}_t\|\ ;\ 0 \le t \le 1/n\}$$

is finite. Indeed, if this were not true one could construct a sequence (t_n) tending to 0 such that $\|\mathsf{T}_{t_n}\|$ tends to $+\infty$. By the Uniform Boundedness Principle (Theorem 3.29), this would mean the existence of $x \in \mathcal{B}$ such that $(\mathsf{T}_{t_n}\, x)_{n\in\mathbb{N}}$ is unbounded. This would contradict the strong continuity in 0^+.

Put $C = C_n^n$, then, by the semigroup property of (T_t) we have $\|\mathsf{T}_t\| \le C$ for all $t \in [0, 1]$. As a consequence, for all $t \in \mathbb{R}^+$, we have

$$\|\mathsf{T}_t\| \le C^{[t]+1} \le C^{\,t+1}\,.$$

This gives (3.9).

In particular, if $\lim_n t_n = t$ in $\mathbb{R}^+$ and $\lim_n f_n = f$ in $\mathcal{B}$ then

$$\begin{aligned}\|\mathsf{T}_{t_n} f_n - \mathsf{T}_t f\| &\leq \|\mathsf{T}_{t_n}(f_n - f)\| + \|\mathsf{T}_{t_n} f - \mathsf{T}_t f\| \\ &\leq M\,\mathrm{e}^{\beta t_n}\ \|f_n - f\| + M\,\mathrm{e}^{\beta\min(t_n,t)}\ \left\|\mathsf{T}_{|t-t_n|} f - f\right\| .\end{aligned}$$

In particular, $\lim_n \|\mathsf{T}_{t_n} f_n - \mathsf{T}_t f\| = 0$. This proves that (T_t) is continuous. □

Note that $\mathsf{S}_t = \mathrm{e}^{-\beta t}\,\mathsf{T}_t$, $t \in \mathbb{R}^+$, is also a semigroup, from which (T_t) is easily deduced. This semigroup is uniformly bounded, that is, $\|\mathsf{S}_t\| \leq M$ for all t. Hence, multiplying by M^{-1}, one can easily reduce the study of strongly continuous semigroups to the case of *contraction semigroups*, that is, such that $\|\mathsf{T}_t\| \leq 1$ for all t.

We end up this discussion with an important result: another improvement of the strong continuity condition.

Theorem 3.31. *Let (T_t) be a semigroup on $\mathcal{B}$. Then the following assertions are equivalent.*

i) (T_t) is strongly continuous.

ii) (T_t) is weakly continuous.

iii) (T_t) is weakly continuous at 0^+.

Proof. Obviously i) implies ii) and ii) implies iii). The only non trivial direction to prove is that iii) implies i). Even better, the semigroup property shows easily that iii) implies ii). There just remains to prove that ii) implies i).

The main argument for proving (3.9) in Theorem 3.30 was the fact that

$$C_n = \sup\{\|\mathsf{T}_t\|\ ;\ 0 \leq t \leq 1/n\}$$

is finite, by an application of the Uniform Boundedness Principle (Theorem 3.29). The rest was just applying the semigroup property. If (T_t) is only weakly continuous at 0^+ then, for all $f \in \mathcal{B}$ and $\phi \in \mathcal{B}^*$, the quantity $\langle \phi\,,\ \mathsf{T}_t\,f\rangle$ tend to $\langle \phi\,,\ f\rangle$ when t tends to 0, hence $t \mapsto \langle \phi\,,\ \mathsf{T}_t\,f\rangle$ is bounded in a neighborhood of 0. By the Uniform Boundedness Principle applied to the linear form $\phi \mapsto \langle \phi\,,\ \mathsf{T}_t\,f\rangle$, the mapping $t \mapsto \mathsf{T}_t\,f$ is then bounded in a neighborhood of 0. By the Uniform Boundedness Principle again we have that $t \mapsto \mathsf{T}_t$ is bounded near 0. Hence, developing the same argument as in Theorem 3.30, we get that (3.9) holds true if (T_t) is weakly continuous at 0^+.

If (T_t) is weakly continuous, then the closed linear span of the set $\{\mathsf{T}_t\,f\ ;\ t \in \mathbb{R}^+\}$ (which we denote by $\mathcal{R}$) is invariant under T_t and separable, for it is the weak (and hence the strong) closure of the linear space generated by $\{\mathsf{T}_t\,f\ ;\ t \in \mathbb{Q}^+\}$ (see Proposition 3.6).

Let $f \in \mathcal{B}$ and $\phi \in \mathcal{B}^*$, by the above arguments and by the weak continuity we see that $t \mapsto \langle \phi\,,\ \mathsf{T}_t f\rangle$ is locally bounded and measurable, hence the integral

$$\frac{1}{\varepsilon}\int_0^\varepsilon \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t$$

converges and the linear form

$$\lambda\ :\ \phi \mapsto \frac{1}{\varepsilon}\int_0^\varepsilon \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t$$

is clearly a continuous linear form on $\mathcal{B}^*$.

