Lovely pairs of models

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Abstract

We introduce the notion of a lovely pair of models of a simple theory T, generalizing Poizat's "belles paires" of models of a stable theory and the third author's "generic pairs" of models of an SU-rank 1 theory. We characterise when a saturated model of the theory T_P of lovely pairs is a lovely pair (that is when the notion of a lovely pair is "axiomatizable"), finding an analogue of the non finite cover property for simple theories. We show that, under these hypotheses, T_P is also simple, and we study forking and canonical bases in T_P. We also prove that assuming only that T is low, the existentially universal models of the universal part of a natural expansion T_P^+ of T_P, are lovely pairs, and "simple Robinson universal domains".

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1 Introduction

In this paper we study a certain class (lovely pairs) of elementary pairs $M \subseteq N$ of models of a complete simple theory T. The language L_P for pairs consists of the language L of T together with a predicate P for the smaller model. The work here can be seen as a common generalization of the third author's work [18] on generic pairs of models of a theory of SU-rank 1, and Poizat's theory [14] of "belles paires" of models of a stable theory. One of our motivations is to find new ways of constructing simple theories. Another motivation is to find the right analogue of the "non finite cover property" for simple theories. The work here also has connections with so-called *Robinson theories* from [8].

Roughly speaking, an elementary pair $M \subset N$ of models of T will be called a *lovely pair* if (i) any complete L-type over a small subset A of N has some nonforking extension over $M \cup A$ realized in N, and (ii) if the complete L-type p over the small subset A of N does not fork over M, then p is realized in M. If T happens to be stable, this essentially agrees with Poizat's notion of a belle paire. Let $\mathcal{C}_{T,P}$ be the class of lovely pairs (as L_P-structures), and T_P the theory of this class. The general idea is that $\mathcal{C}_{T,P}$ should be more or less the class of big models or "universal domains" of a possibly non first order simple theory. The "best" case is when $\mathcal{C}_{T,P}$ is first order, that is when a saturated model of T_P is again a lovely pair, and we prove in this case that T_P is an outright simple first order theory. When T is stable, Poizat proved that $\mathcal{C}_{T,P}$ is first order if and only if T does not have the *fcp*. Our necessary and sufficient condition for $\mathcal{C}_{T,P}$ to be first order is that each of the ranks $D(-,\phi)$ be finite valued and definable. Bearing in mind Shelah's "fcp theorem" [16], this gives what we believe to be the right analogue of the "non fcp" for simple theories. But we leave open the issue of finding a nice combinatorial equivalent. A next best case is when $\mathcal{C}_{T,P}$ is more or less the class of existentially universal models of T_P (or rather of the universal part of T_P). (Here, and even before, it is convenient to work in the definitional expansion of T_P obtained by adding relation symbols for formulas $\exists x \in P(\phi(x,y))$ where $\phi \in L$). In any case, we prove that this second-best case holds just if T is low (in the sense of Buechler [2]). As the first author has observed, in general case the category of lovely pairs can be viewed profitably as a "compact abstract theory" [1], and this will be discussed in a future paper.

When T is a simple one-sorted theory whose universe has SU-rank 1, the class of lovely pairs (which turns out to be first order) was studied in detail by the third author. (He called them generic pairs, but as the word "generic" is becoming rather overused we changed to lovely pairs.) He recognized the importance of condition (ii) in the definition. For example, if T is the theory of the random graph, and $M \subseteq N$ is an elementary pair of models such that M is saturated and N is $|M|^+$-saturated, then the L_P-theory of this pair satisfies condition (i) but also has the strict order property (see [18]).

This paper assumes some knowledge and familiarity with simple theories. Frank Wagner's book [19] is a good source, as well as original papers such as [9], [10], [7], [3]. Notation is standard.

In section 2, we study various properties of formulas (lowness, definability of D_ϕ-rank etc.) which subsequently turn out to be important for the analysis of lovely pairs, but are also possibly important for their own sake. In section 3, we begin our study of lovely pairs, showing existence and examining types. In section 4, we give necessary and sufficient conditions for $\mathcal{C}_{T,P}$ to be first order. In section 5, we make the connection with Poizat's belles paires. In section 6 we prove that, if $\mathcal{C}_{T,P}$ *is* first order then T_P is a simple first order theory. In section 7, under the same assuptions, we describe forking in models of T_P, study and characterize canonical bases in T_P, and prove some related results (such as preservation of 1-basedness). In section 8, we show that assuming just lowness of T, a lovely pair is a Robinson universal domain.

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2 Properties of formulas in simple theories

T denotes a complete first order theory (not necessarily simple) in a language L. We work as usual in a very saturated model $\bar{M}$ of T, and for now we work in T^{eq}. Recall that a formula $\phi(x, b)$ is said to divide over a set A if there is an infinite A-indiscernible sequence $(b_i : i < \omega)$ of realizations of $tp(b/A)$ such that $\{\phi(x, b_i) : i < \omega\}$ is inconsistent. If we also demand k-inconsistency ($\{\phi(x, b_i) : i < k\}$ is inconsistent) we say that $\phi(x, b)$ k-divides over A. (The same definition can be made for a partial type $p(x, b)$ with b in general an infinite tuple, in place of the formula $\phi(x, b)$.) T is simple if and only if every complete type $p(x, b)$ does not divide over some $b_0 \subseteq b$ of cardinality at most $|T|$.) We recall also the $D(-, \phi)$-rank, which we denote $D_\phi(-)$: Let $\phi(x, y)$ be an L-formula, and $\psi(x)$ a formula (possibly with parameters). $D_\phi(\psi(x)) \geq 0$ if $\psi(x)$ is consistent. $D_\phi(\psi(x)) \geq \alpha + 1$ if there is some b such that $\phi(x, b)$ divides over the parameters in ψ, and $D_\phi(\psi(x) \wedge \phi(x, b)) \geq \alpha$. For δ limit, $D_\phi(\psi(x)) \geq \delta$ if it is $\geq \alpha$ for all $\alpha < \delta$. If we demand k-dividing in place of dividing (k a positive integer), we obtain the rank $D(-, \phi, k)$. T is simple if and only if $D(\psi, \phi, k) < \omega$ for all ϕ, ψ, k. On the other hand, for T a simple theory, $D_\phi(-)$ need not even be ordinal valued ([6], [4]).

Next we want to define when a formula $\phi(x, y)$ is low. The notion of a low (simple) theory was introduced by Buechler [2]. Other, notably Shami [15] and Casanovas and Kim [5], talk about low formulas (also in simple theories). The equivalence of various definitions of lowness is well-known, but we will take the liberty to reprove them below in the context of some new observations. It is convenient to take our definition of lowness from [3].

Definition 2.1 *Let $\phi(x, y)$ be an L-formula. We say that $\phi(x, y)$ is low if there is some $k < \omega$ such that whenever $(b_i : i < \omega)$ is an indiscernible sequence in some model of T, and $\{\phi(x, b_i) : i < \omega\}$ is inconsistent, then it is k-inconsistent. T is said to be low if every formula $\phi(x, y)$ is low.*

Remark 2.2 *(i) Note the similarity with the the non finite cover property (*nfcp*). $\phi(x, y)$ has the $nfcp$ if there is $k < \omega$ such that for any set $\{b_i : i \in I\}$ in a model of T, $\{\phi(x, b_i) : i \in I\}$ is consistent if it is k-consistent (that is if every $\{\phi(x, b_{i_1}), ., \phi(x, b_{i_k})\}$ is consistent).*
(ii) If $\phi(x, y)$ is stable (namely there do not exist a_i, b_i for $i < \omega$ such that $\models \phi(a_i, b_j)$ iff $i \leq j$), then $\phi(x, y)$ is low.

Proof. (ii): It is well-known ([16]) that if $\phi(x, y)$ is stable then there is $k < \omega$, such that for any indiscernible sequence $\{b_i : i < \omega\}$, and $a \in \bar{M}$, the set of b_i such that $\models \phi(a, b_i)$, has cardinality at most k, or the set of b_i such that $\models \neg\phi(a, b_i)$, has cardinality at most k. So if $\{\phi(x, b_i) : i \leq k\}$ is consistent, then $\{\phi(x, b_i) : i < \omega\}$ is consistent.

Our definition of a low formula makes sense in any theory, simple or not. In the next lemma we assume simplicity. The implication $(v) \rightarrow (i)$ is new.

Lemma 2.3 *Let T be simple, and $\phi(x, y) \in L$. Then the following are equivalent:*
(i) $\phi(x, y)$ is low,
(ii) There is $k < \omega$ such that $D_\phi(-) = D(-, \phi, k)$.
(iii) $D_\phi(x = x) < \omega$
(iv) There is a uniform bound (depending on ϕ) on $m < \omega$ for which there exist $(b_i : i < m)$ such that $\{\phi(x, b_i) : i < m\}$ is consistent but $\phi(x, b_i)$ divides over $(b_0, .., b_{i-1})$ for all $i < m$.
(v) The condition on (b, c): $\phi(x, b)$ divides over c, is type-definable. That is, for any (possibly infinite) tuple z of variables there is a partial type $\Sigma(y, z)$ such that for any b, and c of length that of z, $\models \Sigma(b, c)$ iff $\phi(x, b)$ divides over c.

Proof. (i) implies (ii): Choose k as in Definition 2.1, and then (ii) is immediate.
(ii) implies (iii), because as T is simple, $D(x = x, \phi, k) < \omega$ for all k.
(iii) implies (iv): Suppose $\phi(x, b_i)$ divides over $\{b_0, .., b_{i-1}\}$ for all $i < m$, and $\{\phi(x, b_i) : i < m\}$ is consistent. Then $D_\phi(\bigwedge_{j<i} \phi(x, b_j)) > D_\phi(\bigwedge_{j \le i} \phi(x, b_j))$ for all $i < m$, whereby $D_\phi(x = x) \ge m$.
(iv) implies (i): If $(b_i : i < \omega)$ is an indiscernible sequence such that $\{\phi(x, b_i) : i < \omega\}$ is inconsistent, then $\phi(x, b_i)$ divides over $\{b_0, b_1, .., b_{i-1}\}$ for all $i < \omega$. So if $m < \omega$ is the bound given by (iv), it follows that $\{\phi(x, b_i) : i < m + 1\}$ must be inconsistent.
(i) implies (v): There is a partial type in (b, c) expressing that there is an indiscernible over c sequence $(b_i : i < \omega)$ of realizations of $tp(b/c)$ such that $\{\phi(x, b_i) : i < k\}$ is inconsistent.
(v) implies (i): Suppose that $\phi(x, y)$ is not low. For each $k < \omega$, let $(b_i^k : i < \omega + \omega)$ be an indiscernible sequence such that that $\{\phi(x, b_i^k) : i < \omega + \omega\}$ is inconsistent but k-consistent. Let $(c_i : i < \omega + \omega)$ realize some limit point (ultraproduct) of the $\{tp((b_i^k)_{i<\omega+\omega}) : k < \omega\}$. Then $(c_i : i < \omega + \omega)$ is indiscernible, and $\{\phi(x, c_i) : i < \omega + \omega\}$ is consistent. Note that $(c_{\omega+n} : n < \omega)$ is a Morley sequence in $tp(c_\omega/\{c_i : i < \omega\})$. We conclude, by Kim's lemma ([9]) that $\phi(x, c_\omega)$ does not divide over $\{c_i : i < \omega\}$. On the other hand, clearly $\phi(x, b_\omega^k)$ divides over $\{b_i^k : i < \omega\}$ for each $k < \omega$. So "$\phi(x, y)$ divides over z" is not type-definable (in (y, z)).

