Permutation groups with metrizable universal minimal flow

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When Topological Dynamics Meets Model Theory
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Recall that the closed subgroups of $S_\infty$ are exactly the automorphism groups of relational Fraïssé structures.

If $K$ is a Fraïssé structure, then $\mathcal{K} = \text{Age}(K)$ is a Fraïssé class. Conversely, if $\mathcal{K}$ is a Fraïssé class, there is up to isomorphism a unique Fraïssé structure $K = \text{Flim}(\mathcal{K})$ with $\text{Age}(K) = \mathcal{K}$. 
For $G$ a topological group, a $G$-flow is a compact Hausdorff space $X$ along with a continuous right action $\tau : X \times G \rightarrow X$. 
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A $G$-flow $X$ is minimal if every orbit is dense, and $X$ is universal if for any minimal $G$-flow $Y$, there is a map of $G$-flows $\pi : X \to Y$. 
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It is a fact that for any topological group $G$, there is up to $G$-flow isomorphism a unique flow $M(G)$ which is minimal and universal. $M(G)$ is called the universal minimal flow.
For $\mathbf{K}$ a Fraïssé structure, there is a fascinating interplay between the dynamical properties of $G = \text{Aut}(\mathbf{K})$ and the combinatorics of $\mathcal{K} = \text{Age}(\mathbf{K})$. 
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Let $\mathcal{K}$ be a class of finite structures, and let $\mathbf{A} \in \mathcal{K}$. We say that $\mathbf{A}$ is a *Ramsey object* if for every $\mathbf{B} \in \mathcal{K}$ with $\mathbf{B} \succeq \mathbf{A}$ and every $k \in \mathbb{N}$, there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{C} \succeq \mathbf{B}$ for which we have

$$\mathbf{C} \hookrightarrow (\mathbf{B})^\mathbf{A}_k$$

We say that $\mathcal{K}$ has the *Ramsey Property* if each $\mathbf{A} \in \mathcal{K}$ is a Ramsey object.
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This says that for every coloring $\gamma : \text{Emb}(A, C) \to [k]$, there is $f \in \text{Emb}(B, C)$ so that $|\gamma(f \circ \text{Emb}(A, B))| = 1$. 
For $K$ a Fraïssé structure, there is a fascinating interplay between the dynamical properties of $G = \text{Aut}(K)$ and the combinatorics of $\mathcal{K} = \text{Age}(K)$.

Let $\mathcal{K}$ be a class of finite structures, and let $A \in \mathcal{K}$. We say that $A$ is a Ramsey object if for every $B \in \mathcal{K}$ with $B \succeq A$ and every $k \in \mathbb{N}$, there is a $C \in \mathcal{K}$ with $C \succeq B$ for which we have

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We say that $\mathcal{K}$ has the Ramsey Property if each $A \in \mathcal{K}$ is a Ramsey object. We can now state the following theorem.
Theorem (Kechris-Pestov-Todorčević)

Let $\mathcal{K}$ be a Fraïssé structure, $\mathcal{K} = \text{Age}(\mathcal{K})$, and $G = \text{Aut}(\mathcal{K})$. Then $\mathcal{K}$ has the Ramsey Property iff $G$ is extremely amenable (i.e. $M(G)$ is a singleton).

Problem

Is there a similar combinatorial characterization of when $M(G)$ is metrizable?

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Let $\mathcal{K}$ be a class of finite structures, and let $A \in \mathcal{K}$. We say that $A$ has \textit{finite Ramsey degree} if there is $\ell \in \mathbb{N}$ so that for every $B \in \mathcal{K}$ with $B \geq A$ and every $k \geq \ell$, there is $C \in \mathcal{K}$ with $C \geq B$ for which we have

