# Model theory in positive and continuous logics

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# 1 Positive logic

## 1.1 Motivation: the semantics of hyperimaginary sorts

**Classical approach: Semantics encoded by syntactic objects**

$$\begin{array}{ccc} \textbf{Syntax} & & \textbf{Semantics} \\ \text{Formulae} & \rightsquigarrow & \varnothing\text{-Definable sets} \\ \mathcal{L}(n) = \{\varphi(x_0, \ldots, x_{n-1})\} & & \mathcal{L}_\varnothing(n) = \mathcal{L}(n)/\equiv \end{array}$$

Let $S_n$ be the Stone space of the Boolean algebra $\mathcal{L}_\varnothing(n)$, i.e., the set of all complete consistent $n$-types in $\mathcal{L}$. By Stone duality:

$$\begin{array}{ccc} S_n & = & \{\text{ultrafilters on } \mathcal{L}_\varnothing(n)\} \\ \{\text{clopen sets in } S_n\} & = & \mathcal{L}_\varnothing(n) \\ \{\text{closed sets in } S_n\} & = & \text{type-definable properties} \end{array}$$

If $T$ is a theory, we have a similar duality between $\mathcal{L}_T(n) = \mathcal{L}(n)/\equiv_T$ and $S_n(T)$.

**Positive model theory: an alternative (and more general) approach to semantics in model theory**

- Idea: Semantics can be coded by topological objects.
- Advantage: it is easier to "relax the hypotheses" on a topological space than on a Boolean algebra.
- Why should we want to relax the hypotheses? For example, hyperimaginary sorts.

**Imaginary elements: first order syntax works**

Let $E(\bar{x}, \bar{y})$ be a definable equivalence relation on $n$-tuples, and:

$$\begin{aligned}
\mathcal{L}^* &= \mathcal{L} \cup \{\pi_E\} \\
M^* &= M \cup M^n/E, \qquad \pi_E^{M^*}(\bar{a}) = \bar{a}/E. \\
T^* &= \mathrm{Th}_{\mathcal{L}^*}\{M^* \colon M \vDash T\} \\
&= T \cup \{\text{“for all } \bar{x},\ \pi_E(\bar{x}) \text{ is the } E\text{-class of } \bar{x}\text{”}\}.
\end{aligned}$$

If $M$ is a monster model of $T$ then $M^*$ is a monster model of $T^*$; if $T$ is model complete so is $T^*$; etc.

**Example: cosets**

Let $M = \langle G, 1, \cdot, \ldots \rangle$ be a group, and $H \leq G$ a definable subgroup. Let $E(x, y)$ be the formula

$$\exists z\, (z \in H \wedge x = yz).$$

Then $E$ is a definable equivalence relation, and the sort $M_E$ is the sort of left cosets of $H$: $\{gH \colon g \in G\}$.

If $H$ is a normal subgroup, then we can define multiplication on $M_E$ by the formula $\varphi(x_E, y_E, z_E)$:

$$\exists xy\, \big(\pi_E(x) = x_E \wedge \pi_E(y) = y_E \wedge \pi_E(xy) = z_E\big).$$

**Hypermaginary elements: first order syntax fails**

Let $E(\bar{x}, \bar{y})$ be a type-definable equivalence relation on $\alpha$-tuples (possibly infinite).

$E$-classes are called *hyperimaginary elements*.

If $\alpha = n < \omega$, we may again try to define:

$$\begin{aligned}
\mathcal{L}^* &= \mathcal{L} \cup \{\pi_E\} \\
M^* &= M \cup M^n/E, \qquad \pi_E^{M^*}(\bar{a}) = \bar{a}/E. \\
T^* &= \mathrm{Th}_{\mathcal{L}^*}\{M^* \colon M \vDash T\}.
\end{aligned}$$

However, even if $M$ is saturated $M^*$ needs not be, and in a saturated model $N^* \vDash T^*$ we may find $\bar{a}$, $\bar{b}$ such that $E(\bar{a}, \bar{b})$ but $\pi_E(\bar{a}) \neq \pi_E(\bar{b})$.

For the purpose of studying $E$-classes, $T^*$ is useless.

**Example: infinitesimals**

Let $RCF$ be the theory of real closed fields in the language of ordered rings. Let $E(x, y)$ be "$x - y$ is infinitesimal". This is type-definable:

$$E(x, y) = \{-1/n < x - y < 1/n \colon n < \omega\}.$$

If $M \vDash RCF$ then the set $\pi_E(x) \neq \pi_E(y) \cup E(x, y)$ is finitely realised in $M^*$, but not realised. It would be realised in a saturated model of $T^*$, which means that $\pi_E$ is not what we want it to be.

**Solution: semantics via types**

Let $U$ be a monster model for $T$, $U_E = U^\alpha/E$ a hyperimaginary sort.

$$\begin{aligned} \text{As sets:} \quad & S_n(T) = U^n/\operatorname{Aut}(U), \\ & S_\alpha(T) = U^\alpha/\operatorname{Aut}(U), \text{ etc.} \\ \\ \rightsquigarrow \quad & S_E(T) = U_E/\operatorname{Aut}(U). \end{aligned}$$

We obtain a projection $S_\alpha(T) \twoheadrightarrow S_E(U)$. We equip $S_E(U)$ with the quotient topology: it is compact and Hausdorff. We can similary construct $S_{n,m\times E}(T)$ ($n$ real elements, $m$ $E$-classes), and do the same with more than one equivalence relation.

**From definable sets to type-definable ones**

We have a new feature: the type-space $S_E(T)$ is not necessarily totally disconnected (no base of clopen sets).

Hence, no Stone duality with a Boolean algebra, and no canonical notion of a formula/definable set. We do have a formal analogue of the notion of type-definable properties, i.e., properties definable by a set of formulae, through the classical correspondence:

$$\text{type-definable properties} \leftrightarrow \text{closed sets of types.}$$

The family of type-definable properties is closed under conjunction, disjunction, existential (and universal) quantification, *but not negation.* It also satisfies:

**Compactness for type-definable properties**

If $\{p_i(\bar{x}) \colon i \in I\}$ is a family of type-definable properties which is finitely satisfiable, then it is satisfiable.

**Recovering some syntax**

Let $\mathcal{L}_E$ consist of a $n$-ary predicate symbol $P_R$ for every closed set $R \subsetneq S_{m\times E}(T)$. Let $\Delta_E$ be the set of all quantifier-free positive $\mathcal{L}_E$-formulae. Then $U_E$ is naturally an $\mathcal{L}_E$ structure.

The type of a tuple in $U_E$ is determined by the $\Delta_E$-formulae it satisfies (its $\Delta_E$-type), and every finitely satisfiable small set of $\Delta_E$-formulae is satisfied (more precise definitions follow).

