# INEQUALITIES BETWEEN LITTLEWOOD-RICHARDSON COEFFICIENTS

FRANÇOIS BERGERON, RICCARDO BIAGIOLI, AND MERCEDES H. ROSAS

Abstract. We prove that a conjecture of Fomin, Fulton, Li, and Poon, associated to ordered pairs of partitions, holds for many infinite families of such pairs. We also show that the bounded height case can be reduced to checking that the conjecture holds for a finite number of pairs, for any given height. Moreover, we propose a natural generalization of the conjecture to the case of skew shapes.

Contents



## 1. Introduction

In the course of their study of Horn type inequalities for eigenvalues and singular values of complex matrices, Fomin, Fulton, Li, and Poon [2] come up with a very interesting conjecture concerning the Schur-positivity of special differences of products of Schur functions. More precisely, they consider differences of the form

$$s_{\mu^*} s_{\nu^*} - s_\mu s_\nu,$$

where $\mu^*$ and $\nu^*$ are partitions constructed from an ordered pair of partitions $\mu$ and $\nu$ through a seemingly strange procedure at first glance. In our presentation, their transformation $(\mu, \nu) \mapsto (\mu^*, \nu^*)$ on ordered pairs of partitions, will rather be denoted

$$(\mu, \nu) \longmapsto (\mu, \nu)^* = (\lambda(\mu, \nu), \rho(\mu, \nu)) \tag{1.1}$$

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and will be called the $*$-operation. As we shall see, this change of notation is essential in order to simplify the presentation of the many nice combinatorial properties of this operation. On the other hand, it underlines that both entries, $\lambda$ and $\rho$ of the image $(\mu,\nu)^*$ of $(\mu,\nu)$, actually depend on both $\mu$ and $\nu$.

With this slight change of notation, the original definition of the $*$-operation is as follows. Let $\mu=(\mu_1,\mu_2,\ldots,\mu_n)$ and $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$ two partitions with the same number of parts, allowing zero parts. From these, two new partitions $\lambda(\mu,\nu)=(\lambda_1,\lambda_2,\ldots,\lambda_n)$ and $\rho(\mu,\nu)=(\rho_1,\rho_2,\ldots,\rho_n)$ are constructed as follows

$$\begin{aligned} \lambda_k &:= \mu_k-k+\#\{j \mid 1\le j\le n,\ \nu_j-j\ge \mu_k-k\}; \\ \rho_j &:= \nu_j-j+1+\#\{k \mid 1\le k\le n,\ \mu_k-k>\nu_j-j\}. \end{aligned} \tag{1.2}$$

Although this definition does not make it immediately clear, both $\lambda(\mu,\nu)$ and $\rho(\mu,\nu)$ are truly partitions, and they are such that

$$|\lambda(\mu,\nu)|\,+\,|\rho(\mu,\nu)|=|\mu|\,+\,|\nu|,$$

where as usual $|\mu|$ denotes the sum of the parts of $\mu$.

Recall that the product of two Schur functions can always be expanded as a linear combination

$$s_\mu s_\nu=\sum_\theta c_{\mu\,\nu}^{\theta} s_\theta,$$

of Schur functions indexed by partitions $\theta$ of the integer $n=|\mu|+|\nu|$, since these Schur functions constitute a linear basis of the homogeneous symmetric functions of degree $n$. It is a particularly nice feature of this expansion that the coefficients $c_{\mu\,\nu}^{\theta}$ are always non-negative integers. They are called the *Littlewood-Richardson coefficients*. More generally, we say that a symmetric function is *Schur positive* whenever the coefficients in its expansion, in the Schur function basis, are all non-negative integers. For more details on symmetric function theory see Macdonald's classical book [3], whose notations we will mostly follow. We can then state the following:

**Conjecture 1.1** (Fomin-Fulton-Li-Poon). *For any pair of partitions $(\mu,\nu)$, if*

$$(\mu,\nu)^*=(\lambda,\rho),$$

*then the symmetric function*

$$s_\lambda s_\rho - s_\mu s_\nu \tag{1.3}$$

*is Schur-positive.*

In other words, this says that $c_{\mu\,\nu}^{\theta}\le c_{\lambda\,\rho}^{\theta}$, for all $\theta$ such that $s_\theta$ appears in the expansion of $s_\mu s_\nu$.

For an example of one of the simplest case of the $*$-operation, let $\mu=(a)$ and $\nu=(b)$, with $a>b$, be two one-part partitions. In this case, we get

$$((a),(b))^*=(a-1,b+1),$$

so that Conjecture 1.1 corresponds exactly to an instance of the classical Jacobi-Trudi identity:

$$\begin{aligned} s_{a-1}s_{b+1} - s_a s_b &= \det \begin{pmatrix} s_{a-1} & s_a \\ s_b & s_{b+1} \end{pmatrix} \\ &= s_{a-1,b+1}. \end{aligned}$$

In this paper we give a new recursive combinatorial description of the $*$-operation. This recursive description allows us to prove many instances of Conjecture 1.1 and to show that it reduces to checking a finite number of instances for any fixed $\nu$, if we bound the number of parts of $\mu$. Moreover we show how to naturally generalize the conjecture to pairs of skew partitions.

## 2. Combinatorial properties of the $*$-operation and implications

We first derive some nice combinatorial properties of the transformation $*$. To help in the presentation of these properties, let us introduce some further notation. For any undefined notation we refer to [3]. We often identify a partition with its (Ferrers) diagram. Diagrams are drawn here using the "French" convention of ordering parts in decreasing order from bottom to top.

We write $\mu = \overrightarrow{\alpha}^i$, if the partition $\mu$ is obtained from the partition $\alpha$ by adding one cell in line $i$; and $\mu = \alpha{\uparrow_k}$, if $\mu$ is obtained from $\alpha$ by adding one cell in column $k$. In other words, $\mu = \overrightarrow{\alpha}^i$ means that $\mu_i = \alpha_i$ for all $i \neq \ell$, and $\mu_\ell = \alpha_\ell + 1$. This is illustrated in Figure 1 in term of diagrams.