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\textbf{Theorem (Z.)}

\textit{Let $\mathcal{K}$ be a Fraïssé structure, $\mathcal{K} = \text{Age}(K)$, and $G = \text{Aut}(K)$. Then $M(G)$ is metrizable iff each $A \in \mathcal{K}$ has finite Ramsey degree.}
So if $M(G)$ is metrizable, what can it look like? KPT-correspondence provides us with many examples:
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- For $V_\infty$ the infinite dimensional vector space over $F_q$, $M(\text{Aut}(V_\infty))$ is the space of lex. orderings of $V_\infty$. 
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- For the tournament $S(2)$, $M(\text{Aut}(S(2)))$ is the space of admissible labelled 2-part partitions of $S(2)$. 
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Let $\mathcal{K}$ be a Fraïssé class in a language $L$ with limit $\mathbf{K}$. Let $\mathcal{K}^*$ be a Fraïssé class in $L^* = L \cup \{S_i : i \in I\}$, where the $S_i$ are countably many new relation symbols of arity $n(i)$, with limit $\mathbf{K}^*$ and with the property that $\mathbf{K}^*|_L = \mathbf{K}$ (i.e. $\mathcal{K}^*$ is a reasonable expansion of $\mathcal{K}$).
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The topological space $X_{\mathcal{K}^*}$ is the collection of all structures of the form $\langle \mathcal{K}, \vec{S}^{\mathcal{K}} \rangle$. If $A \subseteq \mathcal{K}$, $A \in \mathcal{K}$, and $A^* \in \mathcal{K}^*$ with $A^*|_L = A$, then this determines a basic open neighborhood of $X_{\mathcal{K}^*}$ via

$$N(A^*) = \{ \vec{S}^{\mathcal{K}} \in X_{\mathcal{K}^*} : \langle A, \vec{S}^{\mathcal{K}}|_A \rangle = A^* \}$$
\(X_{\mathcal{K}^*}\) is compact iff for each \(A \in \mathcal{K}\), \(\{A^* \in \mathcal{K}^* : A^*|_L = A\}\) is finite (i.e. \(\mathcal{K}^*\) is precompact). \(G = \text{Aut}(\mathcal{K})\) acts on \(X_{\mathcal{K}^*}\) via the logic action, i.e. for \(K' \in X_{\mathcal{K}^*}\), \(g \in G\), and each \(i \in I\), we have

\[
S_i^{K'} \cdot g(x_1, \ldots, x_{n(i)}) = S_i^{K'}(g(x_1), \ldots, g(x_{n(i)}))
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$$S_i^{K'} \cdot g(x_1, \ldots, x_{n(i)}) = S_i^{K'}(g(x_1), \ldots, g(x_{n(i)}))$$

We say that $\mathcal{K}^*$ has the Expansion Property if for any $A \in \mathcal{K}$, there is $B \in \mathcal{K}$ with $A \leq B$ so that for any expansions $A^*$, $B^*$ of $A$ and $B$ respectively, we have $A^* \leq B^*$. 

Theorem (Kechris-Pestov-Todorˇ cevi´ c, Nguyen Van Th´ e)

Let $\mathcal{K}$ be a Fr¨ a¨ıss´ e structure, $\mathcal{K} = \text{Age}(\mathcal{K})$, and $G = \text{Aut}(\mathcal{K})$. Let $\mathcal{K}^*$ be a reasonable, precompact Fr¨ a¨ıss´ e expansion of $\mathcal{K}$. Then $M(G) \cong X_{\mathcal{K}^*}$ iff $\mathcal{K}^*$ has the ExpP and the RP.
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$$S_i^{\mathcal{K}'} \cdot g(x_1, \ldots, x_{n(i)}) = S_i^{\mathcal{K}'}(g(x_1), \ldots, g(x_{n(i)}))$$

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**Theorem (Kechris-Pestov-Todorčević, Nguyen Van Thé)**

*Let $\mathcal{K}$ be a Fraïssé structure, $\mathcal{K} = \text{Age}(\mathcal{K})$, and $G = \text{Aut}(\mathcal{K})$. Let $\mathcal{K}^*$ be a reasonable, precompact Fraïssé expansion of $\mathcal{K}$. Then $M(G) \cong X_{\mathcal{K}^*}$ iff $\mathcal{K}^*$ has the ExpP and the RP.*
Problem

If $G$ is a closed subgroup of $S_\infty$ with $M(G)$ metrizable, can $M(G)$ be described using a logic action as above?
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Yes!