(We can do the same with several sorts simultaneously.)

This is our motivating example for positive logic.

## 1.2 Positive fragments and universal domains

### Syntax for positive logic

Let $\mathcal{L}$ be any first order language.

**Definition.** A *positive fragment* of $\mathcal{L}$ is a subset $\Delta \subseteq \mathcal{L}$ closed under positive Boolean combinations, change of variables and sub-formulae. (Caution: only $\wedge, \vee, \neg, \exists$ are allowed.)

We fix a positive fragment $\Delta$ and define:

$$\Sigma = \exists\Delta = \{\exists\bar{y}\,\varphi(\bar{x},\bar{y})\colon \varphi \in \Delta\} \qquad \text{(positive existential formulae)}$$
$$\Pi = \neg\Sigma = \{\forall\bar{y}\,\neg\varphi(\bar{x},\bar{y})\colon \varphi \in \Delta\} \qquad \text{(negative universal formulae)}$$

Note that $\Sigma$ is also a positive fragment.

### Universal domains

**Definition.** A (partial) $\Delta$-homomorphism between two structures $f\colon M \to N$ ($f\colon M \dashrightarrow N$) is a mapping such that for all $\bar{a} \in \mathrm{dom}(f)$ and $\varphi(\bar{x}) \in \Delta$:

$$M \vDash \varphi(\bar{a}) \Longrightarrow N \vDash \varphi(f(\bar{a})).$$

**Definition.** A *($\kappa$-)universal domain* is an $\mathcal{L}$-structure $U$ satisfying:

- $\Delta$-*Compactness:* Every small ($< \kappa$) set of $\Delta$-formulae which is finitely realised in $U$ is realised in $U$.
- $\Delta$-*Homogeneity:* Every partial $\Delta$-homomorphism $f\colon U \dashrightarrow U$ with small domain extends to an automorphism of $U$.
- *Elimination of* $\exists$*:* Every $\Sigma$-formula $\exists\bar{y}\,\varphi(\bar{x},\bar{y})$ is equivalent in $U$ to some partial $\Delta$-type $p(\bar{x})$.

### Complete positive Robinson theories

Universal domains replace the monster models: big homogeneous models in which the saturation assumption is restricted to our positive fragment $\Delta$. Therefore:

A partial $\Delta$-type (i.e., a set of $\Delta$-formulae) $\Phi(\bar{x})$ is realised in $U$ if and only if it is consistent with its negative universal theory:

$$T = \mathrm{Th}_\Pi(U) = \{\forall\bar{x}\,\neg\varphi(\bar{x})\colon \varphi \in \Delta \text{ and } U \vDash \forall\bar{x}\,\neg\varphi(\bar{x})\}.$$

We therefore say that $U$ is a *universal domain for $T$*.

**Definition.** A *complete positive Robinson theory* is a $\Pi$-theory which has a universal domain.

**Examples**

*Example.* If $\Delta = \mathcal{L}_{\omega,\omega}$, $T$ is a complete first order theory, and $U$ is a monster model of $T$.

We call this the "first order" case.

*Example.* With $U_E$ and $\Delta_E$ as constructed in the example of hyperimaginaries, $U_E$ is a universal domain with respect to $\Delta_E$.

*Example.* If $\Delta$ is closed for negation, $T$ is a Robinson theory and $U$ is its universal domain (Hrushovski [Hru97]), whence the term "positive Robinson theory".

## 1.3 On Δ-inductive theories and e.c. models

**Δ-inductive limits**

Fix a positive fragment $\Delta$ of $\mathcal{L}$. Here $\Delta$-homomorphisms will play the role of embeddings. Note that a $\Delta$-homomorphism $f\colon M \to N$ needs not be injective, so we cannot say that $N$ extends $M$. Instead, we will say that $N$ *continues* $M$.

**Δ-inductive limits**

Let $(I, <)$ be totally ordered, $(M_i \colon i \in I)$ be structures, and for each $i < j$ let $f_{ij}\colon M_i \to M_j$ be a $\Delta$-homomorphism such that $f_{jk} \circ f_{ij} = f_{ik}$.

One can construct in the obvious manner a direct limit $N = \varinjlim M_i$, equipped with $\Delta$-homomorphisms $g_i\colon M_i \to M$, such that $g_j \circ f_{ij} = g_i$ for $i < j$.

**If $\Delta = \{$q.f. formulae$\}$...**

Then a $\Delta$-homomorphism is an embedding, and a $\Delta$-inductive limit is an increasing union.

**Δ-inductive theories**

**Definition.** A first order theory $T$ is $\Delta$-inductive if an inductive limit of models of $T$ is a model of $T$.

For example, every $\Pi$-theory is $\Delta$-inductive. More generally:

**Lemma (Characterisation of Δ-inductive theories).** *A first order theory $T$ is $\Delta$-inductive if and only if it can be axiomatised by sentences of the form*

$$\forall \bar{x} \exists \bar{y}\, \varphi(\bar{x}) \to \psi(\bar{x}, \bar{y}) \qquad \varphi, \psi \in \Delta.$$

**If $\Delta = \{$q.f. formulae$\}$...**

This is the classical characterisation of inductive theories.

## Existentially closed (e.c.) models

**Definition.** A model $M$ of a theory $T$ is *existentially closed* if for every $\Delta$-homomorphism $f\colon M \to N \vDash T$, and every $\bar{a} \in M$ and $\varphi(\bar{x}, \bar{y}) \in \Delta$:

$$N \vDash \exists \bar{y}\, \varphi(f(\bar{a}), \bar{y}) \Longrightarrow M \vDash \exists \bar{y}\, \varphi(\bar{a}, \bar{y}).$$

We can use the classical chain argument to show:

**Lemma (Existence of existentially closed models).** *If $T$ is $\Delta$-inductive and $M \vDash T$, then $M$ continues to an e.c. model $N \vDash T$. (Uses the Axiom of Choice.)*

**If $\Delta = \{$q.f. formulae$\}$...**
This is the classical definition of e.c. models.

## Companions

We will be interested in what happens in the class of e.c. models of a $\Delta$-inductive theory. If two such theories have precisely the same e.c. models, we call them *companions*, and for our purposes they are equivalent.

**Lemma.** *The companionship class of a $\Delta$-inductive theory $T$ is determined by the set of $\Pi$-consequences of $T$, denoted $T_\Pi$.*

Therefore, on the one hand we may restrict our consideration to $\Pi$-theories. On the other, when it is convenient to consider other $\Delta$-inductive theories, we are allowed to do so.