The mapping $t \mapsto \mathsf{T}_t\, f$ is weakly measurable and has a separable range $\mathcal{R}$ (as proved above), it is then Bochner measurable and hence $t \mapsto \|\mathsf{T}_t\, f\|$ is measurable by Lemma 3.14. We have also seen that this map is locally bounded near 0. If (ϕ_n) is a sequence in the unit ball of $\mathcal{B}^*$ converging $*$-weakly to $\phi \in \mathcal{B}^*$, then by the Banach-Alaoglu Theorem (Theorem 3.8) the mapping ϕ belongs to the unit ball of $\mathcal{B}^*$ too. Furthermore, by the Dominated Convergence Theorem we have that

$$\frac{1}{\varepsilon}\int_0^\varepsilon \langle \phi_n\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t$$

converges to

$$\frac{1}{\varepsilon}\int_0^\varepsilon \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t\,.$$

Hence the linear form λ is $*$-weakly continuous on the unit ball of $\mathcal{B}^*$. This means that the intersection of the kernel of λ with the unit ball of $\mathcal{B}^*$ is $*$-weakly closed. By the Krein-Smulian Theorem (Theorem 3.9) this implies that the kernel of λ is $*$-weakly closed, hence λ is $*$-weakly continuous. This means (Theorem 3.7) that there exists $f_\varepsilon \in \mathcal{B}$ such that

$$\langle \phi\,,\, f_\varepsilon\rangle = \frac{1}{\varepsilon}\int_0^\varepsilon \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t\,,$$

for all $\phi \in \mathcal{B}^*$.

As a consequence we have

$$\begin{aligned}
|\langle \phi\,,\, \mathsf{T}_h\, f_\varepsilon - f_\varepsilon\rangle| &= |\langle \mathsf{T}_h^*\, \phi\,,\, f_\varepsilon\rangle - \langle \phi\,,\, f_\varepsilon\rangle| \\
&= \frac{1}{\varepsilon}\left|\int_0^\varepsilon \langle \mathsf{T}_h^*\, \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t - \int_0^\varepsilon \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t\right| \\
&= \frac{1}{\varepsilon}\left|\int_h^{\varepsilon+h} \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t - \int_0^\varepsilon \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t\right| \\
&= \frac{1}{\varepsilon}\left|\int_\varepsilon^{\varepsilon+h} \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t - \int_0^h \langle \phi\,,\, \mathsf{T}_t\, f\rangle\ \mathrm{d}t\right| \\
&\leq \frac{\|f\|\ \|\phi\|}{\varepsilon}\left(\int_\varepsilon^{\varepsilon+h} M\,\mathrm{e}^{\beta t}\,\mathrm{d}t - \int_0^h M\,\mathrm{e}^{\beta t}\,\mathrm{d}t\right)\,.
\end{aligned}$$

Since ϕ is arbitrary, this means that

$$\lim_{h\to 0} \|\mathsf{T}_h\, f_\varepsilon - f_\varepsilon\| \leq \lim_{h\to 0} \frac{\|f\|}{\varepsilon}\left(\int_\varepsilon^{\varepsilon+h} M\,\mathrm{e}^{\beta t}\,\mathrm{d}t - \int_0^h M\,\mathrm{e}^{\beta t}\,\mathrm{d}t\right) = 0\,.$$

Let $\mathcal{L}$ be the set of all $g \in \mathcal{B}$ such that $\lim_{h\to 0} \mathsf{T}_h\, g = g$. It is a subspace and it is norm-closed by (3.9). In particular it is weakly closed. Now, note that the weak limit of f_ε is f, when ε tends to 0. Hence $\mathcal{L} = \mathcal{B}$ and the proof is complete. □

3.3 Generators

We have seen in Theorem 3.27 that uniformly continuous semigroups admit bounded generators. In the case of strongly continuous semigroups there still exists a good notion of associated generator.

3.3.1 Definition

Definition 3.32. Let (T_t) be a strongly continuous semigroup on $\mathcal{B}$. We define the *generator* of (T_t) to be the operator Z on $\mathcal{B}$ such that

$$\mathrm{Dom}\,\mathsf{Z} = \left\{ f \in \mathcal{B}\,;\ \lim_{t\to 0} \frac{1}{t}(\mathsf{T}_t - \mathrm{I})\, f \text{ exists} \right\}$$

and

$$\mathsf{Z}\, f = \lim_{t\to 0} \frac{1}{t}(\mathsf{T}_t - \mathrm{I})\, f$$

for all $f \in \mathrm{Dom}\,\mathsf{Z}$.

Definition 3.33. Recall that, in the same way as for operators on Hilbert space, an operator T on a Banach space is a *closed* operator if it is densely defined operator and if, for all sequence $(f_n) \subset \mathrm{Dom}\,\mathsf{T}$ which converges to some f in $\mathcal{B}$ and such that $(\mathsf{T} f_n)$ converges in $\mathcal{B}$ then $f \in \mathrm{Dom}\,\mathsf{T}$ and $\mathsf{T} f = \lim \mathsf{T} f_n$.

The following result shows that the operator Z defined this way is always an “interesting” operator.

Proposition 3.34. *The space* $\mathrm{Dom}\,\mathsf{Z}$ *is dense in* $\mathcal{B}$ *and the operator* Z *is closed.*

Proof. Set $\mathsf{Z}_h\, f = \frac{1}{h}(\mathsf{T}_h\, f - f)$ and $\mathsf{Y}_s\, f = \frac{1}{s}\int_0^s \mathsf{T}_u\, f\,\mathrm{d}u$. The operators Z_h and Y_s are bounded and

$$\mathsf{Z}_h \mathsf{Y}_s = \mathsf{Y}_s \mathsf{Z}_h = \mathsf{Z}_s \mathsf{Y}_h = \mathsf{Y}_h \mathsf{Z}_s ,$$

as can be checked easily (making use of Proposition 3.26). In particular, for every $s > 0$ and $f \in \mathcal{B}$ we have

$$\lim_{h \to 0} \mathsf{Z}_h \mathsf{Y}_s f = \lim_{h \to 0} \mathsf{Z}_s(\mathsf{Y}_h f) = \mathsf{Z}_s f .$$

Therefore $\mathsf{Y}_s f$ belongs to $\mathrm{Dom}\, \mathsf{Z}$ and since $\lim_{s\to 0} \mathsf{Y}_s f = f$ we get that $\mathrm{Dom}\, \mathsf{Z}$ is dense.