Figure 1

Observe that,

$$\begin{aligned} \mu = \overrightarrow{\alpha}^i \qquad &\text{iff} \qquad \mu' = \overrightarrow{\alpha'}^{\mu_i} \\ &\text{iff} \qquad \mu = \alpha{\uparrow_{\mu_i}} \\ &\text{iff} \qquad \mu' = \alpha'{\uparrow_i} \end{aligned}$$

We can now state our recursive description of the $*$-operation.

**Proposition 2.1** (Recursive formula)**.** *For any partitions $\alpha$ and $\nu$, let $\mu = \overrightarrow{\alpha}^i$ and $(\lambda, \rho) = (\alpha, \nu)^*$, then we have*

$$(\mu, \nu)^* = \begin{cases} (\lambda, \rho{\uparrow_{\mu_i}}) & \text{if there exists } j \text{ such that } \nu_j - j = \alpha_i - i, \\ \\ (\overrightarrow{\lambda}^i, \rho) & \text{otherwise.} \end{cases} \tag{2.1}$$

*Similarly, when* $\nu = \overrightarrow{\beta}^i$ *and* $(\lambda, \rho) = (\mu, \beta)^*$, *we have*

$$(\mu, \nu)^* = \begin{cases} (\lambda\!\uparrow_{\nu_i}, \rho) & \textit{if there exists } j \textit{ such that } \mu_j - j = \nu_i - i, \\ (\lambda, \overrightarrow{\rho}^i) & \textit{otherwise.} \end{cases} \tag{2.2}$$

We can clearly use Proposition 2.1 to recursively compute $\lambda(\mu, \nu)$ and $\rho(\mu, \nu)$. This is discussed more extensively in Section 5. The actual computation of the $*$-operation can be simplified in view of the following property (see Lemma 4.1). For any pair of partitions $(\mu, \nu)$, we have

$$(\mu, \nu)^* = (\lambda, \rho) \qquad \text{iff} \qquad (\nu', \mu')^* = (\lambda', \rho'), \tag{2.3}$$

where, as usual, $\mu'$ stands from the conjugate of $\mu$. Using the fact that the involution $\omega$ (which is the linear operator that maps $s_\mu$ to $s_{\mu'}$) is multiplicative, it easily follows that

**Proposition 2.2.** *Conjecture 1.1 holds for the pair* $(\mu, \nu)$ *if and only if it holds for the pair* $(\nu', \mu')$.

In practice, there are many ways to describe the $*$-operation recursively, since we can freely choose how to make partitions grow. It is sometimes convenient to start from the pair $(0, \nu)$, with 0 standing for the empty partition, whose image under the $*$-operation has a simple description.

**Lemma 2.3.** *Let* $\nu$ *be any partition. Then*

$$\rho(0, \nu) = (\nu_1, \nu_2 - 1, \ldots, \nu_k - (k-1))$$
$$\lambda'(0, \nu) = (\nu'_1 - 1, \nu'_2 - 2, \ldots, \nu'_k - k),$$

*where* $k = \max\{i \mid \nu_i \geq i\}$.

We will sometimes use respectively $\overline{\nu}$ and $\underline{\nu}$ to denote the partitions $\lambda(0, \nu)$ and $\rho(0, \nu)$. For example if $\nu = 866554421$, then

$$\overline{\nu} = 44432211 \qquad \text{and} \qquad \underline{\nu} = 85421$$

as is illustrated in Figure 2.

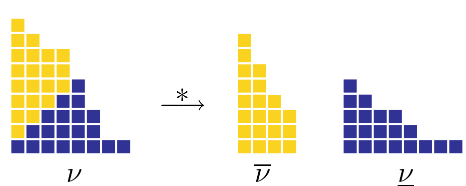


FIGURE 2

In Section 5 we elaborate on the various ways that Proposition 2.1 can be used to compute the $*$-operation. This gives rise to a $*$-operation on pairs of Young tableaux. In Figure 3 we illustrate the effect of the $*$-operation on pairs of the form $((n), \nu)$.

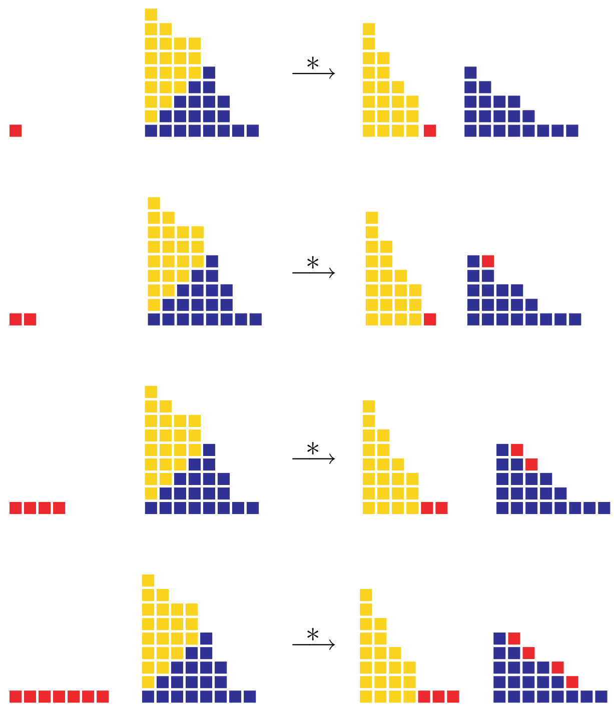

FIGURE 3

Given partitions $\mu$ and $\nu$, define the partition $\mu + \nu$ by

$$(\mu + \nu)_i := \mu_i + \nu_i.$$

and set

$$\mu \cup \nu := (\mu' + \nu')'.$$

For example, if $\mu = 33221$ and $\nu = 531$, then $\mu \cup \nu = 53332211$ and $\mu + \nu = 86321$. For $\mu$ and $\nu$ two partitions of $n$, $\mu$ is said to be *dominated* by $\nu$, written $\mu \preceq \nu$, if for all $k \geq 1$:

$$\mu_1 + \mu_2 + \cdots + \mu_k \leq \nu_1 + \nu_2 + \cdots + \nu_k.$$

Another remarkable property of the $*$-operation is that its image behaves nicely under the dominance order. More precisely:

**Lemma 2.4.** *For any pair of partitions $(\mu, \nu)$, if $(\lambda, \rho) = (\mu, \nu)^*$, then we have*

$$\mu \cup \nu \succeq \lambda \cup \rho, \qquad \text{and equivalently} \tag{2.4}$$

$$\mu + \nu \preceq \lambda + \rho. \tag{2.5}$$

Observe that when $s_\theta$ appears in $s_\mu s_\nu$ with a nonzero coefficient, then

$$\mu \cup \nu \preceq \theta \preceq \mu + \nu.$$

Thus (2.4) and (2.5) imply that

$$\lambda \cup \rho \preceq \theta \preceq \lambda + \rho,$$

which is compatible with Conjecture 1.1.