*Example (Positive Morleyisation).* Let $\Delta \subseteq \mathcal{L}_{\omega,\omega}$ be a positive fragment, and $T$ an $\Delta$-inductive theory (e.g., a $\Pi$-theory).

$$\begin{aligned}
\mathcal{L}' &= \{R_\varphi(\bar{x}) \colon \varphi(\bar{x}) \in \Delta\} \\
\Delta' &= \text{positive quantifier-free } \mathcal{L}'\text{-formulae} \\
T' &= \forall \bar{x} \exists \bar{y}\, R_\varphi(\bar{x}) \to R_\psi(\bar{x}, \bar{y}) && \forall \bar{x} \exists \bar{y}\, \varphi(\bar{x}) \to \psi(\bar{x}, \bar{y}) \in T \\
&\quad\ \forall \bar{x} \exists! y\, R_{f(\bar{x})=y}(\bar{x}, y) && f \in \mathcal{L} \\
&\quad\ \forall \bar{x}\, R_{\neg\varphi}(\bar{x}) \leftrightarrow \neg R_\varphi(x) && \neg\varphi \in \Delta \\
&\quad\ \forall \bar{x}\, R_{\exists \bar{y} \varphi}(\bar{x}) \leftrightarrow \exists \bar{y}\, R_\varphi(\bar{x}, \bar{y}) && \exists \bar{y} \varphi \in \Delta \\
&\quad\ \ldots \text{(same for } \vee,\, \wedge)
\end{aligned}$$

Then $T'$ is $\Delta'$-inductive, and up to a change of language it has the same e.c. models as $T$.

$\therefore$ We may always assume that $\Delta$ consists only of positive quantifier-free formulae.

## Positive Robinson theories, revisited

We can now re-define positive Robinson theories by a "quantifier elimination" property:

**Definition.** Let $T$ be a $\Delta$-inductive theory. It is:

- A *positive Robinson theory* if every $\Sigma$-formula $\exists \bar{y}\, \varphi(\bar{x}, \bar{y})$ is equivalent, in the class of e.c. models of $T$, to a partial $\Delta$-type (a set of $\Delta$-formulae).
- *Complete* if every two models of $T$ continue to a third ("positive JEP").

Our new definition of a complete positive Robinson theory agrees with the previous one:

**Theorem.** *A $\Pi$-theory is a complete positive Robinson theory if and only if it has a $\kappa$-universal domain for some (all) big enough $\kappa$.*

*Proof.* $\Longrightarrow$: The class of e.c. models of $T$ has enough amalgamation properties to construct a sufficiently homogeneous e.c. model of $T$ in which all small e.c. models $\Delta$-embed.

$\Longleftarrow$: Verify the definition for small (cardinality $< \kappa$) e.c. models by $\Delta$-embedding them in the universal domain. The same properties for arbitrary e.c. models follow easily. $\square$

## Types and type spaces

- Classical concepts and methods transfer (mostly) to positive logic:

$$\begin{array}{ccc} \text{Monster model} & & \text{Universal domain} \\ & \rightsquigarrow & \\ \mathcal{L}_{\omega,\omega} & & \Delta \end{array}$$

- Types (over a small set $A \subseteq U$):

$$\operatorname{tp}(\bar{a}/A) = \{\varphi(\bar{x}, \bar{b}) \colon \varphi \in \Delta, \bar{b} \in A, U \vDash \varphi(\bar{a}, \bar{b})\}$$
$$S_n(A) = U^n / \operatorname{Aut}_A(U) = \{\operatorname{tp}(\bar{a}/A) \colon \bar{a} \in U^n\}$$

- Topology: every $\Delta$-formula with parameters in $A$ defines a closed subset:

$$[\varphi(\bar{x}, \bar{a})] = \{p(\bar{x}) \in S_n(A) \colon \varphi(\bar{x}, \bar{a}) \in p\}.$$

  These sets form a base of closed sets for a compact, $T_1$ topology.

## A word of caution

Because there is no canonical notion of a formula, the choice of language is even more arbitrary than in first order logic: $\Delta$-formulae define closed sets of types (i.e., a partial type), but a closed set of types may or not be defined by a $\Delta$-formula. If such a closed set is not definable by a $\Delta$-formula, we can always expand the language so that it is. The semantic contents, coded by the type spaces, would not change. Therefore:

Different positive Robinson theories in different (arbitrary!) languages with the homeomorphic type spaces are to be considered equivalent. The common semantic contents will be referred to as a *compact abstract theory*, or *cat*.

## 1.4 Simplicity

**Closing the circle: simplicity**

Our motivation for positive logic was the notion of hyperimaginary elements. These were introduced by Hart, Kim and Pillay [HKP00] for the construction of canonical bases in first order simple theories.

They had to re-prove for hyperimaginaries most of the theorems concerning independence in simple theories.

One can define indiscernible sequences, non-dividing and simplicity (= local character of non-dividing) in the positive setting.

**Caution:** not all equivalent definitions of first order logic are equivalent.)

**Definition.**

- A sequence of tuples $(\bar{a}_i \colon i < \omega)$ is *indiscernible over* $A$ if for all $i_0 < i_1 < \ldots < i_{n-1} < \omega$: $\mathrm{tp}(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}/A)$ depends solely on $n$.
- A type $p(\bar{x}, \bar{b}, A) \in S_n(\bar{b}A)$ *divides* over $A$ if There exists an $A$-indiscernible sequence $(\bar{b}_i \colon i < \omega)$ such that $\bar{b}_0 = \bar{b}$, and $\bigcup_{i<\omega} p(\bar{x}, \bar{b}_i, A)$ is inconsistent.
- Let $A, B, C \subseteq U$. If $\mathrm{tp}(\bar{a}/\bar{b}C)$ does not divide over $C$ for all $\bar{a} \in A$, $\bar{b} \in B$, we say that $A$ and $B$ are *independent* over $C$, in symbols $A \mathop{\smile\!\!\!\!\!\!|}_C B$.
- A cat $T$ is *simple* if for all $a$ and $A$ there is $A_0 \subseteq A$, $|A_0| \leq |\mathcal{L}|$, such that $a \mathop{\smile\!\!\!\!\!\!|}_{A_0} A$.

**Theorem.** *If $T$ is simple, then $\mathop{\smile\!\!\!\!\!\!|}$ satisfies:*

- *Symmetry:* $A \mathop{\smile\!\!\!\!\!\!|}_C B \Longleftrightarrow B \mathop{\smile\!\!\!\!\!\!|}_C A$.
- *Transitivity:* $A \mathop{\smile\!\!\!\!\!\!|}_B CD \Longleftrightarrow \left[A \mathop{\smile\!\!\!\!\!\!|}_B C \wedge A \mathop{\smile\!\!\!\!\!\!|}_{BC} D\right]$.
- *Invariance, finite character, local character.*
- *Extension only holds for some types (e.g., types over sufficiently saturated models). We call such types extendible.*
- *Independence theorem for extendible Lascar strong types.*

*Conversely, if $\mathop{\smile\!\!\!\!\!\!|}'$ satisfies these hypotheses, then $T$ is simple and $\mathop{\smile\!\!\!\!\!\!|}' = \mathop{\smile\!\!\!\!\!\!|}$.*

*Moreover, extendible Lascar strong types admit hyperimaginary canonical bases.*

As we may add hyperimaginary sorts and remain within positive logic, we may always assume that canonical bases exist in the structure, achieving our original goal.