Theorem (Z.)

Let $K$ be a Fraïssé structure, $\mathcal{K} = \text{Age}(K)$, and $G = \text{Aut}(K)$. Suppose $M(G)$ is metrizable. Then $\mathcal{K}$ admits a reasonable, precompact Fraïssé expansion class $\mathcal{K}^*$ with the Expansion Property and the Ramsey Property.
This has the nice consequence of solving the Generic Point Problem for closed subgroups of $S_\infty$. 
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If $G$ is a topological group and $X$ is a minimal $G$-flow, then $x \in X$ is a \textit{generic point} if $x \cdot G$ is comeager. $G$ is said to have the \textit{Generic Point Property} if each minimal flow has a generic point. This holds iff $M(G)$ has a generic point.
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If $G$, $\mathcal{K}$, and $K$ are as always and $\mathcal{K}^*$ is a reasonable Fraïssé expansion of $\mathcal{K}$ with the Expansion Property, then the orbit of $K^* = \text{Flim}(\mathcal{K}^*)$ is generic.
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If \( G, \mathcal{K}, \text{ and } K \) are as always and \( K^* \) is a reasonable Fraïssé expansion of \( K \) with the Expansion Property, then the orbit of \( K^* = \text{Flim}(K^*) \) is generic.

\textbf{Corollary (Z.)}

\textit{Let \( G \) be a closed subgroup of \( S_\infty \), and suppose \( M(G) \) is metrizable. Then \( G \) has the Generic Point Property.}
However, the Generic Point Problem as originally asked is still open.
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**Problem (Angel, Kechris, Lyons)**

*Let $G$ be a Polish group, and suppose $M(G)$ is metrizable. Then does $M(G)$ have the Generic Point Property?*
The first ingredient in the proof is a different way of thinking about the Ramsey Property.
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Let $D$ be a countably infinite relational structure with $D = \text{Age}(D)$, and let $A \in D$. We say $T \subseteq \text{Emb}(A, D)$ is thick if for every $B \in D$, there is $f \in \text{Emb}(B, D)$ with $f \circ \text{Emb}(A, B) \subseteq T$. 

We consider partial colorings $\gamma : \text{Emb}(A, D) \to [k]$; we say $\gamma$ is full if $\text{dom}(\gamma) = \text{Emb}(A, D)$, and we say $\gamma$ is large if $\text{dom}(\gamma)$ is thick.
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Proposition

Suppose $D$ is a countably infinite relational structure, $D = \text{Age}(D)$, and $C$ is cofinal in $D$. Let $A \in C$ and fix any $k \geq 2$. Then the following are equivalent:

1. $A$ is a Ramsey object in $C$,
2. $A$ is a Ramsey object in $D$,
3. For any full $k$-coloring $\gamma$ of $\text{Emb}(A, D)$, there is some $\gamma_i$ which is thick,
4. For any large $k$-coloring $\gamma$ of $\text{Emb}(A, D)$, there is some $\gamma_i$ which is thick.
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**Proposition**

*Suppose $D$ is a countably infinite relational structure, $D = \text{Age}(D)$, and $C$ is cofinal in $D$. Let $A \in C$ and fix any $r > k$. Then the following are equivalent:*

1. $A$ has Ramsey degree $t \leq k$ in $C$,
2. $A$ has Ramsey degree $t \leq k$ in $D$,
3. Any full $r$-coloring of $\text{Emb}(A, D)$ has some subset of $k$ colors which form a thick subset,
4. Any large $r$-coloring of $\text{Emb}(A, D)$ has some subset of $k$ colors which form a thick subset.
However, there is another characterization of Ramsey degrees we will find useful.
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With $D$, $\mathcal{D}$ as above and $A \in \mathcal{D}$, we say that $S \subseteq \text{Emb}(A, D)$ is *syndetic* if $\text{Emb}(A, D) \setminus S$ is not thick.