Lemma 2.4 immediately implies a statement very similar to that of Conjecture 1.1. As is usual (See [3]), $h_\mu$ denotes the *complete homogeneous* symmetric function:

$$h_\mu := h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k},$$

with $h_a := s_a$.

**Proposition 2.5.** *For any pair of partitions $(\mu,\nu)$, if $(\lambda,\rho) = (\mu,\nu)^*$, then*

$$h_\lambda h_\rho - h_\mu h_\nu \tag{2.6}$$

*is Schur-positive.*

Recalling that $h_\mu h_\nu = h_{\mu\cup\nu}$, this follows from the fact that a difference of two homogeneous symmetric functions $h_\alpha - h_\beta$ is Schur-positive, if and only if $\alpha \preceq \beta$ (see [5, Chapter 2]). A clear link between this proposition and Conjecture 1.1 is established through the classical identity:

$$h_\alpha = s_\alpha + \sum_{\beta \succeq \alpha} K_{\beta\alpha} s_\beta, \tag{2.7}$$

where as usual $K_{\beta\alpha}$, the *Kostka* numbers, count the number of semistandard tableaux of shape $\beta$ and type $\alpha$. In fact, (2.6) contains (1.3) as "top component" via (2.7).

It follows directly from Proposition 2.1 that the $*$-operation is compatible with "inclusion" of partitions. Here, we say that $\alpha$ is *included* in $\mu$, if the diagram of $\alpha$ is included in the diagram of $\mu$. We will simply write

$$(\alpha,\beta) \subseteq (\mu,\nu), \qquad \text{whenever} \qquad \alpha \subseteq \mu \quad \text{and} \quad \beta \subseteq \nu,$$

and we have:

**Lemma 2.6.** *For $\alpha$, $\beta$, $\mu$ and $\nu$ partitions such that $(\alpha,\beta) \subseteq (\mu,\nu)$, the following inclusions hold*

$$\lambda(\alpha,\beta) \subseteq \lambda(\mu,\nu), \qquad \text{and} \qquad \rho(\alpha,\beta) \subseteq \rho(\mu,\nu).$$

An immediate, but interesting, consequence of this lemma is the following observation.

**Observation 2.7.** *Let $(\alpha,\beta)$ and $(\gamma,\delta)$ be two fixed points of the $*$-operation such that $(\alpha,\beta) \subseteq (\gamma,\delta)$. Writing simply $\lambda$ for $\lambda(\mu,\nu)$ and $\rho$ for $\rho(\mu,\nu)$, we see (using Lemma 2.6) that*

$$(\alpha,\beta) \subseteq (\mu,\nu) \subseteq (\gamma,\delta),$$

*implies*

$$(\alpha,\beta) \subseteq (\lambda,\rho) \subseteq (\gamma,\delta).$$

As is underlined in [2], a pair of partitions $(\alpha,\beta)$ is a fixed point of the $*$-operation if and only if

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \cdots \geq \beta_n \geq \alpha_n. \tag{2.8}$$

Let us underline here that, for any $(\mu,\nu)$, it is easy to characterize the "largest" (resp. "smallest") fixed point contained in (resp. containing) the pair $(\mu,\nu)$. We will see below how this observation can be used to link properties of $\lambda$ and $\rho$ to properties of $\mu$ and $\nu$.

Recall that a *hook* is a shape of the form $(a, 1^b)$ with $a, b \geq 0$, a *$n$-line partition* is a shape contained in a rectangle $(a^n)$ with $a, n \geq 0$, a *horizontal strip* is a skew shape $\mu/\alpha$ with no two squares in the same column, and that a *ribbon* is a connected skew shape with no $2 \times 2$ squares (see [6, Chapter 7], for more details). If we drop the condition of being connected in this last definition, we say that we have a *weak ribbon*.

Another striking consequence of Lemma 2.6 is that it allows a natural extension of the $*$-operation to skew partitions. Denoting by $(\mu, \nu)/(\alpha, \beta)$ the pair of skew shapes $(\mu/\alpha, \nu/\beta)$, we can simply define

$$(\mu/\alpha, \nu/\beta)^* := (\mu, \nu)^*/(\alpha, \beta)^*. \tag{2.9}$$

In other words, we have

$$\lambda(\mu/\alpha, \nu/\beta) := \lambda(\mu, \nu)/\lambda(\alpha, \beta), \tag{2.10}$$

and

$$\rho(\mu/\alpha, \nu/\beta) := \rho(\mu, \nu)/\rho(\alpha, \beta). \tag{2.11}$$

The $*$-operation, or its extension as above, preserves (among others) the following families of pairs of (skew) shapes.

**Proposition 2.8.** *The $*$-operation preserves the families of*

- **(1)** *pairs of hooks;*
- **(2)** *pairs of $n$-line partitions;*
- **(3)** *pairs of horizontal strips;*
- **(4)** *pairs of weak ribbons.*

Note that **(1)** and **(2)** follow directly from Observation 2.7, and that the statements **(3)** and **(4)** are made possible in view of our extension of the $*$-operation.

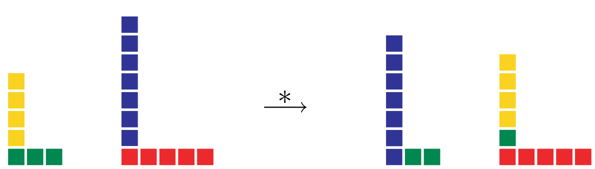


FIGURE 4. The effect of the $*$-operation on hooks.

Results outlined what follows, and extensive computer experimentation suggests that we have the following extension of Conjecture 1.1.