Now, if (f_n) is a sequence in $\mathrm{Dom}\, \mathsf{Z}$ converging to f in $\mathcal{B}$ and such that $(\mathsf{Z} f_n)_{n \in \mathbb{N}}$ converges to $g \in \mathcal{B}$, then

$$\begin{aligned} \mathsf{Y}_s g &= \lim_n \mathsf{Y}_s \mathsf{Z} f_n = \lim_n \mathsf{Y}_s \left(\lim_h \mathsf{Z}_h f_n \right) \\ &= \lim_n \lim_h \mathsf{Z}_s(\mathsf{Y}_h f_n) = \lim_n \mathsf{Z}_s f_n = \mathsf{Z}_s f . \end{aligned}$$

It follows that $\lim_{s\to 0} \mathsf{Z}_s f$ exists, hence $f \in \mathrm{Dom}\, \mathsf{Z}$ and $\mathsf{Z} f = \lim_{s \to 0} \mathsf{Z}_s f = g$. We have proved that Z is closed. □

The following is a set of important relations relating the generator and its semigroup.

Proposition 3.35. *If $f \in \mathrm{Dom}\, \mathsf{Z}$ then the following holds.*

1) For all $t \in \mathbb{R}^+$ we have $\mathsf{T}_t f \in \mathrm{Dom}\, \mathsf{Z}$.

2) The function $f \longmapsto \mathsf{T}_t f$ is strongly differentiable in $\mathcal{B}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{T}_t f = \mathsf{Z}\, \mathsf{T}_t f = \mathsf{T}_t \mathsf{Z} f . \tag{3.10}$$

3) We have

$$\mathsf{T}_t f - f = \int_0^t \mathsf{T}_s \mathsf{Z} f \, \mathrm{d}s = \int_0^t \mathsf{Z}\, \mathsf{T}_s f \, \mathrm{d}s . \tag{3.11}$$

Proof. We have, for $f \in \mathrm{Dom}\, \mathsf{Z}$,

$$\lim_{s \to 0} \frac{1}{s} [\mathsf{T}_s(\mathsf{T}_t f) - \mathsf{T}_t f] = \lim_{s \to 0} \mathsf{T}_t \left[\frac{1}{s} (\mathsf{T}_s f - f) \right] = \mathsf{T}_t \mathsf{Z} f.$$

This proves 1) and $\mathsf{Z}\, \mathsf{T}_t f = \mathsf{T}_t \mathsf{Z} f$. This also proves that $t \mapsto \mathsf{T}_t f$ has right-hand derivative equal to $\mathsf{T}_t \mathsf{Z} f$, in each point $t \in \mathbb{R}$. This right-hand derivative is continuous in t, hence it is a classical theorem of real analysis (see for example [Rud87], Theorem 8.21, to be adapted here to the case of Banach-valued functions) that this implies that $t \mapsto \mathsf{T}_t f$ is differentiable at every point.

Thus $\mathsf{T}_t f = f + \int_0^t \mathsf{T}_s \mathsf{Z} f \, \mathrm{d}s$. This proves 2) and 3). □

In Theorem 3.31 we have seen that the weak continuity of the semigroup is sufficient for proving its strong continuity. In the same spirit, the following result shows that weak and strong generators coincide.

Proposition 3.36. *Let* (T_t) *be a weakly continuous semigroup on* $\mathcal{B}$. *Consider the operator* L *on* $\mathcal{B}$ *defined by*

$$\mathrm{Dom}\,\mathsf{L} = \left\{ f \in \mathcal{B}\,;\ \text{weak-}\lim_{t\to 0} \frac{1}{t}(\mathsf{T}_t - \mathrm{I})f \ \textit{exists} \right\}$$

and

$$\mathsf{L}\, f = \text{weak-}\lim_{t\to 0} \frac{1}{t}(\mathsf{T}_t - \mathrm{I})f$$

for all $f \in \mathrm{Dom}\,\mathsf{L}$. *Then* L *coincides with the generator* Z *of* (T_t).

Proof. As the norm convergence implies the weak convergence, we obviously have $\mathrm{Dom}\,\mathsf{Z} \subset \mathrm{Dom}\,\mathsf{L}$ and $\mathsf{Z} = \mathsf{L}$ on $\mathrm{Dom}\,\mathsf{Z}$. Now we just need to prove that $\mathrm{Dom}\,\mathsf{L} \subset \mathrm{Dom}\,\mathsf{Z}$.