Proposition With $D$, $\mathcal{D}$, and $A \in \mathcal{D}$, then $A$ has Ramsey degree $t \geq k$ (possibly infinite) iff there is a syndetic $k$-coloring of $\text{Emb}(A, D)$.
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**Proposition**

*With $D, \mathcal{D}$, and $A$ as above, then $A$ has Ramsey degree $t \geq k$ (possibly infinite) iff there is a syndetic $k$-coloring of $\text{Emb}(A, D)$.*
Let $A, B \in \mathcal{D}$ with $f \in \text{Emb}(A, B)$. We define $\hat{f} : \text{Emb}(B, D) \to \text{Emb}(A, D)$ via $\hat{f}(x) = x \circ f$. 

We often consider these "dual" maps when dealing with a Fra"{i}ssé structure $K$ with age $\mathcal{K}$. Notice that $K$ is a Fra"{i}ssé structure iff every such $\hat{f}$ is surjective.

Using the amalgamation property, we obtain the following:

**Proposition**

Let $K, K'$ be as above, and fix $A \leq B \in K$ and $f \in \text{Emb}(A, B)$. Then $X \subseteq \text{Emb}(A, K)$ is thick (resp. syndetic) iff $\hat{f}^{-1}(X) \subseteq \text{Emb}(B, K')$ is thick (resp. syndetic).
Let \( A, B \in \mathcal{D} \) with \( f \in \text{Emb}(A, B) \). We define \( \hat{f} : \text{Emb}(B, D) \to \text{Emb}(A, D) \) via \( \hat{f}(x) = x \circ f \).

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**Corollary**

Let $K, K, A \leq B$ be as above. Then if $B$ has Ramsey degree $k$, then $A$ has Ramsey degree $t \leq k$. In particular, if $B$ is a Ramsey object, then so is $A$. 
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This is not in general true for the “substructure” version of the Ramsey property.
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If $(X, x_0)$ and $(Y, y_0)$ are $G$-ambits, then $f : X \to Y$ is a map of $G$-ambits if $f$ is a $G$-map sending $x_0$ to $y_0$. There is at most one map of ambits from $(X, x_0)$ to $(Y, y_0)$; if there is one, we write $(X, x_0) \succeq (Y, y_0)$.
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It is a fact that every topological group $G$ admits up to isomorphism a unique greatest ambit $(S(G), 1)$; any minimal subflow of $S(G)$ is universal, hence isomorphic to $M(G)$. 
From now on, we fix once and for all a Fraïssé structure $K$ with age $\mathcal{K}$. We also set $G = \text{Aut}(K)$. Fix finite substructures $A_1 \subseteq A_2 \subseteq \cdots$ with $K = \bigcup_n A_n$. Write $H_n = \text{Emb}(A_n, K)$. 
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For $m \leq n$, let $i^n_m : A_m \hookrightarrow A_n$ be the inclusion map. This gives rise to a surjective dual map $\hat{i}^n_m : H_n \rightarrow H_m$. Note that if $m \leq n \leq N$, then $\hat{i}^N_n = \hat{i}^n_m \circ \hat{i}^N_n$. 
From now on, we fix once and for all a Fraïssé structure $K$ with age $\mathcal{K}$. We also set $G = \text{Aut}(K)$. Fix finite substructures $A_1 \subseteq A_2 \subseteq \cdots$ with $K = \bigcup_n A_n$. Write $H_n = \text{Emb}(A_n,K)$.

For $m \leq n$, let $i_m^n : A_m \hookrightarrow A_n$ be the inclusion map. This gives rise to a surjective dual map $\hat{i}_m^n : H_n \rightarrow H_m$. Note that if $m \leq n \leq N$, then $\hat{i}_n^n = \hat{i}_m^m \circ \hat{i}_n^N$.