**Conjecture 2.9.** *For any skew partitions $\mu/\alpha$ and $\nu/\beta$, if*

$$(\lambda, \rho) = (\mu/\alpha, \nu/\beta)^*,$$

*then the symmetric function*

$$s_\lambda s_\rho - s_{\mu/\alpha} s_{\nu/\beta} \tag{2.12}$$

*is Schur-positive.*

This has yet to be understood in geometrical terms. One should point out that there are many skew shapes giving the same expression for the symmetric function $s_{\mu/\alpha}s_{\nu/\beta}$. The result of the $*$-operation is dependent on the particular choice of the skew-shape, so that there are many identities encoded in (2.12). On the other hand, it is clear that Proposition 2.2 extends to skew partitions.

Many others combinatorial properties of the $*$-operations can be obtained with approaches similar to those above. As an example, we state the following without proof. Let $\tau$ and $\nu$ be two fixed partitions, and consider all possible $\mu$'s such that $\rho(\mu,\nu)=\tau$. We claim that there is a minimal such $\mu$, if any, and we denote it $\theta(\tau,\nu)$. More precisely, we could easily show that

$$\theta(\tau,\nu)\subseteq\mu.$$

Furthermore, $\theta=\theta(\tau,\nu)$ is exactly the partition

$$\theta=\tau_1^{b_1}\tau_2^{b_2-b_1}\tau_3^{b_3-b_2}\cdots,$$

with $b_j=\tau_j-\nu_j+j-1$.

## 3. Main results

In this section we state our results concerning the validity of Conjecture 1.1 for certain families of pairs, as well as its reduction to a finite number of tests for other families. We will show (in Section 7) the following.

**Theorem 3.1.** *Conjecture 1.1 (or 2.9) holds*

- **(1)** *For any pair $(\mu,\nu)$ of hook shapes.*
- **(2)** *For skew pairs of the form $(\mu/\alpha,\nu/\beta)$, where $\mu$, $\nu$, $\alpha$, $\beta$ are hooks, with $\alpha=\beta$.*
- **(3)** *For skew pairs of the form $(0,\nu/\beta)$, with $\nu/\beta$ a weak ribbon.*

On another note, a careful study of the recursive construction of $\lambda(\mu,\nu)$ and $\rho(\mu,\nu)$ shows that, in a sense, Conjecture 1.1 follows, under some conditions, from a finite number of cases when $\nu$ is fixed and $\mu$ becomes large.

More precisely, we obtain the result below. As usual, the number of nonzero parts of $\mu$ is denoted by $\ell(\mu)$ and called the *height* of $\mu$.

**Theorem 3.2.** *For any positive integer $p$, let $\nu$ be a fixed partition with at most $p$ parts, i.e. $\ell(\nu)\leq p$. Then, the validity of Conjecture 1.1 for the infinite set of all pairs $(\mu,\nu)$, with $\ell(\mu)\leq p$, reduces to checking the validity of the conjecture for the finite set of pairs $(\alpha,\nu)$, with $\alpha$ having at most $p$ parts, and largest part bounded as follows*

$$\alpha_1\leq p\,(\nu_1+p). \tag{3.1}$$

Theorem 3.2 can also be generalized in a straightforward manner to the set of skew shapes pairs $(\mu/\alpha,\nu/\beta)$ of bounded height, with $\nu$ and $\alpha$ fixed.

## 4. Proofs of the combinatorial properties

In what follows, unless it is specifically mentioned, all partitions will be considered to have $n$ (possibly zero) parts. We first observe that the set $A_k(\mu,\nu) := \{j \mid \nu_j - j \geq \mu_k - k\}$, appearing in (1.2), has to be of the form $A_k(\mu,\nu) = \{1, 2, \ldots, a_k\}$ for some $a_k = a_k(\mu,\nu)$, since $\nu_1 - 1 > \nu_2 - 2 > \ldots > \nu_{a_k} - a_k \geq \mu_k - k$, and thus

$$a_k(\mu,\nu) := \#A_k(\mu,\nu). \tag{4.1}$$

In other words,

$$\mu_k - k \leq \nu_m - m \qquad \text{iff} \qquad 1 \leq m \leq a_k. \tag{4.2}$$

Thus definition (1.2) of $\lambda_k(\mu,\nu)$ can be reformulated as

$$\lambda_k := \mu_k - k + a_k(\mu,\nu). \tag{4.3}$$

In the same spirit, we consider the set $B_j(\mu,\nu) := \{k \mid \mu_k - k > \nu_j - j\}$, which also has to be of the form $\{1, 2, \ldots, b_j\}$, with

$$b_j(\mu,\nu) = \#B_j(\mu,\nu), \tag{4.4}$$

In other words,

$$\nu_j - j < \mu_m - m \qquad \text{iff} \qquad 1 \leq m \leq b_j \tag{4.5}$$

and

$$\rho_j := \nu_j - j + 1 + b_j(\mu,\nu). \tag{4.6}$$

**Proof of Proposition 2.1.** To prove our recursive formula for the computation of the $*$-operation, we first analyze the case $\mu = \overrightarrow{\alpha}^i$. As we have already mentioned, this means that $\mu_k = \alpha_k$ for all $k \neq i$, and $\mu_i = \alpha_i + 1$. Let $(\lambda, \rho) = (\alpha, \nu)^*$. Now, suppose that there exists a $j \in \{1, \ldots, n\}$ such that

$$\nu_j - j = \alpha_i - i.$$

This implies that $A_k(\mu,\nu) = A_k(\alpha,\nu)$ for all $k \neq i$ and that

$$A_i(\alpha,\nu) = \{1, 2, \ldots, j\}, \text{ and } A_i(\mu,\nu) = \{1, 2, \ldots, j-1\}.$$

It follows that $\lambda_k(\mu,\nu) = \lambda_k$ for all $k \neq i$, and that

$$\begin{aligned} \lambda_i(\mu,\nu) &= \mu_i - i + a_i(\mu,\nu) \\ &= \alpha_i + 1 - i + a_i(\alpha,\nu) - 1 = \lambda_i \end{aligned}$$

Hence $\lambda(\mu,\nu) = \lambda$.