**Definition.** A cat $T$ is *Hausdorff* if $S_n(T)$ is Hausdorff for all $n$. It follows that $S_\alpha(A)$ is Hausdorff for every $\alpha$ and set of parameters $A$.

*Example.* Adding hyperimaginary sorts to a first order theory (as in the beginning) yields a Hausdorff cat.

**Theorem.** *If $T$ is Hausdorff (in fact, a weaker assumption, "thickness", suffices) then:*

- *Every type has non-dividing extensions to arbitrary sets.*
- *The theory of hyperdefinable groups [Wag01] holds.*
- *Lovely/beautiful pairs construction can be carried out.*
- *Etc.*

# 2 Continuous first order logic

## 2.1 Characterising first order logic

**Going back a bit closer to first order logic**

- It really is fun to look for the most general context in which a certain argument works.
- But our original goal was merely to "relax the conditions defining first order logic", whatever that means.

**So what precisely does define the full first order logic?**

Let $T$ be a complete first order theory (equivalently: a positive Robinson theory with $\Delta = \mathcal{L}_{\omega,\omega}$). Then:

- $S_n(T)$ is totally disconnected for all $T$ (i.e., has a base of clopen sets).
- The restriction mapping $S_{n+1}(T) \to S_n(T)$ is open.

If a cat satisfies the latter property we say it is *open*.

(The mapping $S_{n+1}(T) \to S_n(T)$ is continuous and closed for every cat.)

Conversely, let $T$ be a cat:

- If $S_n(T)$ is totally disconnected for all $n$, then we may choose a language for it such that $\Delta$ is closed for negation.
- Then, $T$ being open is equivalent to the class of e.c. models of $T$ being elementary, or to any universal domain being a saturated model of its own theory.

**First order logic without negation?!**

- When adding a hyperimaginary sort we relaxed the first condition, replacing "totally disconnected" with "Hausdorff" (which is indeed next-best.)
- In the totally disconnected case (i.e., "with negation"), openness of $S_{n+1}(T) \to S_n(T)$ meant the existence of a first order model completion for $T$. In the case without negation, does it still mean that some kind of a first order model completion exists?
- Surprisingly enough: YES.
- By the way, hyperimaginary sorts do satisfy the openness hypothesis.

**Convention.** From now on we assume $T$ is an open Hausdorff cat, i.e., that $S_n(T)$ is Hausdorff and $S_{n+1}(T) \to S_n(T)$ is open for all $n$.

**Continuous predicates**

- A compact totally disconnected topology is given by the family of clopen sets; equivalently: by the family of functions to $\{0,1\}$ (also known as $\{T,F\}$).
- Similarly: A compact Hausdorff topology is given by the family of all continuous functions to $[0,1]$ (or to $\mathbb{C}$).
- In first order logic, a continuous mapping from $S_n(T)$ to $\{0,1\}$ is simply a formula, or a definable $n$-ary predicate.
- By analogy, we define:

**Definition.** A *(definable) continuous $n$-ary predicate* is a continuous mapping $\varphi\colon S_n(T) \to [0,1]$. For a tuple $\bar{a} \in U^n$ we write $\varphi(\bar{a})$ instead of $\varphi(\mathrm{tp}(\bar{a}))$.

**A lemma concerning open mappings**

**Lemma.** *Let $X$ and $Y$ be compact Hausdorff space, $f\colon X \twoheadrightarrow Y$ a continuous ($\Longrightarrow$ closed) and open projection. Let $\varphi\colon X \to [0,1]$ be a continuous mapping, and define*

$$\psi(y) = \inf_{x \in f^{-1}(y)} \varphi(x).$$

*Then $\psi\colon Y \to [0,1]$ is continuous.*

*Proof.* For any $r \in [0,1]$: $\psi^{-1}([0,r]) = f(\varphi^{-1}([0,r]))$ is closed. Assume now that $\psi(y) < r$: then there is $x \in f^{-1}(y)$ such that $\varphi(x) < r$, and there is a neighbourhood $U$ of $x$ such that $\varphi(U) \subseteq [0,r)$. Then $f(U)$ is a neighbourhood of $y$, and $\psi(f(U)) \subseteq [0,r)$. Thus $\psi^{-1}([0,r))$ is open. □

**Operations on continuous predicates**

- **Variable manipulation:** If $\varphi(x,y)$ is a definable predicate, then so are:

$$\begin{aligned} &\psi(x,y,z) := \varphi(x,y) && \text{(dummy variables)} \\ &\chi(x) := \varphi(x,x) && \text{(repetition of variables).} \end{aligned}$$

- **Continuous (instead of Boolean) combinations:** If $f\colon [0,1]^2 \to [0,1]$ is continuous and $\varphi(x,y)$ and $\psi(x,z)$ are definable continuous then so is:

$$\chi(x,y,z) := f(\varphi(x,y), \psi(x,z)).$$

- **Continuous quantification:** If $\varphi(\bar{x},y)$ is a definable continuous predicate, then so are:

$$\psi(\bar{x}) := \inf_y \varphi(\bar{x},y) \qquad \chi(\bar{x}) := \sup_y \varphi(\bar{x},y).$$

  (Applying the Lemma to $S_{n+1}(T) \to S_n(T)$.)

**An (important) example**

Assume $|\mathcal{L}| \le \omega$. As $[x = y] \subseteq S_2(T)$ is closed, it is the zero set of a continuous mapping $\varphi\colon S_2(T) \to [0,1]$.

In other words, we have a definable continuous predicate $\varphi(x,y)$ such that for every $a, b \in U$: $\varphi(a,b) = 0 \iff a = b$.

Let $d(x,y) := \sup_z |\varphi(x,z) - \varphi(y,z)|$. Then $d$ is also a definable continuous predicate, satisfying:

$$\begin{aligned} &d(x,y) = d(y,x) \\ &d(x,y) \le d(x,w) + d(w,x) \\ &d(x,y) = 0 \iff x = y. \end{aligned}$$

In other words, $d$ is a *definable metric.*

**Uniform continuity**

**Notation.** If $\varphi(\bar{x})$ is an $n$-ary definable continuous predicate, let $[\varphi(\bar{x}) \le r]$ denote the (closed) set $\{p \in S_n(T)\colon \varphi(p) \le r\}$.