For all $f \in \mathrm{Dom}\,\mathsf{L}$, all $\phi \in \mathcal{B}^*$ put

$$\mathsf{F}(t) = \left\langle \phi\,,\, (\mathsf{T}_t - \mathrm{I})f - \int_0^t \mathsf{T}_s\,\mathsf{L}\,f\ \mathrm{d}s \right\rangle .$$

We have

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}(\mathsf{F}(t+h) - \mathsf{F}(t)) &= \lim_{h\to 0} \left\langle \mathsf{T}_t^*\,\phi\,,\, \frac{1}{h}(\mathsf{T}_h\, f - f) \right\rangle \\ &\qquad - \lim_{h\to 0} \left\langle \phi\,,\, \frac{1}{h}\int_t^{t+h} \mathsf{T}_s\,\mathsf{L}\,f\ \mathrm{d}s \right\rangle \\ &= \langle \mathsf{T}_t^*\,\phi\,,\, \mathsf{L}\,f\rangle - \langle \phi\,,\, \mathsf{T}_t\,\mathsf{L}\,f\rangle = 0\,.\end{aligned}$$

This proves that $\mathsf{F}(t) = 0$ for all t. As this is true for all $\phi \in \mathcal{B}^*$, this shows that

$$(\mathsf{T}_t - \mathrm{I})f = \int_0^t \mathsf{T}_s\,\mathsf{L}\,f\ \mathrm{d}s$$

so that

$$\lim_{t\to 0} \frac{1}{t}(\mathsf{T}_t - \mathrm{I})f = \mathsf{L}\,f\,.$$

In other words f belongs to $\mathrm{Dom}\,\mathsf{Z}$ and $\mathsf{Z}\,f = \mathsf{L}\,f$. □

3.3.2 Spectrum

Consider the generator Z of a strongly continuous semigroup (T_t) satisfying

$$\|\mathsf{T}_t\| \leq M\, \mathrm{e}^{\beta t}$$

for all t (Theorem 3.30). The following theorem gives information on the spectrum $\sigma(\mathsf{Z})$ of Z and the resolvent operators $\mathsf{R}_\lambda(\mathsf{Z}) = (\lambda \mathrm{I} - \mathsf{Z})^{-1}$.

Theorem 3.37. *If* (T_t) *is a strongly continuous semigroup on* $\mathcal{B}$*, with generator* Z *and satisfying* (3.9)*, then the spectrum of* Z *is included in the set* $\{\lambda \in \mathbb{C}\,;\ Re(\lambda) \leq \beta\}$. *Moreover, for all* λ *such that* $Re(\lambda) > \beta$ *we have*

$$\mathsf{R}_\lambda(\mathsf{Z}) = \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{T}_t \,\mathrm{d}t\,.$$

Proof. If $\mathrm{Re}(\lambda) > \beta$, the integral

$$\mathsf{Y}^\lambda = \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{T}_t \,\mathrm{d}t$$

is convergent in operator norm, by (3.9). Put $g = \mathsf{Y}^\lambda f$, we have, by Proposition 3.26,

$$\begin{aligned}
\frac{1}{h}(\mathsf{T}_h\, g - g) &= \frac{1}{h}\left(\int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{T}_{t+h}\, f \,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{T}_t\, f \,\mathrm{d}t\right) \\
&= \frac{1}{h}\left(\int_h^\infty \mathrm{e}^{-\lambda(t-h)}\, \mathsf{T}_t\, f \,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{T}_t\, f \,\mathrm{d}t\right) \\
&= \frac{1}{h}\left((\mathrm{e}^{\lambda h} - 1)\int_0^\infty \mathrm{e}^{-\lambda t}\, \mathsf{T}_t\, f \,\mathrm{d}t - \mathrm{e}^{\lambda h}\int_0^h \mathrm{e}^{-\lambda t}\, \mathsf{T}_t\, f \,\mathrm{d}t\right).
\end{aligned}$$

Passing to the limit $h \to 0$, we get $\lambda g - f$ on the right hand side.

This proves that g belongs to $\mathrm{Dom}\,\mathsf{Z}$ and $(\lambda \mathrm{I} - \mathsf{Z})g = f$, hence $g = \mathsf{R}_\lambda(\mathsf{Z})f$. □

3.3.3 Dual Semigroup

It is very useful, in particular for semigroups of operators acting on Hilbert spaces, to consider the dual of operator semigroups. We study here their main properties.

First recall the definition of adjoint operators on Banach spaces.

Definition 3.38. Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}^*$ be its dual. Let Z be a densely defined operator on $\mathcal{B}$, with domain $\mathrm{Dom}\,\mathsf{Z}$. One defines the operator Z^* on $\mathcal{B}^*$ as follows. Consider the set

$$\mathrm{Dom}\,\mathsf{Z}^* = \{\phi \in \mathcal{B}^*\,;\ f \mapsto \langle \phi\,,\, \mathsf{Z}\, f\rangle \ \text{ is continuous on } \ \mathrm{Dom}\,\mathsf{Z}\}\,.$$

For every $\phi \in \operatorname{Dom} \mathsf{Z}^*$, the mapping $\lambda : f \mapsto \langle \phi \,,\, \mathsf{Z}\, f \rangle$ is continuous on $\operatorname{Dom} \mathsf{Z}$ which is dense. Hence λ extends to a unique continuous linear functional on $\mathcal{B}$, that is, λ belongs to $\mathcal{B}^*$. We denote by $\mathsf{Z}^*\phi$ this unique element of $\mathcal{B}^*$. To be clear, the element $\mathsf{Z}^*\phi$ is characterized by the relation

$$\langle \phi \,,\, \mathsf{Z}\, f \rangle = \langle \mathsf{Z}^*\, \phi \,,\, f \rangle$$

for all $f \in \operatorname{Dom} \mathsf{Z}$, $\phi \in \operatorname{Dom} \mathsf{Z}^*$.