From now on we assume T to be simple (so we use "forking" and "dividing" interchangeably). Let us introduce another property.

Definition 2.4 *Let $\psi(y,z)$ and $\phi(x,y)$ be L-formulas. $Q_{\phi,\psi}$ is the predicate which is defined to hold of a tuple c (in $\bar{M}$) if for all b satisfying $\psi(y,c)$, $\phi(x,b)$ does not divide over c.*

Remark 2.5 *Note that $\neg Q_{\phi,\psi}(c)$ holds just if there is a complete type $p(y,c)$ over c, containing $\psi(y,c)$ such no nonforking extension of $p(y,c)$ contains a formula of the form $\phi(d,y)$. Note also that by Lemma 2.3, if $\phi(x,y)$ is low, then $\neg Q_{\phi,\psi}$ is type-definable for all ψ.*

Definition 2.6 *Let $\phi(x,y) \in L$. We say that the D_ϕ-rank is finite and definable if for each $\psi(x,y) \in L$,*
(i) $D_\phi(\psi(x,b)) < \omega$ for all b, and
(ii) for each $n < \omega$ there is $\chi_n(y) \in L$ such that for all b, $D_\phi(\psi(x,b)) = n$ iff $\models \chi_n(b)$.

Remark 2.7 *Similarly we can define when the D_ϕ-rank is ordinal valued and definable, and by an easy compactness argument, this is equivalent to D_ϕ-rank being finite and definable.*

Proposition 2.8 *The following are equivalent:*
(i) T is low and $Q_{\phi,\psi}(z)$ is type-definable for all $\phi(x,y), \psi(y,z) \in L$.
(ii) T is low and $Q_{\phi,\psi}(z)$ is definable for all $\phi(x,y), \psi(y,z) \in L$,
(iii) for all $\phi(x,y) \in L$, the D_ϕ-rank is finite and definable.

Proof. (i) implies (ii): Lowness implies already type-definability of $\neg Q_{\phi,\psi}(z)$, so together with type-definability of $Q_{\phi,\psi}$ we get definability of $Q_{\phi,\psi}$.
(ii) implies (iii): As T is low, $D_\phi(\psi(x,c)) < \omega$ for all $\psi(x,z) \in L$ and c.
Let us now prove by induction on n that for all formulas $\theta(x,w) \in L$, there is $\chi(w) \in L$ such that $D_\phi(\theta(x,d)) \geq n$ iff $\models \chi(d)$ for all d. Assume true for n. Now $D_\phi(\theta(x,d)) \geq n+1$ iff there is b such that $D_\phi(\theta(x,d) \wedge \phi(x,b)) \geq n$ and $\phi(x,b)$ divides over d. Let $\psi(y,z)$ be the L-formula defining "$D_\phi(\theta(x,z) \wedge \phi(x,y)) \geq n$". Let $\chi(z)$ be the formula defining $\neg Q_{\phi,\psi}(z)$. Then clearly $\models \chi(d)$ iff $D_\phi(\theta(x,d)) \geq n+1$.
(iii) implies (i). Fix $\phi(x,y)$. First, as $D_\phi(x = x) < \omega$, ϕ is low (by 2.3). We will now prove that $\neg Q_{\phi,\psi}(z)$ is definable for all $\psi(y,z)$. Now $\neg Q_{\phi,\psi}(c)$ holds if and only if there is b such that $\psi(b,c)$ holds and $\phi(x,b)$ divides over c, if

and only if either
(I) there is b such that $\psi(b, c)$ holds and $\phi(x, b)$ is inconsistent, or
(II) there is b such that $\psi(b, c)$ holds, $\phi(x, b)$ is consistent, and $\phi(x, b)$ divides over c.
Now (I) is clearly definable (as a property of c). Let $\phi'(x, z, y)$ the formula $\phi(x, y) \wedge \psi(y, z)$, where the variables are divided into (x, z) and y. We claim that (II) holds iff
(*) $D_{\phi'}(x = x \wedge z = c) \geq 1$.
This is because, for any b,
(a) $\models \psi(b, c)$ and $\phi(x, b)$ is consistent if and only if $(x = x \wedge z = c \wedge \phi(x, b) \wedge \psi(b, z))$ is consistent iff $D_{\phi'}(x = x \wedge z = c \wedge \phi(x, b) \wedge \psi(b, z)) \geq 0$, and
(b) assuming that $\psi(b, c)$ holds, $\phi(x, b)$ divides over c iff $\phi(x, b) \wedge \psi(b, z)$ divides over c.
By our assumption (iii) (for $\phi'(x, z, y)$), (*) is a definable property of c, so we finish.

Remark 2.9 *Let $\psi(y, z)$ be an L-formula. Then $Q_{x=y,\psi(y,z)}(c)$ holds iff $\psi(y, c)$ defines a finite set. Hence type-definability and definability of $Q_{x=y,\psi}(z)$ are equivalent, and amount to there being a finite bound on the cardinalities of finite sets defined by $\psi(x, c)$ as c varies.*

3 Lovely pairs: first properties

Until we say otherwise T will denote a complete simple theory in language L. We will also assume (by Morleyizing) that T has quantifier-elimination, and there is no harm in assuming that L is relational. As a rule we will work inside a big, saturated model $\bar{M}$ of T. That is, any model of T is assumed to be a substructure (so elementary substructure) of $\bar{M}$. We assume for now that T is one-sorted, but we will make subsequent remarks about the many-sorted case (including T^{eq}). Let L_P be the language L together with a new unary predicate P. An L_P-structure has the form $(M, P(M))$ where M is an L-structure, and $P(M)$ is the interpretation of P. For A a subset of M, we let $P(A)$ denote $A \cap P(M)$. Given $(M, P(M))$ and $A \subset M$, we can speak of types over A in the sense of the L-structure M and in the sense of the L_P structure $(M, P(M))$. We will refer to these as L-types ($tp_L(..)$), L_P-types ($tp_{L_P}(..)$) respectively, or will just say type if the meaning is clear from the context. Usually, for any L_P-structure $(A, P(A))$ we consider, A will be a substructure of a model of T, hence a substructure of $\bar{M}$.

Definition 3.1 *Let κ be an cardinal $\geq |T|^+$. By a κ-lovely pair of models of T we mean an L_P-structure $(M, P(M))$ such that both M and $P(M)$ are models of T, and*
$(i)_\kappa$, whenever $A \subset M$ has cardinality $< \kappa$ and p is a complete finitary L-type over A, then there is $a \in M$ realizing p such that $tp_L(a/A \cup P(M))$ does not fork over A.
$(ii)_\kappa$, whenever $A \subset M$ has cardinality $< \kappa$, and p is a complete finitary L-type over A, which does not fork over $P(A)$, then p is realized in $P(M)$.
By a lovely pair of models of T we mean a $|T|^+$-lovely pair. $\mathcal{C}_{T,P}$ denotes the class of lovely pairs, and T_P the theory of this class in L_P.

Remark 3.2 *(a) $(i)_\kappa$ implies that M is κ-saturated as an L-structure. $(ii)_\kappa$ implies that $P(M)$ is κ-saturated. On the other hand if T is stable, and $P(M)$ is κ-saturated then $(ii)_\kappa$ holds.*
(b) Suppose $(M, P(M))$ is a lovely pair, $A \subseteq M$ and $p(x)$ is a complete L-type over A which is realized in M. Then $p(x)$ does not fork over $P(M)$ iff and only if $p(x)$ is finitely satisfiable in $P(M)$. So L-independence inside a lovely pair exhibits some stability-like features.
(c) In the definition of a lovely pair, we need not assume to begin that $P(M)$ is also a model of T (equivalently an elementary substructure of M), as it follows from (ii).

Remark 3.3 *(i) and (ii) of Definition 2.1 are given for types of finite tuples in the home sort. But this implies (i) and (ii) for types and sets in M^{eq} and even in M^{heq} too.*

Explanation. Let $(M, P(M))$ be a (elementary) pair of models of T. Note that $P(M)^{eq}$ is precisely the set of $e \in M^{eq}$ such that $e \in dcl_L(P(M))$. For each sort S_E of T^{eq}, and subset A of $S_E(M)$, let $P(A)$ denote $A \cap P(M)^{eq}$. If A is an arbitrary subset of M^{eq}, let $P(A)$ denote the union of the $P(A \cap S_E)$ as E varies. Suppose now that $(M, P(M))$ is κ-lovely, as in Definition 3.1. It is then completely routine to prove that if $A \subset M^{eq}$ has cardinality $< \kappa$ and p is a complete L-type over A in some imaginary sort, then a nonforking extension of p over $P(M)^{eq} \cup A$ is realized in M^{eq}. Likewise if p does not fork over $P(A)$, then p is realized in $P(M)^{eq}$. (In the latter case for example, first find $A' \subset M$ of cardinality $< \kappa$ such that $A \subseteq dcl_L(A')$ and $P(A) \subseteq dcl_L(P(A'))$, let p' be a nonforking extension of p over A'. Realize p' (in $\bar{M}$) by some e. Suppose $e = a/E$, with a real. Choose a' such that $a'/E = e$ and a' is independent from A' over e. Let $q' = tp_L(a'/A')$. Then q' does not fork over $P(A')$ so is realized in $P(M)$, so p' (and hence p) is realized in $P(M^{eq})$.)

The same thing works for hyperimaginaries. First note that $P(M)^{heq}$ is (up to interdefinability) precisely $\{e \in M^{heq} : e \in dcl(P(M))\}$. But if $(M, P(M))$ is κ-lovely, we should be careful to work with hyperimaginaries which are $< \kappa$-ary (that is of the form a/E where a is a real tuple of length $< \kappa$).

By the above remark, we will freely work in T^{eq}. We suppose that we should formally consider a separate predicate P_E for each sort S_E, but we will in practise work with notation as in the explanation above. Similarly we could formulate the whole set-up for many sorted theories. Also by the above explanation we will work freely with hyperimaginaries.

Remark 3.4 *For $\kappa \geq |T|^+$, axiom $(ii)_\kappa$ in Definition 3.1 above can be restated as: whenever $A \subseteq M$ has cardinality $< \kappa$ and $p(x)$ is a complete L-type over A which does not fork over $P(M)$, then p is realized in $P(M)$. For this reason we call $(ii)_\kappa$ the "coheir property".*

Proof. Suppose $(ii)_\kappa$ is satisfied by $(M, P(M))$. Let a and p be as given. So p has an extension p' to a complete L-type over $A \cup P(M))$ which does not fork over $P(M))$. Let $B \subseteq P(M)$ be of cardinality at most $|T|$ such that p' does not fork over B. Let p'' be the restriction of p' to $A \cup B$. By axiom $(ii)_\kappa$, p'' (and so p) is realized in $P(M)$. The other direction is immdiate.