On the other hand, we clearly have $B_k(\alpha,\nu) = B_k(\mu,\nu)$ for all $k \neq j$, and

$$B_j(\alpha,\nu) = \{1, 2, \ldots, i-1\} \text{ and } B_j(\mu,\nu) = \{1, 2, \ldots, i\}.$$

Hence $\rho_k(\mu,\nu) = \rho_k$ for all $k \neq i$ and,

$$\begin{aligned} \rho_j(\mu,\nu) &= \nu_j - j + 1 + b_j(\mu,\nu) \\ &= \alpha_i - i + 1 + i = \mu_i. \end{aligned}$$

Since $\rho_j = \mu_i - 1$ we conclude that $\rho(\mu,\nu) = \overrightarrow{\rho}^j$, and this settles the first case of (2.1).

If no $j \in \{1,\dots,n\}$ is such that $\nu_j - j = \alpha_i - i$, we have the equalities

$$A_k(\mu,\nu) = A_k(\alpha,\nu) \qquad \text{and} \qquad B_k(\mu,\nu) = B_k(\alpha,\nu)$$

for all $k \in \{1,\dots,n\}$. It follows $\lambda_k(\mu,\nu) = \lambda_k$ for all $k \neq i$ and

$$\lambda_i(\mu,\nu) = \alpha_i + 1 - i + a_i(\alpha,\nu) = \lambda_i + 1,$$

so $\lambda(\mu,\nu) = \overrightarrow{\lambda}^i$. It easily follows that in this case $\rho(\mu,\nu) = \rho$ and this concludes the second case. The part (2.2) of the proposition is shown in a similar manner. □

We use our recursive method to show the next Lemma 2.3.

**Proof of Lemma 2.3.** We proceed by induction on $|\nu|$. If $\nu = 0$ there is nothing to prove. So let $\nu = \overrightarrow{\beta}^i$ be different from 0, and set $(\lambda,\rho) = (0,\beta)^*$, as is now usual. If $i = 1$ or $\nu_i = \beta_i + 1 \geq i$, then we must have $\nu_i - i \geq 0$. This corresponds to the second case of (2.2), since all of the values $\mu_j - j$ are negative. For a partition $\theta$, set

$$k(\theta) := \max\{i \mid \theta_i \geq i\}$$

If $k(\nu) = k(\beta)$, then applying (2.2) and induction we get, $\lambda'(0,\nu) = \lambda' = (\nu'_1 - 1,\dots,\nu'_k - k)$, and $\rho(0,\nu) = \overrightarrow{\rho}^i = (\nu_1, \nu_2 - 1, \dots, \beta_i + 1 - (i-1), \dots, \nu_k - (k-1))$. Otherwise, if $k(\nu) = k(\beta) + 1$, we must have $i = k+1$ and $\nu_i = i = \nu'_i$, and the result again follows by induction and (2.2). On the other hand $\nu_i = \beta_i + 1 < i$, corresponds to the first case of (2.2) with $j = -(\bar{\nu}_i - 1)$. In that case the result follows by an induction similar to that above. □

As announced in (2.3), we have the following lemma.

**Lemma 4.1.** *For any pair of partitions $(\mu,\nu)$, we have*

$$(\mu,\nu)^* = (\lambda,\rho) \qquad \text{iff} \qquad (\nu',\mu')^* = (\lambda',\rho').$$

**Proof.** We proceed by induction on $|\mu|+|\nu|$. The lemma obviously holds when $\mu = \nu = 0$, so suppose that (2.3) holds for all $(\alpha,\nu)$ with $\alpha \subseteq \mu$. That is,

$$\lambda(\alpha,\nu) = \lambda'(\nu',\alpha') \qquad \text{and} \qquad \rho(\alpha,\nu) = \rho'(\nu',\alpha').$$

In view of the recursive description of the $*$-operation, it is easy to verify that we need only show that the "if" part of (2.1) applies to the pair $(\mu,\nu)$ if and only if the "otherwise" part of (2.2) applies to the pair $(\nu',\mu')$. Let us suppose that $\mu = \overrightarrow{\alpha}^i$. We want to show that there is a $j$ such that

$$\nu_j - j = \alpha_i - i$$

if and only if there is no $k$ such that $\nu'_k - k = \alpha'_{\mu_i} - \mu_i + 1$. (See Figure 5.) By (2.1) we have

$$\lambda(\alpha,\nu) = \lambda(\mu,\nu) = \lambda'(\nu',\alpha') = \lambda'(\nu',\mu'),$$

and

$$\rho(\mu,\nu) = \overrightarrow{\rho(\alpha,\nu)}^j.$$

We thus need to check that

$$\rho'(\nu',\mu') = \rho'(\nu',\alpha')\!\uparrow_j .$$

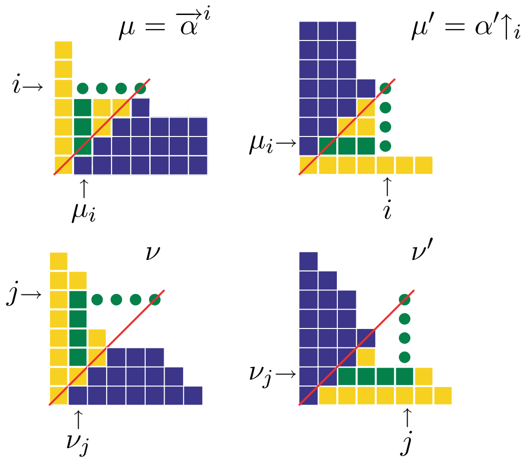


FIGURE 5

Considering the pair $(\nu', \mu')$, we have $\mu' = \overrightarrow{\alpha'}^{\mu_i}$. Thus, the new cell is added to $\alpha'$ in row $\mu_i$, which is of length $i-1$, i.e.: $\alpha'_{\mu_i} = i-1$. We claim that there does not exist a row $k$ of $\nu'$ such that

$$\nu'_k - k = (i-1) - (\alpha_i + 1) + 1 = i - \alpha_i - 1.$$

In fact, if $k = \nu_j$ the difference $\nu'_k - k$ is strictly bigger than $i - \alpha_i - 1$, since

$$\nu'_k - k \geq j - \nu_j = i - \alpha_i.$$

On the other hand, if $k = \nu_j + 1$, the difference $\nu'_k - k$ has to be strictly smaller than $i - \alpha_i - 1$, since $(\nu_{j+1}, j) \notin \nu$, and hence $(j, \nu_{j+1}) \notin \nu'$. Then,

$$\nu'_{\nu_j+1} - (\nu_j + 1) < j - (\nu_j + 1) = i - \alpha_i - 1.$$

□

In preparation for the proof of Lemma 2.4, let us prove the following.