Definition 3.39. If (T_t) is a strongly continuous semigroup of operators on a Banach space $\mathcal{B}$, one can consider the *dual semigroup* (T_t^*) on $\mathcal{B}^*$. Clearly it is a semigroup of operators on $\mathcal{B}^*$. The point is that, in general, the semigroup (T_t^*) may not be strongly continuous. Let us see that with a counter-example.

Consider the operators T_t on $L^1(\mathbb{R})$ defined by $(\mathsf{T}_t\, f)(s) = f(s+t)$ for all $s, t \in \mathbb{R}$. Clearly the T_t's are norm 1 operators and they form a semigroup. Let us show that they constitute a strongly continuous semigroup. The fact that $\lim_{h\to 0} \|\mathsf{T}_t\, f - f\|_1 = 0$ is clear on the dense subspace of continuous functions with compact support, by Lebesgue's Theorem. As the norm of T_t is uniformly bounded in t, it is easy to show that the property $\lim_{h\to 0} \|\mathsf{T}_t\, f - f\|_1 = 0$ extends to all $f \in L^1(\mathbb{R})$.

The dual semigroup of (T_t) is the semigroup $(\mathsf{T}_t^* f)(s) = f(s-t)$ defined on $L^\infty(\mathbb{R})$. It is not strongly continuous on $L^\infty(\mathbb{R})$, as can be seen easily by taking for f any indicator function.

Definition 3.40. The dual semigroup (T_t^*) is not strongly continuous in general, but it is always *-weakly continuous, for

$$\langle \mathsf{T}_h \phi - \phi \,,\, f \rangle = \langle \phi \,,\, \mathsf{T}_h f - f \rangle$$

for all $\phi \in \mathcal{B}^*$, all $f \in \mathcal{B}$, and hence the limit is 0 when h tends to 0.

One defines the *-weak generator Z' of (T_t^*) as follows

$$\operatorname{Dom} \mathsf{Z}' = \left\{ \phi \in \mathcal{B}^* \,;\; *\text{-weak-}\lim_{h\to 0} \frac{1}{h} \left(\mathsf{T}_h^* \phi - \phi\right) \text{ exists} \right\}$$

and

$$\mathsf{Z}'\phi = *\text{-weak-}\lim_{h\to 0} \frac{1}{h} \left(\mathsf{T}_h^* \phi - \phi\right)$$

for all $\phi \in \operatorname{Dom} \mathsf{Z}'$.

We now wish to prove the following theorem characterizing Z'.

Theorem 3.41. *Let (T_t) be a strongly continuous semigroup of operators on $\mathcal{B}$ with generator Z. The *-weak generator Z' of (T_t^*) and its domain $\operatorname{Dom} \mathsf{Z}'$ coincide with the operator Z^* and its domain $\operatorname{Dom} \mathsf{Z}^*$.*

The proof of this theorem will be achieved in several steps. The first step is the definition of a **-weak integral* on $\mathcal{B}^*$.

Definition 3.42. Let ϕ be a function from $(E, \mathcal{E})$ to $\mathcal{B}^*$. We say that ϕ is **-weakly-measurable* if $x \mapsto \langle \phi(x)\,,\, f\rangle$ is measurable for all $f \in \mathcal{B}$.

If ϕ is *-weakly measurable and if we have

$$\int_E |\langle \phi(x)\,,\, f\rangle| \,\mathrm{d}\mu(x) < \infty$$

for all $f \in \mathcal{B}$, we shall say that ϕ is **-weakly integrable.*

Proposition 3.43. *If ϕ is a *-weakly integrable function from $(E, \mathcal{E}, \mu)$ to $\mathcal{B}^*$ then the mapping*

$$f \mapsto \int_E \langle \phi(x)\,,\, f\rangle \,\mathrm{d}\mu(x)$$

is a continuous linear form on $\mathcal{B}$.

Proof. Let T be the operator from $\mathcal{B}$ to $L^1(E, \mathcal{E}, \mu)$ defined by

$$(\mathsf{T}f)(x) = \langle \phi(x)\,,\, f\rangle\ .$$

We claim that T is a closed operator. Indeed, if (f_n) converges to f in $\mathcal{B}$ and if $(\mathsf{T}f_n)$ converges to y in $L^1(E)$, then there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $\mathsf{T}f_{n_k}$ tends to y almost surely. But $\mathsf{T}f_{n_k}(x) = \langle \phi(x)\,,\, f_{n_k}\rangle$ converges to $\langle \phi(x)\,,\, f\rangle$ for all $x \in E$. Hence $y = \mathsf{T}\,f$ and T is closed.

By the Closed Graph Theorem this means that T is bounded, for it is everywhere defined and closed.

The linear form of the proposition is the composition of T and of the mapping $f \mapsto \int_E f \,\mathrm{d}\mu$ from $L^1(E)$ to $\mathbb{C}$. One concludes easily. □

The linear form above is thus an element of $\mathcal{B}^*$. This allows the following definition.

Definition 3.44. One defines the **-weak-integral*

$$\text{*-w-}\int_E \phi(x)\,\mathrm{d}\mu(x)$$

as being the unique element of $\mathcal{B}^*$ which satisfies

$$\left\langle \text{*-w-}\int_E \phi(x)\,\mathrm{d}\mu(x)\,,\, f\right\rangle = \int_E \langle \phi(x)\,,\, f\rangle \,\mathrm{d}\mu(x)$$

for all $f \in \mathcal{B}$.