Lemma 3.5 *κ-lovely pairs exist.*

Proof. Fix any pair $(M_0, P(M_0))$ of models of T of cardinality at least κ. We will construct a chain $(M_i, P(M_i))$ for $i < \kappa^+$ of pairs such that
(a) for any i, any complete L-type over M_i which does not fork over $P(M_i)$ is realized in $P(M_{i+1})$.
(b) $i < j$ implies that $P(M_j)$ is L-independent from M_i over $P(M_i)$
(c) For i successor, M_i is $(\kappa + |P(M_i)|^+)$-saturated (as a model of T).

This is easily accomplished. Given $(M_i, P(M_i))$, let $(p_j : j < \lambda)$ be a list of all complete L-types over M_i which do not fork over $P(M_i)$. Realize them independently (in $\bar{M}$) by $A = (a_j : j < \lambda)$. Then A is independent from M_i over $P(M_i)$. Let N be a model of T (elementary substructure of $\bar{M}$) containing $P(M_i) \cup A$ and independent from M_i over $P(M_i)$. Let M be a model of T containing $M_i \cup N$ which is $(\kappa + |N|^+)$-saturated. Define $M_{i+1} = M$ and $P(M_{i+1}) = N$

Having defined the chain of $(M_i, P(M_i))$, let $(M, P(M))$ be the union. We claim that $(M, P(M))$ is κ-lovely. If A is a subset of M of cardinality $< \kappa$ then $A \subseteq M_i$ for some i. If p is a complete L-type over A then by (c), some (in fact any) nonforking extension of p over $A \cup P(M_i)$ is realized in M_i by d say. By (b), dA is independent from $P(M)$ over $P(M_i)$, and thus d is independent from $A \cup P(M)$ over A.
Now (for the same A), suppose p is a complete L-type over A which does not fork over $P(A)$. Let q be a nonforking extension of p over M_i. Then q does not fork over $P(M_i)$, so by (a) is realised in $P(M_{i+1}) \subset P(M)$.

Remark 3.6 *(i) The construction in Lemma 3.5 is a bit crude and could be refined to bound the cardinality of the κ-lovely pair.*
(ii) The proof shows that any L_P-structure of the form $(A, P(A))$ where A is a subset of a model of T and $P(A)$ is relatively algebraically closed in A (in the sense of $\bar{M}$), embeds in a κ-lovely pair $(M, P(M))$ such that $P(M)$ is L-independent from A over $P(A)$.

We now want to see when two tuples in lovely pairs have the same type. Recall that we are assuming T to have quantifier-elimination. So the quantifier-free L_P-type of a tuple a in a pair of models of T consists precisely of the L-type of a together with the information about which coordinates of a are in P or not in P. The following definition is convenient:

Definition 3.7 *Let A be a subset of a pair $(M, P(M))$ of models of T. We say that A is P-independent if A is independent from $P(M)$ over $P(A)$ (in the sense of L).*

Lemma 3.8 *Let $(M, P(M))$ and $(N, P(N))$ be κ-lovely pairs ($\kappa > |T|$). Let a, b be tuples of the same length $< \kappa$ from M, N, respectively, which are both*

P-independent. Assume that a and b have the same quantifier-free L_P-type. Then a and b have the same L_P-type.

Proof. Let $f : a \rightarrow b$ be the partial L_P-isomorphism given by our hypothesis. It is enough (by back-and-forth, and symmetry)) to show that any $c \in M$ is included in the domain of a partial L_P-isomorphism g extending f. So choose c. Extending c to a suitable small tuple, we may assume that ca is P-independent. Let $c_1 = P(c)$ and let c_2 be the rest of c. Let p be the L-type of c_1 over a, and let p' be its copy over b. Then by P-independence of a and b in $(M, P(M))$ and $(N, P(N))$ respectively, and the axiom (ii) of lovely pairs, p' is realized in $P(N)$ by some d_1. Now let q be the L-type of c_2 over c_1a, and q' the copy over d_1b. Then by the axiom (i) of lovely pairs, some nonforking extension of q' over $P(N) \cup a$ is realized in N, by say d_2. Note that all coordinates of d_2 are outside $P(N)$. Let g extend f by taking c_1 to d_1 and c_2 to d_2. Then g is a partial L_P-isomorphism. The proof is complete.

Corollary 3.9 *All lovely pairs are elementarily equivalent (in fact (∞, ω) equivalent). T_P is a complete L_P-theory.*

Recall that if M is a model of T, and $p(x)$ a complete type over M, then $cl(p)$ denotes the set of L-formulas $\phi(x, y)$ such that $\phi(x, b)$ in $p(x)$ for some $b \in M$. For T stable, $p(x) \in S(A)$, and nonforking extensions q_1, q_2 of p to models, $cl(q_1) = cl(q_2)$ equals the unique smallest class among extensions of p to models. For a simple theory T this is no longer necessarily the case, but in [11] it was pointed out that among nonforking extensions q of $p(x) \in S(A)$ to models there is a unique maximal class. Let us call this class $m\beta(p)$ (THE maximal or maximum class among nonforking extensions of p to models).

Remark 3.10 *Let $(M, P(M))$ be a lovely pair. Then for any $A \subseteq P(M)$ of cardinality at most $|T|$, $p(x) \in S(A)$ and nonforking extension $q(x) \in S(P(M))$ of p which is realized in M, $cl(q) = m\beta(p)$.*

Proof. Let $c \in M$ realize q. Suppose for a contradiction that $cl(q)$ is not maximal (among classes of nonforking extensions of p to models). So there is (somewhere) d independent from c over A, and an L-formula $\phi(x, y)$ such that $\phi(c, d)$ but $\phi(x, y) \notin cl(q)$. As $(M, P(M))$ is a lovely pair we may realize $tp_L(d/Ac)$ by d' in $P(M)$. So $\phi(x, y) \in cl(q)$, contradiction.

It will be convenient for now and quite important for later to work in a certain definitional expansion L_P^+ of L_P. L_P^+ consists of L_P together with new relation symbols $R_\phi(y)$ for each L-formula $\phi(x,y)$. T_P^+ denotes T_P together with the sentences "$\forall y(R_\phi(y) \leftrightarrow \exists x(P(x) \wedge \phi(x,y)))$", which define the new relation symbols. So (by Corollary 3.9), T_P^+ is a complete L_P^+-theory. Note that the unary predicate P is superfluous in T_P^+ as T_P^+ implies $\forall y(P(y) \leftrightarrow R_{x=y}(y))$. We consider any lovely pair $(M, P(M))$ as an L_P^+-structure in the canonical way, that is so that it becomes a model of T_P^+.

We can now rephrase Lemma 2.7 using L_P^+, generalizing Theorem 4 of [14]

Corollary 3.11 *Let $(M, P(M))$, $(N, P(M))$ be lovely pairs. Let a, b be tuples of the same length from M, N, respectively. Then the following are equivalent:*
(i) $cl(tp_L(a/P(M)) = cl(tp_L(b/P(N))$.
(ii) a and b have the same quantifier-free L_P^+ type (in the structures $(M, P(M))$, $(N, P(N))$ viewed as L_P^+ structures as described above).
(iii) The L_P (so also L_P^+)-types of a and b in $(M, P(M)), (N, P(N))$ respectively, are equal.

Proof. The equivalence of (i) and (ii) is immediate (and only requires that the predicates R_ϕ are interpreted according to their defining axioms). (iii) implies (ii) is immediate. So we only have to prove that (ii) implies (iii).
So assume (ii). Note that a and b have the same L-type. Let $A \subseteq P(M)$ be of cardinality at most $|T|$ such that $tp_L(a/P(M))$ does not fork over A. Let $q(z,a)$ be the L-type of A over a. As b has the same quantifier-free L_P^+-type as a, $q(z,b)$ is finitely satisfiable in $P(N)$, and so does not fork over $P(N)$. By Remark 3.4, $q(z,b)$ is realized in $P(N)$, by B say.
We claim that $tp_L(b/P(N))$ does not fork over B. Suppose for a contradiction that $tp_L(b/P(N))$ forks over B. Then there are $\phi(x,y) \in L$, $k < \omega$, $\psi(x,z) \in L$, and $c \in P(N)$ such that $\psi(b,c)$ holds, but $D(\psi(x,c),\phi,k) < n$ where $n = D(tp_L(b/B),\phi,k)$. Now there is a formula $\chi(z) \in tp_L(c)$ such that for any c' realizing χ, $D(\psi(x,c'),\phi,k) < n$. As a and b have the same quantifier-free L_P^+-type, there is $c' \in P(M)$ such that $\psi(a,c')$, implying that $tp_L(a/P(M))$ forks over A, a contradiction.
So $tp_L(b/P(N))$ does not fork over B. Note that aA and bB have the same quantifier-free L_P-type. So by Lemma 3.8 they have the same type (in $(M, P(M)), (N, P(N))$ respectively). This completes the proof.

Definition 3.12 *We will say (with some abuse of language) that the class $\mathcal{C}_{T,P}$ of lovely pairs is first order if any $|T|^+$-saturated model of T_P is a lovely pair.*

In the next section we will find necessary and sufficient conditions on the theory T for $\mathcal{C}_{T,P}$ to be first order. But for now note that it follows from 3.11 that:

Corollary 3.13 *If $\mathcal{C}_{T,P}$ is first order then T_P^+ has quantifier-elimination.*

4 Characterization of when $\mathcal{C}_{T,P}$ is first order

The axioms (i) and (ii) for lovely pairs have quite different features, and we will separate our attempt to find necessary and sufficient conditions for $\mathcal{C}_{P,T}$ to be first order into attempts to express each of (i), (ii) in a "first order manner".

It will be convenient to call property $(i)_\kappa$ the "κ-extension property", and $(ii)_\kappa$ the κ-coheir property. When $\kappa = |T|^+$ we will just say "extension property" and "coheir property". We will say, by abuse of language again, that the extension property is first order, if any $|T|^+$-saturated model of the L_P-theory of all pairs $(M, P(M))$ of models of T satisfying the extension property, also satisfies the extension property. Likewise for "the coheir property is first order", "the κ-extension property is first order" etc.

Lemma 4.1 *The following are equivalent:*
(i) T is low,
(ii) The coheir property is first order,
(iii) for any $\kappa \geq |T|^+$, the κ-coheir property is first order.