Let $\varphi(x,\bar{z})$ be any definable continuous predicate, and $\varepsilon > 0$. Then:

$$[\sup_{\bar{z}} |\varphi(x,\bar{z}) - \varphi(y,\bar{z})| \ge \varepsilon] \cap \bigcap_{\delta>0} [d(x,y) \le \delta] = \varnothing.$$

By compactness, there is $\delta > 0$ such that for all $a, b \in U$:

$$d(a, b) \leq \delta \Longrightarrow \sup_{\bar{z}} |\varphi(a, \bar{c}) - \varphi(b, \bar{c})| < \varepsilon.$$

In other words, every definable continuous predicate is uniformly continuous with respect to $d$.

A similar argument shows that if $f\colon U^n \to U$ is type-definable (i.e., its graph is) then it is uniformly continuous.

If $d'$ is another definable metric, then $d$ and $d'$ are uniformly continuous w.r.t. each other, i.e., they are uniformly equivalent.

We have all we need to motivate continuous logic.

(Another motivation: local stability. Not in this talk.)

## 2.2 Formal definitions

**Continuous logic**

Continuous first order logic as presented here was defined independently by B. and Usvyatsov [BUa]. It is similar to the logic defined by Chang & Keisler [CK66], but is far better adapted to our needs.

- We replace the set of truth values $\{T, F\}$ with the compact interval $[0, 1]$. (It turns out to be more elegant to identify "True" with 0 rather than 1.)
- The distinguished equality symbol $=$ is replaced with a distinguished metric symbol $d$.
- First order logic requires that $=$ be a congruence relation for all predicate and function symbols. Analogously, we require that all symbols be uniformly continuous w.r.t. $d$.

**Definition.** A *continuous signature* is a collection $\mathcal{L}$ of predicate and function symbols, with various arities, where each symbol $s$ is equipped with a uniform continuity modulus $\varepsilon \mapsto \delta_s(\varepsilon)$.

**Continuous structures**

**Definition.** A *continuous pre-$\mathcal{L}$-structure* consists of:

- A set $M_0$.
- For each $n$-ary predicate symbol a mapping $P^{M_0}\colon M_0^n \to [0, 1]$.
- For each $n$-ary function symbol a mapping $f^{M_0}\colon M_0^n \to M_0$.

Such that in addition $d^{M_0}$ is a pseudometric, and all the interpretations of the symbols respect their uniform continuity moduli.

**Definition.** A *continuous structure* is a pre-structure $M$ in which $d^M$ is a complete metric.

For every pre-structure $M_0$ we may construct its *completion* $M = \hat{M}_0$ by modding out $d(a, b) = 0$ and completing.

### Continuous syntax

- **Terms:** constructed using variables and function symbols as usual.
- **Atomic formulae:** defined as usual.
- **Connectives:** In theory any continuous function $f\colon [0,1]^n \to [0,1]$: if $\varphi_i, i < n$ are formulae then so is
$$f(\varphi_0, \ldots, \varphi_{n-1}).$$
- **Quantifiers:** inf and sup:
$$\sup_x \varphi \qquad \inf_x \varphi.$$

### Continuous semantics

Writing $\varphi(\bar{x})$ we mean that all free variables of $\varphi$ are in $\bar{x}$.

If $M$ is a structure, then every term $t(\bar{x})$ and formula $\varphi(\bar{x})$ induce uniformly continuous mappings (by induction on the structure of the term or formula):

$$t^M\colon M^{|\bar{x}|} \to M$$
$$\varphi^M\colon M^{|\bar{x}|} \to [0,1].$$

For example, if $\psi(\bar{x}) = \sup_y \varphi(\bar{x}, y)$, then:

$$\psi^M(\bar{a}) = \sup\{\varphi^M(\bar{a}, b)\colon b \in M\}.$$

### A little more about connectives

The family of all possible continuous connectives is of the size of the continuum. In practise, it suffices to use:

$$\begin{array}{lll} \neg x & = 1 - x & \text{(unary -- like negation)} \\ \frac{1}{2}x & = \frac{x}{2} & \text{(unary -- no analogue)} \\ x \dot{-} y & = \begin{cases} x - y & x \geq y \\ 0 & x < y \end{cases} & \text{(binary -- like ``} y \to x \text{'')} \end{array}$$

We can get most of the connectives we'd like to use by composing these:

$$|x - y| = \neg(\neg(x \dot{-} y) \dot{-} (y \dot{-} x))$$

By lattice Stone-Weierstrass: every continuous function can be uniformly approximated by terms in $\neg, \frac{1}{2}, \dot{-}$.

## 2.3 Theories

### Statements and compactness

**Definition.** A *statement* is something of the form "$r \leq \varphi(\bar{x}) \leq s$". It is closed if $\varphi$ has no free variables.

**Theorem (Łoś's Theorem for continuous logic).** *Let $(M_i \colon i \in I)$ be $\mathcal{L}$-structures, $\mathscr{U}$ an ultrafilter on $I$. Let $M = \prod M_i/\mathscr{U}$. Then for every formula $\varphi(\bar{x})$, and tuples $\bar{a}_i \in M_i$:*

$$\varphi^M([\bar{a}_i \colon i \in I]) = \lim_{\mathscr{U}} \varphi^{M_i}(\bar{a}_i).$$

**Corollary (Compactness for continuous first order logic).** *A set of statements which is finitely satisfiable is satisfiable.*

### Theories

**Definition.** A *theory* is a set of closed statements. (We could restrict to statements of the form $\varphi = 0$, so theories are "ideals" – this boils down to the same thing).

**Definition.** We say that $T$ *eliminates quantifiers* if for every formula $\varphi(\bar{x})$ and $\varepsilon > 0$ there is a quantifier-free formula $\theta(\bar{x})$ such that:

$$T \vDash \sup_{\bar{x}} |\varphi - \theta| \leq \varepsilon$$

i.e.:

$$M \vDash T \Longrightarrow (\sup_{\bar{x}} |\varphi - \theta|)^M \leq \varepsilon.$$

### Example: probability algebras

The language: functional language of Boolean algebras + one unary predicate symbol $\mu$: $\mathcal{L}_{PrAlg} = \{0, 1, \vee, \wedge, \cdot^c, \mu\}$.