**Lemma 4.2.** *The following two statements are equivalent*

**(1)** *For all $(\mu, \nu)$ pair of partitions $\mu \cup \nu \succeq \lambda \cup \rho$.*
**(2)** *For all $(\mu, \nu)$ pair of partitions $\mu + \nu \preceq \lambda + \rho$.*

**Proof.** Assuming **(1)**, we have

$$\begin{aligned}
\mu + \nu &= (\mu' \cup \nu')' \\
&\preceq (\lambda(\nu', \mu') \cup \rho(\nu', \mu'))' \\
&= \lambda'(\nu', \mu') + \rho'(\nu', \mu') \\
&= \lambda(\mu, \nu) + \rho(\mu, \nu).
\end{aligned}$$

A similar computation shows the reverse statement, since conjugation is an anti-automorphism of the dominance order. □

**Proof of Lemma 2.4.** (P. McNamara) Let $(\lambda,\rho)=(\mu,\nu)^*$, for $\mu$ and $\nu$ partitions with $n$ (possibly zero) parts. From the above observation, it is sufficient to show that

$$\lambda+\rho\succeq\mu+\nu,$$

and hence that, for all $i$,

$$\sum_{j=1}^{i}(\lambda_j+\rho_j)\geq\sum_{j=1}^{i}(\mu_j+\nu_j). \tag{4.7}$$

Definitions (4.3) and (4.6) give $\lambda_j=\mu_j-j+a_j$ and $\rho_j=\nu_j-j+1+b_j$, so that (4.7) becomes

$$\sum_{j=1}^{i}(\mu_j+\nu_j+a_j+b_j-(2\,j-1))\geq\sum_{j=1}^{i}(\mu_j+\nu_j), \tag{4.8}$$

which is equivalent to

$$\sum_{j=1}^{i}(a_j+b_j)\geq i^2. \tag{4.9}$$

But the definitions of $a_j$ and $b_j$, can clearly be reformulated as

$$a_j=\#\{(k,j)\mid \nu_k-k\geq\mu_j-j\ \}\qquad\text{and}\qquad b_k=\#\{(k,j)\mid \nu_k-k<\mu_j-j\ \},$$

hence the inequality. □

**Proof of Proposition 2.8.** Among the families of pairs stated to be preserved by the $*$-operation, we have already shown cases **(1)** and **(2)**. The proofs of the other two claims are as follows.

**(3)** Recall that $\mu/\alpha$ and $\nu/\beta$ are horizontal strips if and only if, for all $1\leq k<n$

$$\mu_{k+1}\leq\alpha_k\qquad\text{and}\qquad\nu_{k+1}\leq\beta_k.$$

To show that $(\mu/\alpha,\nu/\beta)^*$ is also an horizontal strip, we need to prove that $\lambda_{k+1}(\mu,\nu)\leq\lambda_k(\alpha,\beta)$ and $\rho_{k+1}(\mu,\nu)\leq\rho_k(\alpha,\beta)$. Once again, by definition, we have

$$\rho_{k+1}(\mu,\nu)=\nu_{k+1}-k+b_{k+1}(\mu,\nu)\quad\text{and}\quad\rho_k(\alpha,\beta)=\beta_k-k+1+b_k(\alpha,\beta).$$

If $b_k(\alpha,\beta)\geq b_{k+1}(\mu,\nu)$ there is nothing to prove. Otherwise, the inequality we want to prove is clearly equivalent to

$$b_{k+1}(\mu,\nu)-b_k(\alpha,\beta)\leq\beta_k-\nu_{k+1}+1. \tag{4.10}$$

Using (4.5), when $b_k(\alpha,\beta)<m\leq b_{k+1}(\mu,\nu)$, we must have $\alpha_m-m\leq\beta_k-k$, and $\nu_{k+1}-(k+1)<\mu_m-m$. Hence for all $m\neq b_{k+1}(\mu,\nu)$

$$\nu_{k+1}-(k+1)<\mu_{m+1}-(m+1)<\alpha_m-m\leq\beta_k-k,$$

since by hypothesis $\mu_{m+1}\leq\alpha_m$. This shows that there are at least $b_{k+1}(\mu,\nu)-b_k(\alpha,\beta)-1$ distinct integers separating $\nu_{k+1}-(k+1)$ from $\beta_k-k$, thus (4.10) follows. Similarly we get the other inequality.

**(4)** The statement that $\mu/\alpha$ and $\nu/\beta$ are two weak ribbons is equivalent to saying that

$$\mu_{k+1} \leq \alpha_k + 1 \qquad \text{and} \qquad \nu_{k+1} \leq \beta_k + 1, \tag{4.11}$$

for all $1 \leq k < n$. We have to show that $\lambda_{k+1}(\mu,\nu) \leq \lambda_k(\alpha,\beta) + 1$. By definition, we have

$$\lambda_{k+1}(\mu,\nu) = \mu_{k+1} - (k+1) + a_{k+1}(\mu,\nu) \quad \text{and} \quad \lambda_k(\alpha,\beta) = \alpha_k - k + a_k(\alpha,\beta).$$

If $a_k(\alpha,\beta) \geq a_{k+1}(\mu,\nu)$ there is nothing to prove. Otherwise, the inequality we want to prove is equivalent to

$$a_{k+1}(\mu,\nu) - a_k(\alpha,\beta) \leq \alpha_k - \mu_{k+1} + 2. \tag{4.12}$$