Proof. (i) implies (ii) and (iii): Assume T to be low. Fix an L-formula $\phi(x, y)$ and a tuple z of variables. Let $\Sigma_{\phi,z}(y, z)$ be the partial L-type given by Lemma 2.3 which expresses that $\phi(x, y)$ forks over z. Let $C_{\phi,z}$ be the set of L_P-sentences:
$\{\forall y, z((P(z) \wedge \neg\exists x(P(x) \wedge \phi(x, y))) \to \psi(y, z)) : \psi(y, z) \in \Sigma_{\phi,z}(y, z)\}$.
It is then more or less immediate that (a) any pair $(M, P(M))$ of models of T with the coheir property is a model of $C_{\phi,z}$ for all ϕ, z, and (b) any

κ-saturated pair $(M, P(M))$ which is a model of $C_{\phi,z}$ for all ϕ, z has the κ-coheir property. So (ii) and (iii) are proved.
Let us now prove (ii) implies (i). Assume (ii) and we have to prove that for any $\phi(x, y) \in L$ and z, the condition (on (y, z)) that $\phi(x, y)$ forks over z is type-definable (in $\bar{M} \models T$). Let b_i, c_i for $i \in I$ be tuples from $\bar{M}$ such that $\phi(x, b_i)$ divides over c_i, and let (b, c) realize some ultraproduct of $\{tp_L(b_i, c_i) : i \in I\}$. We must show that $\phi(x, b)$ forks over c. By Lemma 3.5 we may assume that all (b_i, c_i) are inside a lovely pair $(M, P(M))$, such that, moreover, $c_i \in P(M)$ and $tp_L(b_i/P(M))$ does not fork over c_i for all $i \in I$. We may assume that $(N, P(N), b, c)$ is an ultraproduct of the $(M, P(M), b_i, c_i)$ (for some pair $(N, P(N))$). Now, $\phi(x, b_i)$ is not realized in $P(M)$ for all i. It follows that $\phi(x, b)$ is not realized in $P(N)$. But, as $(M, P(M))$ satisfies the coheir property, it follows from (ii) that $(N, P(N))$ does too. Hence, $\phi(x, b)$ forks over $P(N)$ so over c.

Remark 4.2 *The above proof shows that the following are equivalent:*
(i) T is low,
(ii) for any model $(M, P(M))$ of T_P, $\phi(x, y) \in L$, and $b \in M$, $\phi(x, b)$ does not fork over $P(M)$ if and only if $\phi(x, y)$ is realized in $P(M)$.

We now obtain a preliminary characterization of when $\mathcal{C}_{T,P}$ is first order. Let us say that an extension $(M, P(M)) \subseteq (N, P(N))$ of pairs of models of T is *free* if $P(N)$ is L-independent from M over $P(M)$.

Corollary 4.3 *The following are equivalent:*
(i) $\mathcal{C}_{T,P}$ is first order,
(ii) T is low and T_P^+ has quantifier-elimination,
(iii) Any extension $(M, P(M)) \subseteq (N, P(N))$ of models of T_P is elementary if and only if it is free.

Proof. (i) implies (ii) is by Remark 4.2 and Corollary 3.13.
(ii) implies (iii). Assume (ii). Suppose first that $(M, P(M))$ is an elementary substructure of $(N, P(N))$. Then $tp_L(P(N))/M)$ is finitely satisfiable in $P(M)$, hence $P(N)$ is independent from M over $P(M)$. Now assume that $(M, P(M)) \subseteq (N, P(N))$ is free. Let $(M, P(M))^+$, $(N, P(N))^+$ be the canonical expansions of $(M, P(M)), (N, P(N))$ to models of T_P^+
Claim. $(M, P(M))^+$ is a substructure of $(N, P(N))^+$
Proof of claim. Suppose $\phi(x, y) \in L$, $b \in M$ and $(N, P(N)) \models \exists x(P(x) \wedge$

$\phi(x,b))$. Then $\phi(x,b)$ does not fork over $P(N)$ (in $\bar{M}$). As $P(N)$ is independent from M over $P(M)$), $\phi(x,b)$ does not fork over $P(M)$ in $\bar{M}$. By Remark 4.2, $(M,P(M)) \models \exists x(P(x) \wedge \phi(x,b))$. The claim is proved.

By the claim and the assumption that T_P^+ has quantifier-elimination, $(M,P(M))^+$ is an elementary substructure of $(N,P(N))^+$. In particular $(M,P(M))$ is an elementary substructure of $(N,P(N))$.

(iii) implies (i). Let $(M,P(M))$ be a $|T|^+$-saturated model of T_P. By Remark 3.6(ii), we can find a lovely pair $(N,P(N))$ which is a free extension of $(M,P(M))$. By (ii) $(N,P(N))$ is an elementary extension of $(M,P(M))$. It follows easily that $(M,P(M))$ is lovely too.

Remark 4.4 *It is clear that $\mathcal{C}_{T,P}$ is first order iff for any (some) $\kappa \geq |T|^+$, any κ-saturated model of T_P is κ-lovely.*

Corollary 4.3 is somewhat unsatisfactory as we really seek conditions on the original theory T (not T_P) which are equivalent to $\mathcal{C}_{T,P}$ being first order. We proceed to do this now. The key issue is being able to axiomatize property (i) (the extension property) in the definition of lovely pairs.

We now bring into play the predicates $Q_{\phi,\psi}$ introduced in section 2.

Proposition 4.5 *The following are equivalent:*
(i) $Q_{\phi,\psi}$ is type-definable (in $\bar{M}$) for all $\phi(x,y), \psi(y,z) \in L$.
(ii) The extension property is first order.
(iii) Any saturated model of T_P satisfies the extension property.

Proof. (i) implies (ii).
Assume type-definablity of $Q_{\phi,\psi}$ for all $\phi, \psi \in L$, and identify notationally $Q_{\phi,\psi}(z)$ with this partial L-type.
For each ϕ, ψ let $\Sigma_{\phi,\psi}$ be the following collection of L_P-sentences: $\{\forall z(\chi(z) \rightarrow \exists y(\psi(y,z) \wedge \neg\exists x(P(x) \wedge \phi(x,y)))) : \neg\chi(z) \in Q_{\phi,\psi}(z)\}$.
Claim 1. If $(M,P(M))$ is a pair of models of T which has the extension property, then $(M,P(M)) \models \Sigma_{\phi,\psi}$ for all ϕ, ψ.
Proof. This is immediate, but we go through the translation. Fix $\phi(x,y), \psi(y,z) \in L$, $\neg\chi(z) \in Q_{\phi,\psi}(z)$, and $c \in M$ such that $\chi(c)$. So there is a complete L-type $p(y,c)$ over c such that for any d, $\neg\phi(d,y)$ is in all nonforking extensions

of $p(y,c)$ over cd. As $(M,P(M))$ satisfies the extension property, $p(y,c)$ is realized by some $b \in M$ such that $tp_L(b/P(M)c)$ does not fork over c. In particular $\neg\phi(d,b)$ holds for all $d \in P(M)$). This proves Claim 1.

Claim 2. If $(M,P(M))$ is a κ saturated pair of models of T, which is also a model of all $\Sigma_{\phi,\psi}$ then $(M,P(M))$ satisfies the κ-extension property.
Proof of claim 2. Let $p(y)$ be a complete L-type over a subset A of M of cardinality $< \kappa$. Let $\phi(x,y) \in L$ be such that $\phi(x,b')$ forks over A for some (any) realization b' of p. Let $\psi(y,c)$ be any formula in $p(y)$. So $\neg Q_{\phi,\psi}(c)$ holds, so $\chi(c)$ holds for some $\neg\chi(z) \in Q_{\phi,\psi}(z)$. As $\Sigma_{\phi,\psi}$ is true in $(M,P(M))$, there is $b \in M$ realizing $\psi(y,c)$ such that $\neg\phi(d,b)$ holds for all $d \in P(M)$. κ-saturation of $(M,P(M))$ ensures that some nonforking extension of $p(y)$ over $A \cup P(M))$ is realized in M.

Claims 1 and 2, show (ii).
(ii) implies (iii) is immediate.
(iii) implies (i): Let us fix $\phi(x,y), \psi(y,z) \in L$. Let c_i for $i \in I$ be tuples in $\bar{M}$ such that $Q_{\phi,\psi}$ is true of c_i for all i. Let c be an ultraproduct of the c_i. We must show $Q_{\phi,\psi}$ is true of c too. We may assume all c_i are in $P(M)$ for $(M,P(M))$ some lovely pair and that $c \in P(N)$ where $(N,P(N),c)$ is an ultraproduct of the $(M,P(M)),c_i)$. Now as $(M,P(M))$ is a lovely pair, the formula $\rho(z) : \forall y \exists x(\psi(y,z) \rightarrow (P(x) \wedge \phi(x,y)))$ is true of c_i in $(M,P(M))$ for all i, so true of c in $(N,P(N))$. On the other hand, by the assumption (iii), $(N,P(N))$ has a saturated elementary extension $(N',P(N'))$ say which satisfies the extension property. Choose any complete L-type $p(y,c)$ containing $\psi(y,c)$ and let $b \in N'$ realize a nonforking extension of $p(y,c)$ over $P(N')c$. As $(N',P(N')) \models \rho(c)$, there is $d \in P(N')$ such that $\phi(d,b)$. But then $\phi(x,b)$ does not fork over c. We have shown that $Q_{\phi,\psi}$ is true of c. The proof is complete.

Let us summarize the results we have obtained in this section, making use also of 2.8.

Corollary 4.6 *The following are equivalent:*
(i) $\mathcal{C}_{T,P}$ is first order,
(ii) For any (some) $\kappa \geq |T|^+$, any κ-saturated model of T_P is a κ-lovely pair,
(iii) T is low and T_P^+ has quantifier-elimination,
(iv) Every free extension of pairs of models of T_P is elementary.

(v) T is low and $Q_{\phi,\psi}(z)$ is definable for all $\phi, \psi \in L$.
(vi) For each $\phi(x, y) \in L$, the D_ϕ-rank is finite and definable (in T).

The following axiomatization is then clear (and already figured in proofs above). Let us recall notation: $\Sigma_{\phi(x,y),z}(y, z)$ is the partial L-type expressing that $\phi(x, y)$ divides over z (which exists assuming lowness of T).

Remark 4.7 *Assuming $\mathcal{C}_{T,P}$ to be first order, T_P can be axiomatized as follows:*
(i) $(\forall z)(\forall y)(P(z) \wedge \neg\psi(y, z) \rightarrow \exists x(P(x) \wedge \phi(x, y)))$, whenever $\psi(y, z) \in \Sigma_{\phi,z}(y, z)$.
(ii) $(\forall z)(\neg Q_{\phi,\psi}(z) \rightarrow \exists y(\psi(y, z) \wedge (\forall x)(P(x) \rightarrow \neg\phi(x, y)))$, for all $\phi, \psi \in L$.

5 The stable case and Poizat's "Belles Paires"

Let T be an arbitrary complete first order theory (in language L). Poizat [14] defined a *belle paire* of models of T to be an elementary pair $M \subseteq N$ of models of T such that (i) for any finite $A \subset M$ and complete (L) type $p(x)$ over $M \cup A$, p is realized in N, and (ii) M is $|T|^+$-saturated. Poizat proves, among other things:

Fact 5.1 *(i) Any two belles paires of models of T are elementarily equivalent (in the language L_P),*
(ii) Assuming T stable, T has the nonfcp if and only if every $|T|^+$-saturated model of the theory of belles paires is a belle paire if and only if the theory of belles paires has quantifier elimination in the language L_P^+.

