CONVEX BODIES WITH A DENSE PROJECTIVE ORBIT

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Abstract. For every n, there exists an n-dimensional convex body such that the set of its images under projective transformations is dense.

Denote by K_n the space of nonempty convex compact subsets of the Euclidean space $\mathbf{R}^n$. When equipped with the Hausdorff distance, K_n is a complete metric space [2, Chapter 1.8].

An homography on $\mathbf{R}^n$ is a partially defined map of the form $x \mapsto A(x)/f(x)$, where $A : \mathbf{R}^n \to \mathbf{R}^n$ and $f : \mathbf{R}^n \to \mathbf{R}$ are affine maps of maximal rank. If we think of $\mathbf{R}^n$ as a subset of the projective plane $\mathbf{P}^n$, every homography extends to an automorphism of $\mathbf{P}^n$. For $K \in \mathsf{K}_n$, we consider the *projective orbit* of K

$$\mathrm{Orb}(K) \coloneqq \{\Phi(K) \ : \ \Phi \text{ homography which is well-defined on } K\} \subset \mathsf{K}_n.$$

Theorem 1. *For any $n \geqslant 2$, there is a convex body $K \in \mathsf{K}_n$ with the property that $\mathrm{Orb}(K)$ is dense in K_n.*

Theorem 1 can be reformulated without mentioning explicitly homographies. Identify $\mathbf{R}^n$ with the hyperplane $H \coloneqq \{x_{n+1} = 1\} \subset \mathbf{R}^{n+1}$ and consider the $(n+1)$-dimensional cone $\mathcal{C} \coloneqq \{(tx, t) \ : \ t \geqslant 0, \ x \in K\}$, with K given by Theorem 1. The cone $\mathcal{C}$ has the following property: the family of convex compact subsets which appear as $T(\mathcal{C}) \cap H$ for some $T \in \mathsf{GL}(n+1, \mathbf{R})$ is dense in K_n.

We show actually that a generic K (in the Baire category sense) satisfies the conclusion of Theorem 1 (for a survey of Baire category methods in convexity, see [1]). The key point is the following lemma.

Lemma 2. *Let U, V be two non-empty open subsets in K_n. Then there exists $K \in U$ such that $\mathrm{Orb}(K)$ intersects V.*

It is easy to derive Theorem 1 from Lemma 2. For every nonempty open subset $V \subset \mathsf{K}_n$, the set $\{K \in \mathsf{K}_n \ : \ \mathrm{Orb}(K) \cap V \neq \emptyset\}$ is open (this is easy) and dense (this follows from Lemma 2). Since K_n is secound-countable, Baire category theorem implies that the set $\{K \in \mathsf{K}_n \ : \ \mathrm{Orb}(K) \text{ dense}\}$ is a dense G_δ subset of K_n; in particular it is not empty.

Let $B(0,1)$ be the unit Euclidean ball, and consider also the family of ellipsoids

$$\mathcal{E}_t \coloneqq \{(x_1, \dots, x_n) \in \mathbf{R}^n \ : \ (x_1 - (1+t))^2 + t x_2^2 + \cdots + t x_n^2 \leqslant t^2\}$$

Date: January 22, 2019.
We thank Bo'az Klartag for stimulating discussions on related topics.

which converge to $\{(1,0,\dots,0)\}$ as t tends to 0. For $0<t<1$, we define an homography Φ_t by

$$\Phi_t(x_1,\dots,x_n)=\frac{(1+3t-(1+t)x_1,2tx_2,\cdots,2tx_n)}{1+t-(1-t)x_1}.$$

The following properties of Φ_t can be checked by elementary computations

(1) Φ_t is defined on $\mathrm{conv}(B(0,1)\cup\mathcal{E}_t)$,
(2) $\Phi_t=\Phi_t^{-1}$,
(3) $\Phi_t(B(0,1))=\mathcal{E}_t$, and therefore $\Phi_t(\mathcal{E}_t)=B(0,1)$.

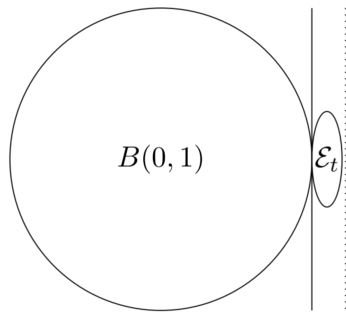

FIGURE 1. The unit disk $B(0,1)$ and the ellipse $\mathcal{E}_t$ for $t=1/10$. The involutive homography Φ_t exchanges $\mathcal{E}_t$ and $B(0,1)$, has the line $x=1$ as a set of fixed points, and sends the dotted line $x=\frac{1+t}{1-t}$ to infinity.

Proof of Lemma 2. Fix $K_1\in U$ and $K_2\in V$. By replacing K_1 and K_2 with an affine image, we may assume that

$$(1,0,\dots,0)\in K_i\subset B(0,1).$$

For $t>0$, consider the convex body $L(t):=\mathrm{conv}(K_1\cup\Phi_t(K_2))$. Since $\lim_{t\to 0}L(t)=K_1$, we have $L(t)\in U$ for t small enough. Similarly, $\Phi_t(L(t))=\mathrm{conv}(\Phi_t(K_1)\cup K_2)$ belongs to V for t small enough. Therefore, for t small enough, the choice $K=L(t)$ satisfies the conclusion of the lemma. □

References

[1] Peter M. Gruber. Baire categories in convexity. In *Handbook of convex geometry, Vol. A, B*, pages 1327–1346. North-Holland, Amsterdam, 1993.

[2] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.