As a second step of the proof of Theorem 3.41 we derive useful identities for the semigroup (T_t^*), using the *-weak integral. They are the one for (T_t^*) corresponding to the (3.11) for (T_t).

Proposition 3.45. *Let (T_t^*) be the dual semigroup of a strongly continuous semigroup (T_t) on $\mathcal{B}$. Then for all $\phi \in \mathcal{B}^*$ we have*

$$*\text{-w-}\int_0^t \mathsf{T}_s^*\phi \,\mathrm{d}s \in \operatorname{Dom}\mathsf{Z}^*$$

and

$$\mathsf{Z}^*\left(*\text{-w-}\int_0^t \mathsf{T}_s^*\phi \,\mathrm{d}s\right) = \mathsf{T}_t^*\phi - \phi\,. \tag{3.12}$$

Furthermore, if ϕ *belongs to* $\operatorname{Dom}\mathsf{Z}^*$ *then*

$$\mathsf{Z}^*\left(*\text{-w-}\int_0^t \mathsf{T}_s^*\phi \,\mathrm{d}s\right) = *\text{-w-}\int_0^t \mathsf{T}_s^*\,\mathsf{Z}^*\phi \,\mathrm{d}s\,. \tag{3.13}$$

Proof. For all $f \in \operatorname{Dom}\mathsf{Z}$ we have

$$\begin{aligned}\left\langle *\text{-w-}\int_0^t \mathsf{T}_s^*\phi \,\mathrm{d}s\,,\, \mathsf{Z}\,f\right\rangle &= \int_0^t \langle \mathsf{T}_s^*\phi\,,\, \mathsf{Z}\,f\rangle \,\mathrm{d}s\\ &= \int_0^t \langle \phi\,,\, \mathsf{T}_s\,\mathsf{Z}\,f\rangle \,\mathrm{d}s\\ &= \left\langle \phi\,,\, \int_0^t \mathsf{T}_s\,\mathsf{Z}\,f\,\mathrm{d}s\right\rangle\\ &= \langle \phi\,,\, \mathsf{T}_t f - f\rangle\\ &= \langle \mathsf{T}_t^*\phi - \phi\,,\, f\rangle\,.\end{aligned}$$

This proves that $*\text{-w-}\int_0^t \mathsf{T}_s^*\phi \,\mathrm{d}s$ belongs to $\operatorname{Dom}\mathsf{Z}^*$ and this proves (3.12).

If furthermore ϕ belongs to $\operatorname{Dom}\mathsf{Z}^*$ then the same computation gives

$$\begin{aligned}\left\langle \mathsf{Z}^*\left(*\text{-w-}\int_0^t \mathsf{T}_s^*\phi \,\mathrm{d}s\right)\,,\, f\right\rangle &= \int_0^t \langle \phi\,,\, \mathsf{T}_s\,\mathsf{Z}\,f\rangle \,\mathrm{d}s\\ &= \int_0^t \langle \mathsf{T}_s^*\,\mathsf{Z}^*\phi\,,\, f\rangle \,\mathrm{d}s\\ &= \left\langle *\text{-w-}\int_0^t \mathsf{T}_s^*\,\mathsf{Z}^*\phi\,\mathrm{d}s\,,\, f\right\rangle\,.\end{aligned}$$

This proves (3.13). □

We can now conclude for the proof of Theorem 3.41; this is our third step.

Proof (of Theorem 3.41). Let ϕ belong to $\operatorname{Dom}\mathsf{Z}^*$ and $f \in \mathcal{B}$. We have, by Proposition 3.45

$$\begin{aligned}\frac{1}{h}\langle \mathsf{T}_h^*\phi - \phi\,,\, f\rangle &= \frac{1}{h}\left\langle \mathsf{Z}^*\left(*\text{-w-}\int_0^h \mathsf{T}_s^*\,\phi\,\mathrm{d}s\right)\,,\, f\right\rangle\\ &= \frac{1}{h}\int_0^h \langle \mathsf{T}_s^*\,\mathsf{Z}^*\,\phi\,,\, f\rangle \,\mathrm{d}s\,.\end{aligned}$$

Taking the limit when h tends to 0 we obtain $\langle \mathsf{Z}^* \phi\,,\, f\rangle$. This proves that ϕ belongs to $\mathrm{Dom}\, \mathsf{Z}'$ and that $\mathsf{Z}'\phi = \mathsf{Z}^*\phi$. We have proved that $\mathsf{Z}^* \subset \mathsf{Z}'$.

Conversely, if ϕ belongs to $\mathrm{Dom}\, \mathsf{Z}'$ and if $f \in \mathrm{Dom}\, \mathsf{Z}$, then

$$\begin{aligned}\langle \mathsf{Z}'\,\phi\,,\, f\rangle &= \lim_{h\to 0} \frac{1}{h}\, \langle \mathsf{T}^*_h\,\phi - \phi\,,\, f\rangle \\ &= \lim_{h\to 0} \frac{1}{h}\, \langle \phi\,,\, \mathsf{T}_h f - f\rangle \\ &= \langle \phi\,,\, \mathsf{Z}\, f\rangle\ .\end{aligned}$$

This proves that ϕ belongs to $\mathrm{Dom}\, \mathsf{Z}^*$ and that $\mathsf{Z}^*\phi = \mathsf{Z}'\phi$. The theorem is proved. □

The case of reflexive Banach space is much more easy. The following theorem is now easy to prove and left to the reader.