It is pretty clear that for T stable, belles paires are essentially lovely pairs. Let us make this more precise: Define a pair $(M, P(M))$ to be a κ-belle paire if (i) any complete type L-type over $A \cup P(M))$ for $A \subseteq M$ of cardinality $< \kappa$, is realized in M, and (ii) $P(M)$ is κ-saturated.

Remark 5.2 *Suppose $\kappa \geq |T|^+$.*
(i) A κ-belle paire is a belle paire.
(ii) For T stable, $(M, P(M))$ is a κ-belle paire iff it is a κ-lovely pair.

Proof. (ii): Axiom (i) for κ-belles paires immediately implies axiom (i) for κ-lovely pairs. On the other hand suppose $(M, P(M))$ satisfies axiom (i) for

κ-lovely pairs. Let $A \subseteq M$ be of cardinality $< \kappa$ and let $p(x)$ be a complete L-type over $P(M) \cup A$. There is no harm, by stability, in assuming $p(x)$ to be stationary. Let $B \subset P(M) \cup A$ be of cardinality $\leq |T|$ such that p does not fork over B and $p|B$ is stationary. In particular p is the unique nonforking extension of $p|(A \cup B)$ over $A \cup P(M)$. Axiom (i) for κ-lovely pairs implies p is realized in M.
The equivalence of Axiom (ii) for κ-lovely pairs and κ-belles paires is already in Remark 3.2 (a).

Let us assume for this paragraph that T is stable. By Fact 5.1 (i) and Remark 5.2 (and our earlier results) the theory of belles paires coincides with the theory T_P of lovely pairs. As any stable theory is low, by Fact 5.2(ii) and Corollary 4.6 the "first-orderness" of the class of belles paires coincides with that of the class of lovely pairs. So we see that T has the nonfcp if and only if the predicates $Q_{\phi,\psi}$ are definable if and only if the D_ϕ-ranks are definable. So we already obtain some "new" equivalents to the nonfcp for stable theories.

Poizat in [14] goes on to show (for stable T) that, assuming T has nonfcp, T_P is stable (with the nonfcp). We will generalize this to the simple case in the next section.

6 Simplicity of T_P, when $\mathcal{C}_{T,P}$ is first order

Let us begin with a little lemma which enables us to check simplicity of a theory by looking at types over models:

Lemma 6.1 *(T any complete first order theory.)*
(i) T is simple if and only if any (finitary) type over a model M of T does not divide over some subset A of M of cardinality at most $|T|$.
(ii) Likewise T is supersimple if and only if every type over a model M does not divide over some finite subset of M.

Proof. We know the left to right directions. Let us start with the right to left direction of (i). If T is not simple, then there is a formula $\phi(x, y) \in L$, and a sequence $(b_i : i < |T|^+)$, such that $\phi(x, b_i)$ divides over $\{b_j : j < i\}$ for all i, and $\{\phi(x, b_i) : i < |T|^+\}$ is consistent. Let $B_i = \{b_j : j < i\}$.
Claim. There is an increasing chain of models $M_i \supseteq B_i$ ($i < |T|^+$), each of

cardinality at most $|T|$, such that $\phi(x, b_i)$ divides over M_i for all i.
Proof of claim. We construct the chain $M_i \supseteq B_i$ inductively, satisfying the stronger conditions: for all $j \geq i$, $\phi(x, b_j)$ divides over $M_i \cup B_j$. Let us show how to find M_0 and leave the rest to the reader. Let $r(z)$ be the type of an enumeration of some model of cardinality at most $|T|$. Let $(b_0^j : j < \omega)$ be an indiscernible sequence such that $b_0^0 = b_0$ and $\{\phi(x, b_0^j) : j < \omega\}$ is inconsistent (by our assumption that $\phi(x, b_0)$ divides over $\emptyset$). By Erdos-Rado, we can find a complete type $r_1(z)$ over $\{b_0^j : j < \omega\}$ such that if m realizes $r_0(z)$ then $(b_0^j : j < \omega)$ is m-indiscernible. Let $r_1'(z)$ be the restriction of $r_1(z)$ to b_0. So if m realizes $r_1'(z)$, then $\phi(x, b_0)$ divides over m. Now, in exactly the same way, extend $r_1'(z)$ to a complete type $r_2'(z)$ over $\{b_0, b_1\}$ such that if m realizes $r_2'(z)$, then $\phi(x, b_1)$ divides over mb_0. Continue, taking unions at limit stages, to find $r_i'(z)$ over B_i for $i < |T|$. If M_0 realizes the union of these types, then it does the job.

Let M the union of the M_i. Let $p(x) \in S(M)$ be a completion of $\{\phi(x, b_i) : i < |T|^+\}$. Then by construction, $p(x)$ divides over each M_i. But any subset of M of cardinality $\leq |T|$ is contained in some M_i. Hence the right hand side of (i) fails.
Now we prove the right to left direction of (ii). By part (i) we may assume T to be simple. Suppose T is not supersimple, so there is some chain $p_i(x) \in S(A_i)$ of types over finite sets such that p_{i+1} divides over A_i for $i < \omega$. Either use the method of (i), or choose inductively models $M_i \supset A_i \cup M_{i-1}$ such that M_i is independent from $\bigcup_j A_j$ over A_i. Note that then $p_{i+1}(x)$ divides over M_i. So if M is the union of the M_i and $p(x)$ a completion of $\bigcup_i p_i$ over M, then p divides over each M_i so divides over every finite subset of M.

For the rest of this section we asume that T is simple and $\mathcal{C}_{T,P}$ is first order. Let us situate ourselves in a very saturated model of T_P which we may assume to be $(\bar{M}, P(\bar{M}))$ where $\bar{M}$ is our big saturated model of T. So $(\bar{M}, P(\bar{M}))$ is a $\bar{\kappa}$-lovely pair.

Proposition 6.2 *T_P is simple. Moreover if T is supersimple, so is T_P.*

Proof. By the lemma above it is enough to consider types over models. Let $(M, P(M))$ be an L_P-elementary substructure of the universe. and a a finite tuple. We will find a subset A of M of cardinality at most $|T|$ such that $tp_{L_P}(a/M)$ does not divide over A. In the case where T is supersimple, we'll

see from the proof that A could be chosen finite.
We will just talk about P rather than $P(\bar{M})$. When we say something like "$tp_L(a/B)$ does not divide over A" we actually mean that $tp(a/B)$ does not divide over A in the L-structure $\bar{M}$. (We hope that there is no ambiguity here.) Note that $tp_L(c/M)$ does not divide over $P(M)$ for any $c \in P$ (in fact $tp_L(c/M)$ is finitely satisfiable in $P(M)$).
By simplicity of T we can find $C \subset P$ and $A \subset M$ both of cardinality at most $|T|$ such that
(i) $tp_L(a/M \cup P)$ does not divide over $A \cup C$,
(ii) $tp_L(C/M)$ does not divide over $P(A)$.
Claim. $tp_{L_P}(a/M)$ does not divide over A.
Proof of claim. Let $\{M_i : i < \omega\}$ be an A-indiscernible (in the L_P sense) sequence of realizations of $tp_{L_P}(M/A)$ with $M_0 = M$. Let $p_0(y) = tp_L(C/M_0)$, and $p_i(y)$ its copy over M_i. By (ii) there is C' realizing $\bigcup_i p_i(y)$ such that $tp_L(C'/\cup M_i)$ does not divide over A, and $(M_i : i < \omega)$ is AC'-indiscernible in the sense of L. Note that $tp_L(C'/\bigcup_i M_i)$ does not divide over $P(A)$, hence by the $\bar{\kappa}$-lovelieness of $(\bar{M}, P(\bar{M}))$, we may assume that C' is contained in P. As both $M \cup C$ and $M \cup C'$ are P-independent and have the same quantifier-free L_P-type, we conclude from 3.8 that they have the same L_P-type. So, (by changing the sequence of M_i's) we may assume that $C' = C$. By (i), $tp_L(a/M \cup C)$ does not divide over $A \cup C$. Let $r(x) = tp_L(a/MC)$. Remember that $(M_i : i < \omega)$ is AC-indiscernible in the sense of L. So letting $r_i(x)$ be the copy of $r(x)$ over $M_i \cup C$, $\bigcup_i r_i(x)$ can be realized by some a' which is L-independent from $\bigcup_i M_i \cup C$ over $A \cup C$. By lovelieness of $(\bar{M}, P(\bar{M}))$, we may assume that a' is L-independent from P over $\bigcup_i M_i \cup C$. In particular each $a'M_iC$ is P-independent. Finally note that all $a'M_iC$ have the same quantifier-free L_P-type, so by 3.8, they have the same L_P-type. But for the same reason this is the L_P-type of aMC. We have shown that $tp_{L_P}(a/M)$ does not divide over A.

7 Forking in T_P, when $\mathcal{C}_{T,P}$ is first order

We assume throughout this section that $\mathcal{C}_{T,P}$ is first order, unless we say otherwise. This long and central section initiates the analysis of the simple theory T_P. We characterize forking, give some information on canonical bases, show that 1-basedness is preserved (in passing from T to T_P), and finally examine the implications of T_P being ω-categorical. There are many more things to be done and questions to be settled. A few such problems will be stated at the end of the paper.

We begin by characterizing forking in $(\bar{M}, P(\bar{M}))$. In fact this would also be another route to Proposition 6.2, but in the proofs below it is convenient to know the simplicity of T_P in advance.
Let us introduce some notation:

Definition 7.1 *Let a be a (possibly infinite) tuple. By a^c we mean the canonical base of $tp_L(a/P(\bar{M}))$. By $\hat{a}$ we mean (a, a^c).*

Remark 7.2 *(i) a^c is contained in $dcl_{L_P}(a)$.*
(ii) a is L_P-independent from P over a^c.

Proof. (i) is clear, as any automorphism of $(\bar{M}, P(\bar{M}))$ fixes P setwise so if it also fixes a it fixes a^c.
(ii). Let $B \subset P$, B containing a^c. Let $(B_i)_i$ be an a^c-indiscernible sequence in the sense of L_P, with $B_0 = B$ (so all $B_i \subset P$). By lovelieness we can find a' such that $tp_L(a'B_i) = tp_L(aB)$ for all i and a' is L independent from P over $\bigcup_i B_i$. It follows (by 3.8) that $tp_{L_P}(a'B_i) = tp_{L_P}(aB)$ for all i, proving (ii).

Proposition 7.3 *Let $A \subseteq B$ be sets and c a tuple. Then the following are equivalent:*
(i) $tp_{L_P}(c/B)$ does not fork over A.
(ii) $tp_L(c/B \cup P)$ does not fork over $A \cup P$ and $tp_L((cA)^c/B^c)$ does not fork over A^c,
(iii) $tp_L(c/B \cup P)$ does not fork over $A \cup P$ and $tp_L(\widehat{cA}/\hat{B})$ does not fork over $\hat{A}$.
(iv) There is (small) $a \subset P$ such that $tp_L(c/B \cup P)$ does not fork over $A \cup a$ and $tp_L(a/\hat{B})$ does not fork over $\hat{A}$.