Using (4.2), when $a_k(\alpha,\beta) < m \leq a_{k+1}(\mu,\nu)$, we must have $\beta_m - m < \alpha_k - k$, and $\mu_{k+1} - (k+1) \leq \nu_m - m$. Hence for all $m \neq a_{k+1}(\mu,\nu)$

$$\mu_{k+1} - (k+1) \leq \nu_{m+1} - (m+1) \leq \beta_m - m - 1 + 1 < \beta_m - m + 1 < \alpha_k - k + 1,$$

since by hypothesis $\nu_{m+1} \leq \beta_m + 1$. This shows that there are at least $a_{k+1}(\mu,\nu) - a_k(\alpha,\beta) - 1$ distinct integers separating $\mu_{k+1} - (k+1)$ from $\alpha_k - k + 1$, thus (4.12) follows. Similarly we get the other inequality. This last case concludes the proof. □

## 5. Extension of the $*$-operation to tableaux

Since we are using the french notation for partitions, standard tableaux have increasing entries along rows from left to right, and increasing entries along columns from bottom to top. As usual, a semistandard tableau is one in which we relax the requirement along rows to weakly increasing. The *reading word* of a tableau is obtained by reading the entries of the tableau starting with the top row, from left to right, and going down the rows. For instance, the reading word of

| | | | |
|---|---|---|---|
| 5 | 5 | | |
| 2 | 3 | 4 | |
| 1 | 2 | 3 | 3 |

is 552341233. It is well known that a semistandard tableau corresponds to a chain in the Young lattice, $0 \subseteq \mu_1 \subseteq \mu_2 \subseteq \ldots \subseteq \mu_k = \mu$, such that $\mu_{i+1}/\mu_i$ is a horizontal strip. The chain associated to the tableau above is

$$0 \subseteq 1 \subseteq 21 \subseteq 42 \subseteq 43 \subseteq 432.$$

Using this correspondence, *standard tableaux* correspond to maximal chains. The shape of a tableau is the final partition in the corresponding chain. All of these notions extend to skew shapes. In particular, a semistandard tableau of skew shape $\lambda/\mu$ is a chain starting at shape $\mu$ and ending at shape $\lambda$. For this to be possible, we clearly need $\mu \subseteq \lambda$. We sometimes say that a semistandard tableau, of shape $\lambda/\mu$, is a *filling* of $\lambda/\mu$. The *natural* filling of a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, is the semistandard tableau corresponding to the chain

$$0 \subseteq (\mu_1) \subseteq (\mu_1,\mu_2) \subseteq (\mu_1,\mu_2,\mu_3) \subseteq \ldots \subseteq (\mu_1,\mu_2,\ldots,\mu_k)$$

Thus, each cell is *filled* by the number of the row in which it lies in. The *type* of a semistandard (possibly skew shaped) tableau $t$ is the sequence $(m_1, m_2, \ldots)$ of multiplicities of its entries. This is to say that $m_i = m_i(t)$ is the number of entries that are equal to $i$. When

$m_1 \geq m_2 \geq \ldots$, this type can be identified with a partition. The natural filling of $\mu$ is the only semistandard tableau of shape $\mu$ that also has type $\mu$.

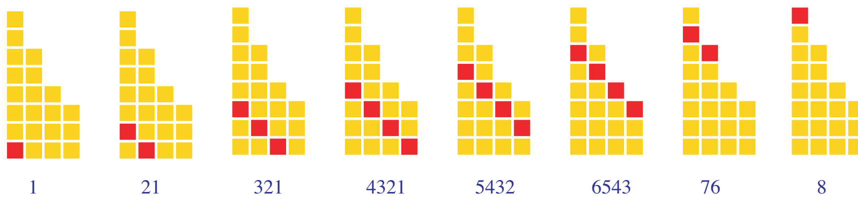


FIGURE 6. $\Delta(\mu)$: the antidiagonal reading of the natural filling of $\mu$.

The *antidiagonal reading*, $\Delta(t)$, of a tableau $t$, of shape $\mu$, is obtained by recording the entries of $t$ following the diagonals $x+y=k$ in the partition $\mu$, from left to right, and from top to bottom, for $k=1,2,\ldots,\ell(\mu)$. We simply denote $\Delta(\mu)$ the antidiagonal reading of the natural filling of $\mu$. For $\mu = 44432211$, we have

$$\Delta(\mu) = 1\,21\,321\,4321\,5432\,6543\,76\,8$$

as is illustrated in Figure 6.

To describe more consequences of the properties of $*$, we consider the *double Young lattice*, $\mathcal{D}$, which is just the direct product of two copies of the usual Young lattice. The double Young lattice already plays an explicit role in [2], see also [1].

There is a natural grading for $\mathcal{D}$ given by $(\mu,\nu) \mapsto |\mu|+|\nu|$. A *standard (tableau) pair* of *shape* $(\mu,\nu)$ is a maximal chain in this graded poset that starts at $(0,0)$ and ends at $(\mu,\nu)$. For example, we have

$$(0,0) \subseteq (0,1) \subseteq (0,2) \subseteq (1,2) \subseteq (1\,1,2) \subseteq (2\,1,2) \subseteq (2\,1,3). \tag{5.1}$$

As in the usual case, such a chain can be identified with a pair $(t,r)$ of standard tableaux, of respective shapes $\mu$ and $\nu$, with non-repeated entries from the the set $\{1,2,\ldots,n\}$, $n=|\mu|+|\nu|$. The number $f_{(\mu,\nu)}$ of standard pairs of shape $(\mu,\nu)$ is thus

$$f_{(\mu,\nu)} = \binom{|\mu|+|\nu|}{|\mu|} f_\mu\, f_\nu \tag{5.2}$$

where $f_\mu$ and $f_\nu$ are both given by the usual hook formula. In terms of tableaux, the standard pair (5.1) corresponds to:

$$\left( \begin{array}{|c|c} \hline 4 & \\ \hline 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \; , \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline \end{array} \right).$$

The double Young lattice occurs naturally in the study of representations of the hyperoctahedral groups. This suggests that there might be a link between that subject and the study of properties of the transformation $*$.