Theorem 3.46. *If (T_t) is a strongly continuous semigroup on a reflexive Banach space $\mathcal{B}$, with generator Z, then $(\mathsf{T}^*_t)_{t\in\mathbb{R}^+}$ is a strongly continuous semigroup on $\mathcal{B}^*$, with generator Z^*.*

3.3.4 Predual Semigroups

Another situation which is very useful is the case where the Banach $\mathcal{B}$ admits a *predual space*.

Definition 3.47. The *predual* of a Banach space $\mathcal{B}$, when it exists, is a Banach space $\mathcal{B}_*$ such that $\mathcal{B}$ is the dual of $\mathcal{B}_*$.

This is typical of the situation where $\mathcal{B}$ is $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_* = \mathcal{L}_1(\mathcal{H})$ or more generally when $\mathcal{B}$ is a, so-called, von Neumann algebra.

Lemma 3.48. *If L is a bounded operator on $\mathcal{B}$ and if L is also $*$-weakly continuous, then there exists a unique bounded operator L_* on $\mathcal{B}_*$ such that*

$$\langle \mathsf{L}\, f\,,\, \phi\rangle = \langle f\,,\, \mathsf{L}_*\, \phi\rangle$$

for all $\phi \in \mathcal{B}_*$, all $f \in \mathcal{B}$. In particular L is the adjoint of L_*.

Proof. The linear form $\lambda : f \mapsto \langle \mathsf{L}\, f\,,\, \phi\rangle$ is $*$-weakly continuous on $\mathcal{B}$, by the $*$-weak continuity of L. Hence by Theorem 3.7, there exists an element $\phi_* \in \mathcal{B}_*$ such that $\lambda(f) = \langle f\,,\, \phi_*\rangle$ for all $f \in \mathcal{B}$. The element ϕ_* depends linearly on ϕ, we denote it by $\mathsf{L}_*\,\phi$. The mapping L_* is bounded on $\mathcal{B}_*$ for

$$|\langle f\,,\, \mathsf{L}_*\,\phi\rangle| = |\langle \mathsf{L}\, f\,,\, \phi\rangle| \le \|\phi\|\, \|\mathsf{L}\|\, \|f\|\ .$$

Uniqueness is obvious and the relation $\mathsf{L} = (\mathsf{L}_*)^*$ is also immediate now. □

Definition 3.49. The map L_* is call the *predual map* of L.

Theorem 3.50. *Let $\mathcal{B}$ be a Banach space which admits a predual $\mathcal{B}_*$. Let (T_t) be a strongly continuous semigroup of operators on $\mathcal{B}$ such that all the mappings T_t are $*$-weakly continuous. Consider the semigroup (T_{*t}) on $\mathcal{B}_*$, defined by*

$$\langle f\,,\,\mathsf{T}_{*t}\,\phi\rangle = \langle \mathsf{T}_t\,f\,,\,\phi\rangle\ ,$$

for all $\phi \in \mathcal{B}_*$ and all $f \in \mathcal{B}$. Then (T_{*t}) is a strongly continuous semigroup of bounded operators on $\mathcal{B}_*$. The semigroup (T_t) is the dual of the semigroup (T_{*t}).

Proof. By Lemma 3.48 each of the maps T_t admit a predual map T_{*t}. It is easy to check that (T_{*t}) is a semigroup on $\mathcal{B}_*$. Furthermore, for all $f \in \mathcal{B}$ and all $\phi \in \mathcal{B}_*$ we have

$$\langle \mathsf{T}_t\,f - f\,,\,\phi\rangle = \langle f\,,\,\mathsf{T}_{*t}\,\phi - \phi\rangle\ .$$

Hence the semigroup (T_{*t}) is weakly continuous. By Theorem 3.31 it is strongly continuous. The rest of the properties have been already proved in Lemma 3.48. □

3.3.5 Stone's Theorem

We end this section and this lecture with the very important theorem of Stone, characterizing unitary groups of operators.

Definition 3.51. On a Hilbert space $\mathcal{H}$, consider a *strongly continuous unitary semigroup*, that is, a strongly continuous semigroup of operators $(\mathsf{U}_t)_{t\in\mathbb{R}^+}$ on $\mathcal{H}$ such that each of the operators U_t is unitary. For every $t \in \mathbb{R}^+$, put

$$\mathsf{U}_{-t} = \mathsf{U}_t^{-1} = \mathsf{U}_t^*\,. \tag{3.14}$$

Then it is very easy to check that the family $(\mathsf{U}_t)_{t\in\mathbb{R}}$ now satisfies

$$\mathsf{U}_t\,\mathsf{U}_s = \mathsf{U}_{s+t}$$

for all $s, t \in \mathbb{R}$. That is, the unitary semigroup can be extended into a *unitary group*, indexed by $\mathbb{R}$ now.

Conversely, if $(\mathsf{U}_t)_{t\in\mathbb{R}}$ is a unitary group then the relation

$$\mathsf{U}_{-t}\,\mathsf{U}_t = \mathsf{U}_0 = \mathrm{I}$$

leads to

$$\mathsf{U}_{-t} = \mathsf{U}_t^{-1} = \mathsf{U}_t^* \tag{3.15}$$

for all $t \in \mathbb{R}$.