Proof. (i) implies (ii): Assume (i). So $tp_{L_P}(cA/B)$ does not fork over A. By Remark 7.2(i), $tp_{L_P}(\widehat{cA}/\hat{B})$ does not fork over $\hat{A}$. In particular $tp_{L_P}((cA)^c/\hat{B})$ does not fork over $\hat{A}$. By Remark 7.2(ii), $\hat{A}$ is L_P-independent from P over A^c. We conclude that $(cA)^c$ is L_P-independent from B^c over A^c. As these sets all live in P we can replace L_P-independence by L-independence. This

gives the second part of (ii).
We now prove the first part of (ii), that is $tp_L(c/B \cup P)$ does not fork over $A \cup P$. We may assume that $B = \hat{B}$. Suppose for a contradiction that $tp_L(c/B \cup P)$ forks over $A \cup P$. By lovelieness, let $(B_i)_{i<\lambda}$ be a very big sequence of realisations of $tp_L(B/A \cup B^c)$ such that B_i is L-independent from $\bigcup_{j<i} B_j$ over $A \cup P$ and $B_0 = B$. Note that each B_i is P-independent, so $tp_{L_P}(B_i) = tp_{L_P}(B)$ for all i. By Erdos-Rado we may assume that $(B_i : i < \lambda)$ is $A \cup B^c$-indiscernible in the sense of L_P, so in particular $(B_i : i < \lambda)$ is A-indiscernible in the sense of L_P. By (i) we may find c' such that $tp_{L_P}(c'B_i) = tp_{L_P}(cB)$ for all i. Now $\{B_i : i < \lambda\}$ is L-independent over $A \cup P$, and c' L-forks with each B_i over $A \cup P$. As we chose $\lambda \geq |T|^+$, this contradicts simplicity of T.

(ii) implies (iii): Assuming (ii) all we have to do is prove
(*) $\widehat{cA}$ is L-independent from $\hat{B}$ over $\hat{A}$.
We begin to make some observations:
First $(cA)^c$ is L-independent from $\hat{B}$ over B^c. Together with the second part of (ii), we obtain:
(I) $(cA)^c$ is L-independent from $\hat{B}$ over $\hat{A}$.
On the other hand, As cA is L-independent from P over $(cA)^c$, we have:
(II) cA is L-independent from $A \cup P$ over $A \cup (cA)^c$.
But the first part of (ii) yields that cA is L-independent from $B \cup P$ over $A \cup P$, so together with (II) this gives:
(III) cA is L-independent from $\hat{B} \cup (cA)^c$ over $\hat{A} \cup (cA)^c$.
(I) and (III) give (*).

(iii) implies (iv): Assume (iii) and take a to be $(cA)^c$. So by the second part of (iii) $tp_L(a/\hat{B})$ does not fork over $\hat{A}$. But also as $tp_L(cA/P)$ does not fork over a, $tp_L(c/AP)$ does not fork over Aa. Together with the assumption that $tp_L(c/BP)$ does not fork over AP this implies that $tp_L(c/BP)$ foes not fork over Aa.

(iv) implies (iii): Let $a \subset P$ be as given by (iv). So we see immediately that $tp_L(c/BP)$ does not fork over AP.
On the other hand, we also have that $tp_L(c/AP)$ does not fork over Aa, and together with the fact that $tp_L(A/P)$ does not fork over A^c we conclude that:
(*) $tp_L(cA/P)$ does not fork over $A^c a$.
As c is L-independent from $\hat{B}a$ over $\hat{A}a$ and a is L-independent from $\hat{B}$ over

$\hat{A}$, we see that ca is L-independent from $\hat{B}$ over $\hat{A}$ and thus:
(**) $tp_L(ca\hat{A}/B)$ does not fork over $\hat{A}$.
By (*) $\widehat{cA}$ is contained in $ca\hat{A}$, and so from (**) we conclude that $tp_L(\widehat{cA}/\hat{B})$ does not fork over $\hat{A}$. (iii) is proved.

Finally we prove (iii) implies (i). The proof is like that of 6.2, so we are brief. First as any tuple d is L_P-interdefinable with $\hat{d}$, it suffices to prove:
(***) $\widehat{cA}$ is L_P-independent from $\hat{B}$ over $\hat{A}$.
Let $(\hat{B}_i)_i$ be L_P-indiscernible over $\hat{A}$ with $\hat{B}_0 = \hat{B}$. As (by (iii)) $(cA)^c$ is L-independent from $\hat{B}$ over $\hat{A}$, we can, by lovelieness, assume that $(\hat{B}_i)_i$ is L-indiscernible over $\hat{A} \cup (cA)^c$. Now (iii) implies that $\widehat{cA}$ is L-independent from $\hat{B}$ over $\hat{A} \cup (cA)^c$. Let $p(x) = tp_L(\widehat{cA}/\hat{B} \cup (cA)^c)$, and p_i the copy over $\hat{B}_i$. So (using also lovelieness) we can realise $\bigcup_i p_i$ by some d which is L-independent from P over $\bigcup_i(\hat{B}_i) \cup (cA)^c$. It follows that $tp_{L_P}(d\hat{B}_i(cA)^c) = tp_{L_P}(\widehat{cA}\hat{B}(cA)^c)$ for all i. We have proved (***).

We now turn our attention to canonical bases of Lascar strong types in T_P. Understanding canonical bases in a simple (in particular stable) theory is important, as by [12] all hyperimaginaries are essentially canonical bases of types of real tuples over models.

It is worth noting to begin with that T_P may contain really new (hyper-) imaginaries. For example, let T be the theory of a vector space V over a field F, in the module language. T is strongly minimal and every element of V^{eq} is interalgebraic with a real tuple. Let $(V, P(V))$ be a model of T_P and $a \in V \setminus P$. Then the coset $a + P$ is an element of $(V, P(V))^{eq}$ but is not interalgebraic with any real tuple. Note that $a + P$ is the canonical base of $tp_{L_P}(a/a + c)$ where $c \in P$ is generic.

We first work towards proving that the only canonical bases we need consider in T_P are of the form $Cb(tp_{L_P}(d/B))$ where B is a model, d is a hyperimaginary in the sense of L, and $d \in bdd_L(B \cup P)$. (Of course the base set can always be chosen to be a model.)

We start with a preliminary lemma:

Lemma 7.4 *Suppose that $d = Cb(Lstp_L(c/B \cup P))$. Then*
(i) $tp_{L_P}(c/B \cup P)$ does not fork over d.
(ii) $d \in bdd_{L_P}(cB)$.

Proof. (i) By Proposition 7.3, all we have to prove is
(*) $\widehat{cd}$ is L-independent from $\widehat{BP}$ over $\hat{d}$.

Note that $\widehat{BP} = BP$, and so our assumptions imply that c is L-independent from $\widehat{BP}$ over $\hat{d}$, and thus $c\hat{d}$ is L-independent from $\widehat{BP}$ over $\hat{d}$. So it is enough to prove that $\widehat{cd} = c\hat{d}$. But the latter follows as c is L-independent from dP over dd^c and d is L-independent from P over d^c (hence cd is L-independent from P over d^c).
(ii) Note that c is L-independent from BP over $B \cup (cB)^c$, and thus $d \in bdd_L((cB)^c)$. Now use Remark 7.2(i).

Proposition 7.5 *Let B be an elementary substructure of $(\bar{M}, P(\bar{M}))$, and c a real tuple. Let $d = Cb(Lstp_L(c/B \cup P))$. Then $Cb(tp_{L_P}(c/B))$ is L_P-interdefinable with $Cb(tp_{L_P}(d/B))$.*

Proof. Let $p_B = tp_{L_P}(c/B)$ and $r_B = tp_{L_P}(d/B)$. Let $e = Cb(p_B)$ and $e' = Cb(r_B)$. So $e, e' \in dcl_{L_P}(B)$ are hyperimaginaries in the L_P-sense (objects which we have not considered before).
Claim 1. c is L_P-independent from B over e', and $tp_{L_P}(c/e')$ is an amalgamation base.
Proof. We will make repeated use of Proposition 7.3 ((i) if and only if (iii)), and Lemma 7.4. These first imply that c is L_P-independent from BP over d, and so c is L_P-independent from Bd over d, whereby c is L_P-independent from Bd over $e'd$. As d is L_P-independent from B over e' we conclude that cd is L_P-independent from B over e', yielding the first part of claim I. The second part is left to the reader.

Claim II. d is L_P-independent from B over e.
Proof. It will be enough to show that if B' realizes $tp_{L_P}(B/e)$ with B' L_P-independent from B over e, then $r_B \cup r_{B'}$ have a common nonforking extension. By the assumption on e (and the independence theorem) Let c' realize $p_B \cup p_{B'}$, such that c' is L_P-independent from BB' over e (and so over each of B, B'). By 7.3, $Cb(Lstp_L(c'/BB'P)) = Cb(Lstp_L(c'/BP)) = Cb(Lstp(c'/B'P)) = d'$ say where d' realizes $r_B \cup r_{B'}$. By Lemma 7.4(ii), $d' \in bdd_{L_P}(c'B) \cap bdd_{L_P}(c'B')$. Thus d' is L_P-independent from BB' over each of B, B'.

Claims I and II show that e and e' are interdefinable, proving the Proposition.

Now suppose that B is a model say, and $d \in bdd_L(B \cup P)$. We will give an explicit description of the type-definable equivalence relation E on $tp_{L_P}(B)$

such that $B/E = Cb(tp_{L_P}(d/B))$. Let $a = (dB)^c$. So $d \in bdd_L(Ba)$. Let $q_{Ba}(z) = tp_L(d/Ba)$. Define E_0 on $tp_{L_P}(B)$ by:
$(B', B'') \in E_0$ iff there are a', a'' in P such that
(i) $tp_L(B'a') = tp_L(B''a'') = tp_L(Ba)$,
(ii) a' is L-independent from $\widehat{B'B''}$ over B, and a'' is L-independent from $\widehat{B'B''}$ over B',
(iii) there is d' realizing $q_{B'a'}(z) \cup q_{B''a''}(z)$.

Proposition 7.6 *Let E be the 2-iterate of E_0. Then E is L_P-type-definable over $\emptyset$, and $B/E = Cb(tp_{L_P}(d/B))$.*

Proof. Let $p_B(z) = tp_{L_P}(d/B)$. We show first that if $(B', B'') \in E_0$ then $p_{B'}(z)$ and $p_{B''}(z)$ have a common nonforking extension. Let a', a'', d' be as in the definition of E_0. Then a', being in P, is L_P-independent of $\widehat{B'B''}$ over B, hence d' is L_P-independent over $B'B''$ over B'. Likewise d' is L_P-independent of $B'B''$ over B'. It is easy to see that d' realizes $p_{B'} \cup p_{B''}$.