The theory $T_{PrAlg}$: "Boolean algebra with an additive probability measure; distance is the measure f the symmetric difference".

$$\begin{aligned}
&\sup\nolimits_{x,y} d(x^c \wedge y^c, (x \vee y)^c) = 0 \qquad (\text{i.e., } \forall xy\, x^c \wedge y^c = (x \vee y)^c)\\
&\ldots \text{axioms of Boolean algebras} \ldots\\
&\mu(1) = 1\\
&\mu(0) = 0\\
&\forall xy\, \mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y) \qquad (\sup |\ldots - \ldots| = 0)\\
&\forall xy\, d(x, y) = \mu((x \wedge y^c) \vee (y \wedge x^c))
\end{aligned}$$

$T_{AtlsPrAlg} = T_{PrAlg} +$ "no atoms":

$$\sup\nolimits_x \inf\nolimits_y \left| \mu(x \wedge y) - \frac{1}{2}\mu(x) \right| = 0$$

The theory $T_{AtlsPrAlg}$ is complete and eliminates quantifiers (a back-and-forth argument). It the "model completion" of the "universal theory" $T_{PrAlg}$ (note that $M \subseteq N \vDash T_{PrAlg} \Longrightarrow M \vDash T_{PrAlg}$).

**Types**

- A *complete $n$-type* is a maximal (finitely) satisfiable set $p(\bar{x})$ of statements of the form $\varphi(\bar{x}) = r_{p,\varphi} \in [0,1]$. We may also view it as a mapping $\varphi(\bar{x}) \mapsto \varphi(\bar{x})^p = r_{p,\varphi}$.
- The type of an $n$-tuple $\bar{a} \in M$ is given by:

$$\mathrm{tp}^M(\bar{a}) = \{\varphi(\bar{x}) = \langle \text{value of } \varphi^M(\bar{a}) \rangle \colon \varphi(\bar{x}) \in \mathcal{L}\}.$$

- The space of all $n$-types consistent with a theory $T$ is denoted $S_n(T)$. It is equipped with the minimal topology in which the sets:

$$[r \leq \varphi(\bar{x}) \leq s] := \{p \colon r \leq \varphi^p \leq s\}$$

  are closed (equivalently: such that all the mappings $p \mapsto \varphi^p$ are continuous).

- $T$ eliminates quantifiers if and only if every $p \in S_n(T)$ is determined by the mappings $\varphi \mapsto \varphi^p$ for quantifier-free $\varphi$.

**Back to the motivation**

The expressive power of an open Hausdorff cat $T$ is the same as that of a continuous first order theory $T'$:

- To get from $T$ to $T'$: define a continuous language

$$\begin{gathered}\mathcal{L}' = \{P_\varphi \ n\text{-ary predicate} \colon \text{continuous } \varphi \colon S_n(T) \to [0,1]\} \\ T' = \mathrm{Th}_{\mathcal{L}'}(U') \qquad \text{viewing } U \text{ as an } \mathcal{L}'\text{-structure } U'.\end{gathered}$$

  Then $T'$ eliminates quantifiers and $U$ is a monster model.

- To go back, using a monster model $U' \vDash T$:

$$\begin{gathered}\mathcal{L} = \{P_R \ n\text{-ary predicate} \colon \text{closed } R \subseteq S_n(T')\} \\ \Delta = \{\text{quantifier-free positive formulae}\} \\ T = \mathrm{Th}_\Pi(U) \qquad \text{viewing } U' \text{ as an } \mathcal{L}'\text{-structure } U.\end{gathered}$$

  Then $U$ is a universal domain for $T$.

- Going in either direction we have

$$S_n(T) \cong S_n(T').$$

# 3 The metric

## 3.1 In continuous logic, every space is metric...

**What now?**

We can proceed to look at:

- Existentially closed models, inductive theories, model completions, model companions...
- Definability: Beth, Svenonius.
- Saturation, homogeneity. Monster models.
- Omitting types, isolated (and semi-isolated) types, Ryll-Nardzewski.
- Stability, and *local* stability (counting types $\iff$ non order property $\iff$ notion of independence).
- Superstability.

Many of these require us to take into account the metric on the universal domain, and metrics it induces on other spaces.

**My message for the day: do not forget the metric**

In continuous logic, each time we have an object we might think of as a "set with additional structure", it is in fact a *metric space* with additional structure (in classical logic the metric is discrete, so these ARE just sets...):

- In a continuous structure, the interpretations of the symbols are uniformly continuous: this is a (complete) metric space with additional structure.
- Spaces of types admit a natural metric:

  $$d(p, q) = \inf\{d(\bar{a}, \bar{b}) \colon \bar{a} \vDash p, \bar{b} \vDash q\}.$$

  (Here $d(\bar{a}, \bar{b}) = \max_{i<n} d(a_i, b_i)$.) Although this may be less obvious, one should view type spaces as metric spaces (rather than sets) with a topological structure: shows up when considering superstability, omitting types, and even $\omega$-saturation.

**And yet another metric (to be looked at...)**

If $M$ is a countable discrete structure, then $\mathrm{Aut}(M)$ is a polish group in the pointwise convergence topology. The uniform convergence metric is discrete.

In the separable metric case, the *topological* structure of pointwise convergence comes on top of the (non-trivial) *metric* structure of uniform convergence:

| | $M$ is discrete | Hilbert space: $U(H)$ |
|---|---|---|
| Pointwise (polish): | ... | Strong operator topology |
| Uniform (metric): | discrete | Operator norm. |

*Question.* Can we simply take a categorical definition of a topological space in the category of sets, and transfer it to the category of metric spaces?

## 3.2 Omitting types

**"Any fool can realise a type, but it takes a model-theorist to omit one." G.S.**

**Theorem (Omitting Types Theorem, classical version).** *Let $T$ be a first order theory in a countable language, and for each $n < \omega$, let $X_n \subseteq S_n(T)$ be meagre. Then there is a model $M \vDash T$ omitting every type in $\bigcup X_n$.*

*Proof.* The set of all $p \in S_\omega(T)$ which are types of enumerations of models of $T$ is co-meagre (Tarski-Vaught test). Removing countably many meagre sets we still have something co-meagre, and in particular non-empty. □

**Corollary.** *A closed set $X \subseteq S_n(T)$ is realised in every model if and only if it has non-empty interior.*

**Theorem.** *Let $T$ be a continuous first order theory in a countable language, and for each $n < \omega$, let $X_n \subseteq S_n(T)$ be meagre. Then there is a pre-model $M_0 \vDash T$ (i.e., a dense subset of a model $M \vDash T$) omitting every type in $\bigcup X_n$.*

*Proof: same.* The set of all $p \in S_\omega(T)$ which are types of enumerations of pre-models of $T$ is co-meagre (Tarski-Vaught test)... □

**Corollary.** *A closed set $X \subseteq S_n(T)$ is realised in every dense subset of model if and only if it has non-empty interior.*

**So what?!** Well, we forgot the metric

**A "classical" example: Isolated (principal) types**

Recall: $d(p, q) = \inf\{d(\bar{a}, \bar{b}) : \bar{a} \vDash p, \bar{b} \vDash q\}$.

The metric limits the resolution: we cannot "see" single points, only balls of positive radius:

**Definition.** A type $p \in S_n(T)$ is isolated if for every $\varepsilon > 0$, the $\varepsilon$-ball $p^\varepsilon$ is a neighbourhood of $p$.