Form $\beta H_n$, the space of all ultrafilters on $H_n$. Each $\hat{i}_m^n$ extends to a continuous surjective $\tilde{i}_m^n : \beta H_n \rightarrow \beta H_m$. If $p \in \beta H_n$ and $S \subseteq H_m$, then $S \in \tilde{i}_m^n$ iff $(\hat{i}_m^n)^{-1}(S) \in p$. 
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Now form the inverse limit $\lim \beta H_n$ along the maps $\tilde{i}_m^n$. A basic open neighborhood of $\alpha \in \lim \beta H_n$ is given by $
abla \{ \alpha' \in \lim \beta H_n : S \in \alpha'(m) \}$ for some $m \in \mathbb{N}$ and $S \subseteq H_m$, $S \in \alpha(m)$. 
G acts on $\lim_{\leftarrow} \beta H_n$ as follows: if $\alpha \in \lim_{\leftarrow} \beta H_n$, $g \in G$, and $S \in H_m$, then $S \in \alpha g(m)$ iff for some $n \geq m$, \[ \{x \in H_n : x \circ g|_m \in S\} \in \alpha(n). \] This action is jointly continuous!
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**Theorem (Pestov)**

$(\lim \beta H_n, 1)$ is the greatest $G$-ambit.
We can now give a new proof of the extreme amenability result from KPT. In fact, the proofs of many of the other results work by mimicking the methods in the proof I will present here, so this proof is in some ways representative.
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We say that $p \in \beta H_n$ is *thick* if every member of $p$ is thick. Let $R_n \subseteq \beta H_n$ be the set of thick ultrafilters.

**Proposition**

If $m \leq n$, $A_n$ is a Ramsey object, and $p \in R_m$, then there is $q \in R_n$ with $\tilde{i}_n^m(q) = p$. 

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Permutation groups with metrizable universal minimal flow
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$R_n \neq \emptyset$ iff $A_n$ is a Ramsey object.
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**Proposition**

If \( m \leq n \), \( \mathbf{A}_n \) is a Ramsey object, and \( p \in R_m \), then there is \( q \in R_n \) with \( \tilde{i}_m^n(q) = p \).
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**Theorem**

$\alpha \in \lim_{\leftarrow} \beta H_n$ is a fixed point iff $\alpha \in \lim_{\leftarrow} R_n$. In particular, $G$ is extremely amenable iff $\mathcal{K}$ has the Ramsey Property.
Let $\alpha \in \lim\limits_{\leftarrow} R_n$; suppose for sake of contradiction that there were some $g \in G$ with $\alpha g \neq \alpha$. In particular, there is $m \in \mathbb{N}$ and $S \subseteq H_m$, $S \in \alpha(m)$ with $S \not\in \alpha g(m)$. 
Let $\alpha \in \varprojlim R_n$; suppose for sake of contradiction that there were some $g \in G$ with $\alpha g \neq \alpha$. In particular, there is $m \in \mathbb{N}$ and $S \subseteq H_m, S \in \alpha(m)$ with $S \not\in \alpha g(m)$.

For some $n \geq m$, we have $T_1 := \{x \in H_n : x \circ i^n_m \in S\} \in \alpha(n)$ and $T_2 := \{x \in H_n : x \circ g|_m \not\in S\} \in \alpha(n)$. So $T_1 \cap T_2 \in \alpha(n)$. 
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For some \( n \geq m \), we have \( T_1 := \{ x \in H_n : x \circ i^m_n \in S \} \in \alpha(n) \) and \( T_2 := \{ x \in H_n : x \circ g|_m \notin S \} \in \alpha(n) \). So \( T_1 \cap T_2 \in \alpha(n) \).

Now for large \( N \geq n \), find \( h \in H_N \) with \( h \circ \text{Emb}(A_n, A_N) \subseteq T_1 \cap T_2 \). Now consider \( h \circ g|_n \circ i^m_n = h \circ i^N_n \circ g|_m \). The left side is in \( S \), but the right side is not, a contradiction.
Suppose $\alpha(m)$ is not thick, with $S \in \alpha(m)$ not thick. Find $n \geq m$ with $f \circ \text{Emb}(A_m, A_n) \not\subseteq S$ for every $f \in H_n$. 
Suppose $\alpha(m)$ is not thick, with $S \in \alpha(m)$ not thick. Find $n \geq m$ with $f \circ \text{Emb}(A_m, A_n) \not\subseteq S$ for every $f \in H_n$.

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Pick \( r \) with \( T_r \not\subseteq \alpha(n) \). Then for any \( g \in G \) with \( g|_{m} = r \), then \( \alpha g \neq \alpha \).
Thanks!