On the other hand, let $e = Cb(p_B)$. We will show that if B' is L_P-independent of B over e, with the same L_P-type as B over e, then $(B, B') \in E_0$. Let d' realize a common nonforking extension of $p_B \cup p_{B'}$. Then $(d'B)^c \in dcl_{L_P}(d'B)$ is L_P-independent of BB' over B, and likewise $(d'B')^c$ is L_P-independent of BB' over B'. Let $a' = (d'B)^c$ and $a'' = (d'B')^c$.

Next we will apply the above results to show that 1-basedness of T implies 1-basedness of T_P. Recall that the simple theory T is said to be 1-based iff for any c and B, $Cb(Lstp(c/B)) \subseteq bdd(c)$. Other equivalent conditions are:
(a) for any c, b, c is independent from b over $bdd(c) \cap bdd(b)$,
(b) whenever c' realises a nonforking extension of $Lstp(c/B)$ over Bc then $tp(c'/Bc)$ does not fork over c.
In (b) B can be taken to be a model.

Proposition 7.7 *Suppose that T is 1-based. Then T_P is 1-based.*

Proof. By 7.5 and 1-basedness of T it is enough to consider types of the form $tp_{L_P}(d/B)$ where B is a model, and $d \in bdd_L(B \cup P)$. Let d' realize an L_P-nonforking extension of $tp_{L_P}(d/B)$ over $B \cup d$. We must show that $tp_{L_P}(d'/Bd)$ does not fork over d. We may assume that $d = \hat{d}$ (and so $d' = \hat{d}'$.
Claim 1. $\widehat{d'd}$ is L-independent from $\widehat{Bd}$ over $\hat{d}$.

Proof of claim 1. By our assumption, as well as Proposition 7.3, d is L-independent from d' over B, and note that d and d' have the same L-type over B and that B is also a model in the L-sense. By 1-basedness of T, $d'd$ is independent from Bd over d. Thus $bdd_L(d'd)$ is independent from $bdd_L(Bd)$ over d. But as T is 1-based, $Cb_L(tp(a/P)) \subseteq bdd(a)$ for any a. In particular $\widehat{d'd} \subseteq bdd_L(d'd)$ and $\widehat{Bd} \subseteq bdd_L(Bd)$. So we get claim 1.

Claim 2. $d' \in bdd_L(Pd)$
Proof of Claim 2. Let a be a tuple in P such that $d \in bdd_L(Ba)$, and let a' be such that $tp_{L_P}(ad/B) = tp_{L_P}(a'd'/B)$. So also $a' \in P$. Let $b_0 = Cb(tp_L(ad/B)) = Cb(tp_L(a'd'/B))$. Then $d \in bdd_L(a, b_0)$ and $d' \in bdd_L(a', b_0)$. By 1-basedness of T, $b_0 \in bdd_L(ad)$. It follows that $d' \in bdd_L(d, a, a')$, proving Claim 2.

It trivially follows from Claim 2 that $tp_L(d'/BP)$ does not fork over dP. Together with Claim 1 and Proposition 7.3, we conclude that $tp_{L_P}(d'/Bd)$ does not fork over d. The proof is complete.

Let us finally discuss the strength of the hypothesis that T_P is ω-categorical. Let T now be a one-sorted theory in a countable language L, and we make no a priori assumptions on the first-orderness of $\mathcal{C}_{T,P}$. It should be remarked that it is not known whether the ω-categoricity of T implies that T is low. On the other hand, if T is ω-categorical and low, then clearly the D_ϕ-ranks are finite and definable for all $\phi \in L$ whereby $\mathcal{C}_{T,P}$ is first order. Also note that by [7] any ω-categorical 1-based simple theory is supersimple of finite SU-rank (in particular low).

Proposition 7.8 *(T countable, simple).*
(i) Suppose that T is ω-categorical and 1-based. Then T_P is ω-categorical.
(ii) Suppose that $\mathcal{C}_{T,P}$ is first order and that T_P is ω-categorical. Then T is 1-based and ω-categorical.

Proof. (i) Fix n. We will show that there are only finitely many L_P-types of real n-tuples in $(\bar{M}, P(\bar{M}))$. Fix an n-tuple a. $tp_L(a/Cb_P(a))$ is realized in P (as P is a saturated model of T), by a' say. By 1-basedness of T, $Cb_P(a) \in bdd_L(a')$, hence (a, a') is P-independent, so its type is determined by its quantifier-free L_P-type. Clearly there are only finitely many quantifier-free L_P-types of $2n$-tuples.
(ii) Assume that $\mathcal{C}_{T,P}$ is first order and that T_P is ω-categorical. As usual we

work in a very saturated model $(\bar{M}, P(\bar{M}))$ of T_P. We will make a series of easy claims.
Claim I. T is supersimple.
Proof. Suppose for a contradiction that there are, in $\bar{M}$, complete 1-types $p_i(x) \in S(A_i)$ for $i < \omega$, such that $i < j$ implies $p_j(x)$ is a forking extension of $p_i(x)$. So for each $i < j$ there is an L-formula $\phi_{i,j}(x, y_{i,j})$ such that the $D_{\phi_{i,j}}$-ranks of p_i and p_j differ. We may assume that all A_i are in P and then for each i some L-nonforking extension of p_i over P is realized in $\bar{M}$ by a_i say. There are only finitely many $tp_{L_P}(a_i)$. But $tp_{L_P}(a_i) = tp_{L_P}(a_j)$ implies that the D_ϕ-ranks of $tp_L(a_i/P)$ are the same as those of $tp_L(a_i/P)$. This is a contradiction.
Claim II. T has finite SU-rank.
Proof. If not, we can find elements $a \in \bar{M}$ such that $tp_L(a/P)$ has arbitrarily large finite SU-rank. But then we obtain infinitely many 1-types in L_P.
Claim III. Let $p(x) \in S(A)$ be any type of SU-rank 1 in T. Then p is pseudo-linear. That is there is a finite bound on the SU ranks of $Cb(tp(a_1, a_2/B))$, for a_1, a_2 realizing p, $B \supset A$, and with $SU(tp(a_1, a_2/B)) = 1$.
Proof. If not we can find infinitely many L_P types over $\emptyset$ extending $p \times p$, as above.

Note that T is ω-categorical. By Claims II and III, and [17], every type of SU-rank 1 in T is "locally modular" (that is 1-based), hence by [7], T is 1-based.

8 The existentially closed/Robinson case

We saw in section 3 that if $\mathcal{C}_{T,P}$ is first order then T_P^+ has quantifier-elimination, and so is the model completion (companion) of its universal part $(T_P^+)_\forall$. In particular existentially universal (see [13]) models of $(T_P^+)_\forall$ coincide with saturated models of T_P^+ and so are lovely pairs.

In this section we consider the category of existentially closed models of $(T_P^+)_\forall$ without the assumption that $\mathcal{C}_{T,P}$ is first order. (It is clear that in any such model, $(\forall y)(R_\phi(y) \leftrightarrow \exists x \in P(\phi(x, y))$ is true.) In particular we examine the condition:
(EC): Any existentially universal model of $(T_P^+)_\forall$ is a lovely pair.

Bearing in mind the remarks in the first paragraph, this is a weaker condition than $\mathcal{C}_{P,T}$ being first order. We will show that (EC) holds if and only if T is low. Moreover we will show that under these conditions an existentially universal $(M, P(M))^+$ model of $(T_P^+)_\forall$ is a "Robinson universal domain", that is it is saturated and homogeneous for quantifier-free types. The results of the previous section go through. In particular $(M, P(M))^+$ is a simple Robinson universal domain.

The theory of existentially closed and existentially universal models is standard, but we will follow the notation of [13].

We fix a simple complete theory T (with quantifier-elimination in language L) as usual. T_P^+ is the theory of lovely pairs in the language L_P^+, as described in section 3. Namely new relations $R_\phi(y)$ for $\phi(x, y) \in L$ are introduced together with defining axioms:
A_ϕ: $\forall y(R_\phi(y) \leftrightarrow \exists x(P(x) \wedge \phi(x, y))$.

We will denote an L_P-structure $(M, P(M), R_\phi)_\phi$ by $(M, P(M))^+$ (so the $+$ refers to the interpretation of the R_ϕ's, which may or may not be in accordance with the axioms A_ϕ).

Lemma 8.1 *Let $(M, P(M))^+$ be an existentially closed model of $(T_P^+)_\forall$. Then $(M, P(M))^+$ is an elementary pair of models of T, as well as being a model of each A_ϕ*

Proof. The model concerned is a substructure of a model of T_P^+. The sentences A_ϕ as well the sentences expressing that the pair is an elementary pair of models of T are $\forall\exists$, hence are true in $(M, P(M), R_\phi)_\phi$ by existential closure.

Definition 8.2 *Let $(M, P(M))$ be an elementary pair of models of T (in the language L_P) and $(M, P(M))^+$ some expansion to an L_P^+-structure. We will say that $(M, P(M))^+$ is correct if for each $\phi(x, y) \in L$ and $b \in M$, $R_\phi(b)$ holds in $(M, P(M))^+$ iff $\phi(x, b)$ does not divide over $P(M)$ (in the model M of T) iff $(M, P(M)) \models \exists x \in P(\phi(x, b))$.*

Here is our main result. The equivalence of (i) and (ii) has more or less the same content as Remark 4.2.

Proposition 8.3 *The following are equivalent:*
(i) T is low,

(ii) Any existentially closed model of $(T_P^+)_\forall$ is correct.
(iii) For some (any) $\kappa \geq |T|^+$, if $(M, P(M))^+$ is a κ-existentially universal model of $(T_P^+)_\forall$, then $(M, P(M))$ is a κ-lovely pair of models of T.

Proof. (i) implies (ii). By lowness of T, for each $\phi(x, y) \in L$ and tuple z of variables, there is a partial L-type $\Sigma_{\phi,z}(y, z)$ expressing (in a model of T) that $\phi(x, y)$ divides over z. As T has quantifier-elimination we may take this partial type to be quantifier-free. Let $\Gamma_\phi = \{(\forall y)(\forall z)((\neg(R_\phi(y) \wedge P(z)) \rightarrow \psi(y, z))) : \psi(y, z) \in \Sigma_{\phi,z}\}$. Then Γ_ϕ is a set of universal L_P^+-sentences which are in T_P^+ hence in $(T_P^+)_\forall$.

Now let $(M, P(M))^+$ be an existentially closed model of $(T_P^+)_\forall$. So by Lemma 8.1, $(M, P(M))$ is an elementary pair of models of T. Let $\phi(x, y) \in L$ and $b \in M$. We will show that
$\neg R_\phi(b) \Rightarrow \phi(x, b)$ divides over $P(M) \Rightarrow \phi(x, b)$ is not satisfied in $P(M) \Rightarrow \neg R_\phi(b)$ (so $(M, P(M))^+$ wil be correct).
First, if $\neg R_\phi(b)$, then as $(M, P(M))^+$ is a model of Γ_ϕ, we have $M \models \Sigma_{\phi,z}(b, c)$ for all z and $c \in P(M)$. As M is a model of T, $\phi(x, b)$ divides over $P(M)$. If $\phi(x, b)$ divides over $P(M)$ it is clearly not satisfied in $P(M)$. Finally, by Lemma 8.1 (that is, $(M, P(M))^+$ is a model of A_ϕ), if $\phi(x, b)$ is not satisfied in $P(M)$ then $\neg R_\phi(b)$.