**Theorem (Henson).** *A type can be omitted in a model if and only if it is non-isolated.*

*Proof.* If $p$ is not isolated, it's that $p^\varepsilon$ is nowhere dense for some $\varepsilon > 0$. Then there is a pre-model $M_0$ which omits $p^\varepsilon$, and the model $M = \hat{M}_0$ omits $p$. Conversely: construct a convergent sequence... □

**Omitting more than a single type: $0^+$-interior**

**Definition.** • Let $X \subseteq S_n(T)$, $\varepsilon > 0$ The $\varepsilon$-interior of $X$, denoted $X^{\varepsilon,\circ}$, is the interior of $X^\varepsilon = \{p \colon d(p,X) \leq \varepsilon\}$ (if $X$ is closed, so is $X^\varepsilon$).

- The $0^+$-interior of $X$ is defined as:

$$X^{0^+,\circ} = \bigcap_{\varepsilon>0} X^{\varepsilon,\circ}.$$

- $X$ is nowhere $0^+$-dense if $\bar{X}^{0^+,\circ} = \varnothing$.
- $X$ is $0^+$-meagre if it is a countable union of nowhere $0^+$-dense sets.

**Real Omitting Types Theorem**

*Remark.* A type $p$ is isolated if and only if $\{p\}^{0^+,\circ} \neq \varnothing$, if and only $\{p\}^{0^+,\circ} = \{p\}$.

Therefore, the omission of non-principal types is a special case of:

**Theorem.** *Let $T$ be a continuous theory in a countable language. Let $X_n \subseteq S_n(T)$ be $0^+$-meagre for all $n$. Then there exists a model $M \vDash T$ omitting $\bigcup X_n$.*

*Proof.* We may assume we only need to omit one $X \subseteq S_n(T)$, that $X$ is closed and $X^{0^+,\circ} = \varnothing$. For $m < \omega$ define

$$Y_m = \partial(X^{\frac{1}{m}}) = X^{\frac{1}{m}} \smallsetminus X^{\frac{1}{m},\circ}.$$

Each $Y_m$ is closed and nowhere dense, so there is a pre-model $M_0$ omitting them all.

Assume that $p \in X$. Then there is $\varepsilon > 0$ such that $p \notin X^{2\varepsilon,\circ}$.

Let $W = X^{\varepsilon,\circ}$. Then $W$ is open, and it follows that $W^{<\varepsilon}$ is also open, so $W^{<\varepsilon} \subseteq X^{2\varepsilon,\circ}$. Therefore $d(p,W) \geq \varepsilon$. Let $m > \frac{1}{\varepsilon}$. Then $p^{\frac{1}{m}} \subseteq X^{\frac{1}{m}}$, and

$$p^{\frac{1}{m}} \cap X^{\frac{1}{m},\circ} \subseteq p^{<\varepsilon} \cap W = \varnothing.$$

Therefore $p^{\frac{1}{m}} \subseteq Y_m$.

As $Y_m$ is omitted by $M_0$, $p$ is omitted by $M = \hat{M}_0$. □

## 3.3 Semi-isolation and superstability

**Semi-isolation**

**Definition (Classical case: Pillay [Pil83]).** Let $\bar{a}$, $\bar{b}$ be finite tuples, $p(\bar{x},\bar{y}) = \mathrm{tp}(\bar{a},\bar{b})$, $q(\bar{y}) = \mathrm{tp}(\bar{b})$.

We say that $\bar{a}$ *semi-isolates* $\bar{b}$ (over $\varnothing$) if there exists $\varphi(\bar{x},\bar{y}) \in p$ such that $\varphi(\bar{a},\bar{y}) \vdash q(\bar{y})$.

In other words: let $X = [q(\bar{y})] \subseteq S_n(\bar{a})$. Then $\bar{a}$ semi-isolates $\bar{b}$ if and only if $p(\bar{a},\bar{y}) \in X^\circ$.

**Definition (Semi-isolation, continuous version).** Let $X = [\mathrm{tp}(b)] \subseteq S_n(\bar{a})$ as above. Then $\bar{a}$ semi-isolates $\bar{b}$ if $\mathrm{tp}(\bar{b}/\bar{a}) \in X^{0^+,\circ}$.

**Semi-isolation and realising types**

**Corollary.** *Assume that $p$ and $q$ are types over $\varnothing$, and every model of $T$ realising $p$ also realises $q$. Then there exists $\bar{a} \vDash p$ which semi-isolates a realisation $\bar{b} \vDash q$.*

*Proof.* Let $\bar{a} \vDash p$ in a large model $M$, and let $T' = T(\bar{a}) = \mathrm{Th}(M, \bar{a})$. Let $X = [q] \subseteq S_m(\bar{a}) = S_m(T')$. Then by assumption every model of $T'$ realises $X$, whereby $X^{0^+,\circ} \neq \varnothing$.

Let $\bar{b}$ realise a type $\mathrm{tp}(\bar{b}/\bar{a}) \in X^{0^+,\circ}$. Then $\bar{a}$ semi-isolates $\bar{b}$. □

**Corollary (Towards Lachlan's Theorem).** *Assume $T$ has finitely many separable models, and let $(\bar{a}_i : i < \omega)$ be any sequence. Then there is $n < \omega$, and a copy $\bar{b}_{\leq n}$ of $\bar{a}_{\leq n}$, such that $\bar{a}_{<n}$ semi-isolates $\bar{b}_{\leq n}$*

Properties of semi-isolation still hold with this definition:

- If $\mathrm{tp}(\bar{b}/\bar{a})$ is isolated then $\bar{a}$ semi-isolates $\bar{b}$.
- If $\bar{a}$ semi-isolates $\bar{b}$, and $\bar{b}$ semi-isolates $\bar{c}$, then $\bar{a}$ semi-isolates $\bar{c}$.
- In a stable theory, if $\bar{a}$ semi-isolates $\bar{b}$, but $\mathrm{tp}(\bar{b})$ is not isolated, then $\bar{a} \not\perp \bar{b}$.

*Proof of last item.* Let $q(\bar{y}) = \mathrm{tp}(\bar{b})$, $X = [q]^{S_n(\bar{a})}$.

Assume we did have $\bar{a} \perp \bar{b}$. We know there is $\varepsilon > 0$ such that $(q^\varepsilon)^\circ$ is empty. But then: $\mathrm{tp}(\bar{b}/\bar{a}) \in X^{\varepsilon,\circ} \subseteq X^\varepsilon$, so by the Open Mapping Theorem: $q^\varepsilon$ has non-empty interior.