Theorem 3.52 (Stone's Theorem). *A family $(\mathsf{U}_t)_{t\in\mathbb{R}}$ of operators on $\mathcal{H}$ is a strongly continuous unitary (semi)group if and only its generator is of the form* $\mathrm{i}\,\mathsf{H}$ *where* H *is self-adjoint.*

In that case we have

$$\mathsf{U}_t = \mathrm{e}^{\mathrm{i}t\,\mathsf{H}}\,,$$

for all $t \in \mathbb{R}$, in the sense of the functional calculus for self-adjoint operators.

Proof. Let H be a self-adjoint operator on $\mathcal{H}$. If, for each $t \in \mathbb{R}$ we put $\mathsf{U}_t = \mathrm{e}^{\mathrm{i}t\,\mathsf{H}}$ for some self-adjoint operator H, in the sense of the functional calculus of self-adjoint operators, then $(\mathsf{U}_t)_{t\geq 0}$ is clearly a unitary group by the functional calculus. It is strongly continuous for, using the spectral measure μ_φ associated to H, we have

$$\|\mathsf{U}_t\,\varphi - \varphi\|^2 = \int_{\mathbb{R}} |\mathrm{e}^{\mathrm{i}t\lambda} - 1|^2\,\mathrm{d}\mu_\varphi(\lambda)$$

which converges to 0 as t tends to 0. Finally, if φ is such that

$$\lim_{t\to 0} \frac{\mathsf{U}_t\,\varphi - \varphi}{t}$$

exists then define the operator B by

$$\mathsf{B}\,\varphi = \frac{1}{\mathrm{i}}\lim \frac{\mathsf{U}_t\,\varphi - \varphi}{t}\,.$$

We then easily check that

$$\langle \varphi, \mathsf{B}\,\psi\rangle = \langle \mathsf{B}\,\varphi, \psi\rangle$$

and thus B is symmetric.

Now, if φ belongs to $\mathrm{Dom}\,\mathsf{H}$ we have by the Spectral Theorem again

$$\left\|\frac{\mathsf{U}_h\varphi - \varphi}{h} - \mathrm{i}\,\mathsf{H}\varphi\right\|^2 = \int_{\mathbb{R}} \left|\frac{\mathrm{e}^{\mathrm{i}h\lambda} - 1}{h} - \mathrm{i}\,\lambda\right|\,\mathrm{d}\mu_\varphi(x)\,.$$

The right-hand side converges to 0 by Lebesgue's Theorem (use $|\mathrm{e}^x - 1| \leq |x|$). This proves that $\mathsf{H} \subset \mathsf{B}$. As B is symmetric, this gives finally $\mathsf{B} = \mathsf{H}$.

Conversely, let $(\mathsf{U}_t)_{t\geq 0}$ be a strongly continuous unitary semigroup. It has a generator K which is a closed operator (Proposition 3.34).

Put $\mathsf{V}_t = \mathsf{U}_t^*$ for all $t \in \mathbb{R}^+$. This defines another semigroup, the dual semigroup of (U_t), with generator equal to K^* by Theorem 3.46. If f belongs to $\mathrm{Dom}\,\mathsf{K}^*$ then, for all $t > h > 0$, we have

$$\frac{1}{h}(\mathsf{U}_h - \mathrm{I})\mathsf{V}_t\,f = \frac{1}{h}(\mathsf{V}_{t-h} - \mathsf{V}_t)f\,.$$

Hence, when h tends to 0, the limit of the above exists and is equal to $-\mathsf{K}^*\mathsf{V}_t\, f$. This proves that $\mathsf{V}_t \operatorname{Dom} \mathsf{K}^* \subset \operatorname{Dom} \mathsf{K}$ and that $\mathsf{K} = -\mathsf{K}^*$ on $\mathsf{V}_t \operatorname{Dom} \mathsf{K}^*$.

Every $g \in \operatorname{Dom} \mathsf{K}^*$ can be written as $g = \mathsf{U}_t\, \mathsf{V}_t\, g$. Hence $\mathsf{V}_t\, g$ belongs to $\operatorname{Dom} \mathsf{K}$ and finally $g = \mathsf{U}_t\, \mathsf{V}_t\, g$ belongs to $\operatorname{Dom} \mathsf{K}$ by Proposition 3.35. We have proved that $\operatorname{Dom} \mathsf{K}^* \subset \operatorname{Dom} \mathsf{K}$.

Inverting the roles of (U_t) and (V_t), that is, considering the relation $\mathsf{U}_t = \mathsf{V}_t^*$ we would obtain in the same way: $\operatorname{Dom} \mathsf{K} \subset \operatorname{Dom} \mathsf{K}^*$.

We have proved that $\mathsf{K} = -\mathsf{K}^*$, that is, $\mathsf{K} = \mathrm{i}\,\mathsf{H}$ for some self-adjoint operator H. The theorem is proved. □

Notes

In order to write this chapter we have used the following famous reference books on Functional Analysis and Operator Semigroups: the book by Davies [Dav80] on semigroups, Dunford-Schwarz's famous bible on linear operators [DS88], the book by Yosida [Yos80]. Complements, in particular on weak topologies for Banach spaces, can be found in Rudin's book [Rud91] or Conway's book [Con85].

There are other references that may be consulted on the subject of Operator Semigroups. The first volume of Bratelli-Robinson's book [BR87] contains a nice chapter on semigroups. The book by Clement et al [CHA+87] is quite advanced in the subject. The second volume of Reed-Simon's series [RS75] contains a summary of the main results.

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