(ii) implies (iii): Let $(M, P(M))^+$ be a κ-existentially universal model of $(T_P^+)_\forall$. By Remark 3.6 (ii), there is a κ-lovely $(N, P(N))$ which extends $(M, P(M))$ and such that moreover $P(N)$ is L-independent from M over $P(M)$. Note that M is an elementary substructure of N. Let $(N, P(N))^+$ be the canonical expansion of $(M, P(M))$ to a model of T_P^+.
Claim. $(M, P(M))^+$ is an (L_P^+-)substructure of $(N, P(N))^+$.
Proof of claim. Let $\phi(x, y) \in L$ and $b \in M$. We have to show that $R_\phi(b)$ holds in $(M, P(M))^+$ iff it holds in $(N, P(N))^+$. Suppose first that $(M, P(M))^+ \models R_\phi(b)$. By (ii), $\phi(x, b)$ is satisfied in $P(M)$, so in $P(N)$. Thus (as $(N, P(N))^+$ is a model of A_ϕ), $(N, P(N))^+ \models R_\phi(b)$.
Conversely, suppose that $(N, P(N))^+ \models R_\phi(b)$. So $\phi(x, b)$ is satisfied in $P(N)$ by c say. But c is L-independent from b over $P(M)$, so $\phi(x, b)$ does not fork over $P(M)$. By correctness of $(M, P(M))^+$, we have $(M, P(M))^+ \models R_\phi(b)$. The claim is proved.
We proceed to prove that $(M, P(M))$ is a κ-lovely pair. Let $B \subset M$ have cardinality $< \kappa$, and let $p(z)$ be a complete L-type over B. We want to realize $p(z)$ in M by some a such that $tp_L(a/B \cup P(M))$ does not L-fork

over B. Using local D-ranks for example, we can find a set Ψ of L-formulas $\phi(z, y, x)$, such that an extension $q(z)$ of $p(z)$ to a complete L-type over a set $C \supset B$ is a nonforking extension of p if and only if for no $\phi(z, y, x) \in \Psi$, $c \in C$ and $b \in B$, is $\phi(z, b, c) \in q(z)$. As $(N, P(N))^+$ is κ-lovely, we can find $a' \in N$ such that $tp_L(a'/B \cup P(N))$ does not L-fork over B. Hence, for each $\phi(z, y, x) \in \Psi$, and $b \in B$ we have $(N, P(N))^+ \models \neg R_\phi(a', b)$. As $(M, P(M))^+$ is κ-existentially closed, we can find $a \in M$ satisfying $p(z)$ and also such that $(M, P(M)) \models \neg R_\phi(a, b)$ for all $\phi \in \Psi$ and $b \in B$. So a realises a nonforking extension of $p(z)$ over $B \cup P(M)$.
We have shown that $(M, P(M))$ satisfies the κ-extension property. The κ-coheir property follows as $(M, P(M))^+$ is correct (and κ-existentially universal).

(iii) implies (i): Suppose that T is not low, and let $\phi(x, y) \in L$ be non low. As in the proof of Lemma 4.1, we can find tuples b_i, c_i (i in some index set I), inside a lovely pair $(M, P(M))$ such that $c_i \in P(M)$, b_i is independent from $P(M)$ over c_i and $\phi(x, b_i)$ divides over c_i for $i \in I$, but for some ultraproduct (b, c) of $(b_i, c_i)_i$, $\phi(x, b)$ does not divide over c. Let $(M, P(M))^+$ be the canonical expansion of $(M, P(M))$ to a model of T_P^+. We may assume that b, c live in an L_P^+-structure $(N, P(N))^+$ such that $(N, P(N), b, c)^+$ is an ultraproduct of the $(M, P(M), b_i, c_i)^+$. Note that $(M, P(M))^+ \models \neg R_\phi(b_i)$ for all $i \in I$, as $\phi(x, b_i)$ is not realized in $P(M)$. Thus $(N, P(N))^+ \models \neg R_\phi(b)$. But $c \in P(N)$, so $\phi(x, b)$ does not divide over $P(N)$. As $(N, P(N))^+$ is a model of T_P^+ it extends (as an L_P^+-structure) to a κ-existentially universal model $(N', P(N'))^+$ of $(T_P^+)_\forall$. Now $(N, P(N))^+ \models \neg R_\phi(b)$, so by Lemma 8.1 (or simply the fact that $(N', P(N'))^+$ is a model of $(T_P^+)_\forall$), $\phi(x, b)$ is not satisfied in $P(N')$. But as N' is an elementary extension of N, and $P(N) \subseteq P(N')$, $\phi(x, b)$ does not divide over $P(N')$. Hence $(N', P(N'))$ is not a lovely pair.

Assume now that the equivalent conditions of Proposition 8.3 are satisfied. Take large κ and let $(\bar{M}, P(\bar{M}))^+$ be a κ-existentially universal model of $(T_P^+)_\forall$. Then as in [13], $(\bar{M}, P(\bar{M}))^+$ is an e-universal domain of cardinality κ, that is, κ-saturated and κ-homogeneous for existential types. But $(\bar{M}, P(\bar{M}))^+$ is also a lovely pair, hence by 3.11, types are determined by quantifier-free types. It follows that the structure is κ-saturated and κ-homogeneous for quantifier-free (L_P^+-) types. Namely $(\bar{M}, P(\bar{M}))^+$ is a universal domain in the "Robinson" sense (see [8]). All the results of section 6:

proof of simplicity, characterization of forking, go through for this Robinson universal domain. So:

Proposition 8.4 *Suppose T is low. Then any κ-existentially universal model of $(T_P^+)_\forall$ is a simple Robinson universal domain.*

Finally, let us give axioms for $(T_P^+)_\forall$ when T is low. (This is analogous to Remark 4.7 where we gave axioms for T_P assuming $\mathcal{C}_{T,P}$ first order.)

Remark 8.5 *Assume T is low. Then $(T_P^+)_\forall$ can be axiomatized by*
(i) $T_\forall$,
(ii) "$\forall y_1...y_n \forall z_1..z_m (\bigwedge_{i=1,..,n} R_{\phi_i}(y_i) \wedge \bigwedge_{j=1,..,m} \neg R_{\psi_j}(z_j)) \rightarrow \exists x_1..x_n (\bigwedge_i \phi_i(x_i, y_i) \wedge \bigwedge_j (\psi_j(t_j, z_j)$ divides over $(x_1, x_2, .., x_n))))$", for all sequences $\phi_1(x_1, y_1),.., \phi_n(x_n, y_n)$, $\psi_1(t_1, z_1),.., \psi_m(t_m, z_m)$ of L-formulas.

Explanation. (ii) is of course a set of sentences (for each sequence ϕ_i, ψ_j): The expression "there exists x such that $\chi(x, y)$ and $\delta(t, z)$ divides over z" is, by lowness and QE of T, equivalent (in models of T) to a quantifier-free partial L-type in variables y and z. So (ii) is a set of universal L_P^+-sentences.

Proof. (i) and (ii) are clearly consequences of T_P^+. (In (ii) take x in P realising the $\phi_i(x_i, y_i)$.) For the converse, we will show that any existentially closed model of (i) and (ii) is correct (as in Definition 8.2). (Note that an ec model of (i) *will be* an elementary pair of models of T.) So let $(M, P(M))^+$ be an ec model of (i) and (ii). Remember that by convention (or by adding a universal axiom), $P(M)$ coincides with the interpretation of $R_{x=y}(y)$. Consider M as living in the big model $\bar{M}$ of T. By axiom (ii) we can find some set B in $\bar{M}$ such that (a) $\phi(x, b)$ is realized in B for any $\phi(x, y) \in L$ and $b \in M$ such that $(M, P(M))^+ \models R_\phi(b)$, and (b) $\psi(t, b)$ divides over B for every $\psi(t, z) \in L$ and $b \in M$ such that $(M, P(M))^+ \models \neg R_\psi(b)$. Let $M' = M \cup B$ (which is equipped with an L-structure by being a substructure of $\bar{M}$). Expand M' to an L_P^+-structure, by defining R_ϕ to hold of $c \in M'$ if $\phi(x, c)$ does not divides over B (in $\bar{M}$). Let us call this expansion $(M', P(M'))^+$.
It follows immediately from the construction that
Claim 1. $(M', P(M'))^+$ is an L_P^+-extension of $(M, P(M))^+$.

Claim 2. $(M', P(M'))^+$ is a model of axioms (i) and (ii).
Proof of claim 2. Consider axiom (ii) and $\phi_i(x_i, y_i)$ $(i = 1, .., n)$, $\psi(t_j, z_j)$ $(j =$

$1, .., m$) as there, and suppose that $(M', P(M'))^+ \models \bigwedge_i R_{\phi_i}(c_i) \wedge \bigwedge_j \neg R_{\psi_j}(d_j)$. So each $\phi_i(x_i, c_i)$ does not divide over B and each $\psi_j(t_j, d_j)$ divides over B. Let $e_i \in \bar{M}$ realize $\phi_i(x_i, c_i)$ such that e_i is L-independent from M' over B, and $\{e_1, .., e_n\}$ is M'-independent. Then each $\psi_j(t_j, d_j)$ divides over e. Claim 2 is proved.

By Claims 1, 2 and the assumption that $(M, P(M))^+$ is an existentially closed model of axioms (i) and (ii), we see that if $R_\phi(b)$ in $(M, P(M))^+$ then $\phi(x, b)$ is realized in $P(M)$. It easily follows that $(M, P(M))^+$ is correct. The proof of (ii)→(iii) in 8.3, shows that any existentially universal model of axioms (i) and (ii) is a model of T_P^+. Thus any model of (i) and (ii) embeds in a model of T_P^+, completing the proof of the remark.

Let us conclude with some questions. T still denotes a complete simple first order theory with quantifier-elimination.
Problem 1. Suppose the simple theory T has finiteness and definability of the D_ϕ-ranks. Is the same true for T_P?

Problem 2. What can be said about T_P assuming just that that T_P^+ has quantifier elimination? (For example, is it simple?)

Problem 3. Find a combinatorial equivalent to the finiteness and definability of all D_ϕ-ranks.

Problem 4. Suppose $\mathcal{C}_{T,P}$ is first order and T has elimination of hyperimaginaries. Does T_P have elimination of hyperimaginaries?

Problem 5. Prove that finiteness and definability of all D_ϕ-ranks implies that T has elimination of hyperimaginaries.

Problem 6. Describe the imaginaries (up to to interdefinability) in T_P when T is stable without the fcp, in particular when T is theory of algebraically closed fields of a given characteristic.

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