Contradiction. □

**Approximate dependence/independence**

In the previous argument we wasted information: assume that $\varepsilon > 0$ is such that $(q^{2\varepsilon})^\circ = \varnothing$. Then the same argument shows that if $d(\bar{b}', ^\prime) \leq \varepsilon$ then $\bar{b}' \not\perp \bar{a}$ as well.

**Definition (Approximate independence).** Let $\bar{b}$ be a finite tuple, and $\varepsilon > 0$. We say that $\bar{b}^\varepsilon \perp_B C$ ("$\bar{b}$ is independent up to $\varepsilon$ from $C$ over $B$") if there is a tuple $\bar{b}'$ such that $d(\bar{b}, \bar{b}') \leq \varepsilon$ and $\bar{b}' \perp_C B$.

So in fact we proved:

**Lemma.** *Assume that* $\mathrm{tp}(\bar{b})$ *is not isolated. Then there is $\varepsilon > 0$ such that for every tuple $\bar{a}$, if $\bar{a}$ semi-isolates $\bar{b}$, then $\bar{b}^\varepsilon \not\perp \bar{a}$.*

**A further consequence**

**Lemma (Towards Lachlan's Theorem).** *Assume $T$ is stable and has finitely many separable models, and that $p \in S_1(T)$ is non-isolated. Let $(a_i : i < \omega)$ be a Morley sequence in $p$. Then there is $\varepsilon > 0$, $n < \omega$, and an independent sequence $(b_i : i < \omega)$ of realisations of $p$, such that $b_j^\varepsilon \not\perp a_{<n}$ for all $j < \omega$.*

*Proof.* By a previous lemma, there are $n < \omega$, and a copy $c^0_{\leq n}$ of $a_{\leq n}$ such that $a_{<n}$ semi-isolates $c^0_{\leq n}$. Then $a_{<n}$ semi-isolates $b_0 = c^0_n$, and $a_{<n}$ semi-isolates $c^0_{<n}$.

Now find $c^1_{\leq n}$ such that $c^1_{\leq n} c^0_{<n} \equiv c^0_{\leq n} a_{<n}$ and $c^1_{\leq n} c^0_{<n} \perp b_0$. Let $b_1 = c^1_n$. Proceed by induction.

Existence of $\varepsilon$ is by non-isolation of $p$. □

### Superstability

**Definition (Iovino, for Banach spaces).**

- $T$ is $\lambda$-stable, if for every set $|A| \leq \lambda$, the space of types $S_n(A)$ has metric density character $\lambda$.
- $T$ is superstable if it is $\lambda$-stable for all $\lambda \geq 2^{|\mathcal{L}|}$.

**Theorem.** *$T$ is superstable if and only if $T$ is stable and for every singleton (or finite tuple) $a$ and set $A$:*

$$\forall \varepsilon > 0 \quad \exists A_0 \subseteq A \text{ finite s.t.} \quad a^{\varepsilon} \underset{A_0}{\perp} A.$$

If we do not take the metric into account, no "truly continuous" theory can be superstable. With these definitions, most known stable examples are superstable.

### Putting some of this together: Lachlan's Theorem

**Theorem (improved version of B.,Usvyatsov [BUb]).** *Assume that: (1) $T$ is superstable; and (2) there are "enough" elements $a$ such that $\forall\varepsilon\exists\delta$ such that if $a^{\delta} \perp b$ then $b^{\varepsilon} \perp a$. Then $T$ has either one or infinitely many separable models.*

*Sketch of proof.* Assume the contrary. Then there is a non-isolated type, and there is therefore one $p$ whose realisations satisfy (2).

Find an independent sequence $\bar{a} = (a_i : i < n)$ in $p$, and an infinite independent sequence $(b_i : i < \omega)$ in $p$, and $\varepsilon$ such that $b_j^{\varepsilon} \not\perp a_{<n}$ or all $j$.

Since $\bar{a}$ consists of independent realisations of $p$, it also satisfies (2), and choose $\delta$ accordingly, so $\bar{a}^{\delta} \not\perp b_j$ for all $j$.

As $(b_j : j < \omega)$ are independent, it follows that $\bar{a}^{\delta} \not\perp_{b_{<j}} b_j$ for all $j$, contradicting superstability. □

### Remarks about the proof

- Assumption (2) is not redundant. For example, the theory of $L_p$ Banach lattices is superstable, but if we name a single parameter it has precisely two models:

  The models of $\mathrm{Th}\langle L_1([0,1]), \chi_{[0,1]}\rangle$ are

  - $\langle L_1([0,1]), \chi_{[0,1]}\rangle$ ($\chi_{[0,1]}$ supports everything).
  - $\langle L_1([0,2]), \chi_{[0,1]}\rangle$ ($\chi_{[0,1]}$ does not support everything).

  Indeed, $\chi_{[0,1]}$ does not satisfy the property in (2) and it cannot even be approximated by elements which do.

- Pillay's proof of Lachlan's Theorem [Pil83] used the fact that if $T$ has finitely many countable models then a prime model $M$ exists over every finite tuple $\bar{a}$ (and $\bar{a}$ isolates, and therefore semi-isolates, every tuple in $M$). This is not true in continuous logic, but we can get semi-isolation directly using the Omitting Types Theorem.

**Further questions**

- Find a necessary and sufficient condition for a closed set $X \subseteq S_n(T)$ to be always realised (in first order: $X^\circ \neq \varnothing$):

  We showed that if $X$ is closed and $X^{0^+,\circ} = \varnothing$, it can be omitted.

  It is fairly easy to show that if $X$ is closed and $X^{0^+,\circ} = X$, then $X$ is always realised (suffices: $X$ closed in the metric).

  What if $\varnothing \subsetneq X^{0^+,\circ} \subsetneq X$? Boils down to: what if $X \neq X^{0^+,\circ}$ but $X = \overline{X^{0^+,\circ}}$?

**Further questions, cntd.**

- Structure perturbation:

  Theorem: if $T$ is a superstable (supersimple) continuous theory, so is the theory $T_P$ of its beautiful (lovely) pairs.

  But: there exists a superstable continuous theory $T$ such that $T_A$ ($T$ + generic automorphism) is not supersimple: for example, the theory of probability algebras [BH, Ben04].

  Maybe this is because we should only look at an automorphism up to the uniform convergence metric (i.e., allow minor perturbations of the automorphism)?

  Related to a theorem of Henson (unpublished...) characterising theories of pure Banach spaces which are separably categorical up to small perturbation of the norm. A more general version has recently been proved [Ben05b].

**Some background references:**

- Positive logic: [Ben05a] (BSL survey).
- Continuous logic:
  - Mostly oriented towards local stability: B., Usvyatsov [BUa].
  - Text from Henson and Berenstein's tutorial: `http://www.math.uiuc.edu/~henson/newton/newton.pdf`
  - An older, too-general-and-yet-not-general-enough approach: [CK66].

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