A *semistandard pair* is a chain

$$(0,0) = \pi_0 \subseteq \pi_1 \subseteq \cdots \subseteq \pi_k = (\mu,\nu)$$

in $\mathcal{D}$, such that $\pi_{j+1}/\pi_j$ is an horizontal strip pair for each $1 \leq j \leq k-1$. For example, the pair of semistandard tableaux

$$\left(\begin{array}{lll} 3 & & \\ 2 & 3 & 3 \end{array}\ ,\ \begin{array}{lll} 2 & 3 & \\ 1 & 1 & 3 \end{array}\right),$$

corresponds to the path

$$(0,0) \subseteq (0,2) \subseteq (1,21) \subseteq (31,32).$$

It follows from Proposition 2.8 that

**Lemma 5.1.** *The function $* : \mathcal{D} \longrightarrow \mathcal{D}$ is an increasing transformation that preserves both standard and semistandard pairs.*

Thus the $*$-operation extends to semistandard (and standard) pairs. For example,

$$\left(\begin{array}{lllll} 26 & & & & \\ 22 & 23 & 24 & & \\ 16 & 17 & 18 & 19 & 20 \\ 9 & 10 & 11 & 12 & 13 \end{array}\ ,\ \begin{array}{llllllll} 25 & & & & & & & \\ 21 & & & & & & & \\ 14 & 15 & & & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}\right)^{*} = \left(\begin{array}{lllll} 26 & & & & \\ 22 & 24 & & & \\ 16 & 17 & 19 & 20 & \\ 9 & 10 & 11 & 12 & 13 \end{array}\ ,\ \begin{array}{llllllll} 25 & & & & & & & \\ 21 & 23 & & & & & & \\ 14 & 15 & 18 & & & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}\right)$$

We emphasize that the resulting filling of $(\lambda, \rho)$ heavily depends on the particular filling of $(\mu, \nu)$. Fixed points, for standard pairs, are easily characterized as follows.

**Lemma 5.2.** *A standard pair $(t, r)$, of shape $(\mu, \nu)$, is fixed point of the $*$-operation, if and only if $(\mu, \nu)$ is fixed, and the tableau, obtained by alternating rows of $r$ and rows of $t$, is standard.*

Recall that, if the pair $(\mu, \nu)$ is a fixed point, then (2.8) implies that the alternating lengths of the rows are in decreasing order.

## 6. Background on Littlewood-Richardson coefficients

In order to prove that Conjecture 1.1, or our extension of it, holds for some given pairs, we clearly need one of the many classical descriptions of Littlewood-Richardson coefficients. From a broader perspective, let us briefly recall some classical facts about these coefficients (see [3] or [6] for more details). For $\mu$ and $\nu$ two partitions, and $\theta$ such that $|\theta| = |\mu| + |\nu|$, the coefficient $c^{\theta}_{\mu\nu}$ of $s_\theta$ in $s_\mu s_\nu$ is given as

$$c^{\theta}_{\mu\nu} = \langle s_\mu s_\nu, s_\theta \rangle \tag{6.1}$$
$$= \langle s_{\theta/\mu}, s_\nu \rangle, \tag{6.2}$$

where $\langle -, - \rangle$ denotes the usual scalar product on symmetric function, for which Schur functions are orthonormal.

The following is the explicit formulation of the Littlewood-Richardson rule that we are going to use to compute the $c^{\theta}_{\mu\,\nu}$'s. In order to state it, let us recall some terminology. A *lattice permutation* is a sequence of positive integers $a_1 a_2 \cdots a_n$ such that in any initial factor $a_1 a_2 \cdots a_j$ the number of $i$'s is at least as great as the number of $i+1$'s, for all $i$. The type of a lattice permutation is (naturally) the sequence of multiplicities of the integers

$1, 2, \ldots$ that appear in it. Note that $\Delta(\mu)$ (the antidiagonal reading of the natural filling of $\mu$) is always a lattice permutation of type $\mu$. The *reverse reading word* of a tableau, is the reading word of a tableau, read backwards. For a proof of the following assertion see [6].

> **Littlewood-Richardson Rule.** *The Littlewood-Richardson coefficient $c^{\theta}_{\mu\,\nu}$ is equal to the number of semistandard tableaux of shape $\theta/\nu$ and type $\mu$ whose reverse reading word is a lattice permutation.*

When a semistandard tableaux of shape $\theta/\nu$ has a lattice permutation as its reverse reading word, we say that it is a LR-*filling* of shape $\theta/\nu$.

For $\theta = 4421$, $\nu = 21$ and $\mu = 431$, we have $c^{\theta}_{\mu\,\nu} = 2$ since there are exactly two LR-fillings of $\theta/\nu$. These are described in Figure 7. The two corresponding reverse reading words 11221312 and 11221213. They are clearly lattice permutations of type $\mu$.

Figure 7. The two LR-fillings of 4421/21 of type 431.

## 7. Proof of special instances

In this section we show that Conjecture 1.1 holds for pairs of hooks, pairs of two-row (or two-columns) shapes, and in a special case corresponding to our generalization of the conjecture to skew partitions. We first prove that Conjecture 1.1 holds when one of the partitions is empty.

**Lemma 7.1.** *For any partition $\nu$, setting $\overline{\nu} := \lambda(0,\nu)$ and $\underline{\nu} := \rho(0,\nu)$, $\Delta(\overline{\nu})$ is the reverse reading word of a LR-filling of $\nu/\underline{\nu}$ of type $\overline{\nu}$.*

**Proof.** We show that $\Delta(\overline{\nu})$ encodes a LR-filling of $\nu/\underline{\nu}$ of type $\overline{\nu}$. To this end, we proceed as follows. We "slide" the natural filling of $\overline{\nu}$ up the columns of $\nu$. This gives a partial filling of $\nu$ with empty cells for the portion of $\nu$ that corresponds to $\underline{\nu}$. We will suppose that these empty cells are filled with zeros. We then sort each row in increasing order to get a filling of the skew shape $\nu/\overline{\nu}$. By construction, we obtain a filling of $\nu/\underline{\nu}$ whose reverse reading word is the lattice permutation $\Delta(\overline{\nu})$. An example is given in Figure 8.

To show the lemma, we need only show that the resulting tableau is semistandard. We already have strict increase along rows, so we need only check that this is also true along columns. By construction, the right-most entry in the $(k+1)^{\text{th}}$-row of the final filling of $\nu/\underline{\nu}$ is $k$. Since the integers in a row are consecutive by construction, the difference between two entries in the same column of $\nu/\underline{\nu}$, one in the $i^{\text{th}}$ row and the other in the $(i+1)^{\text{th}}$ row, has to be equal to $\nu_i - \nu_{i+1} + 1$, which is larger than zero. □

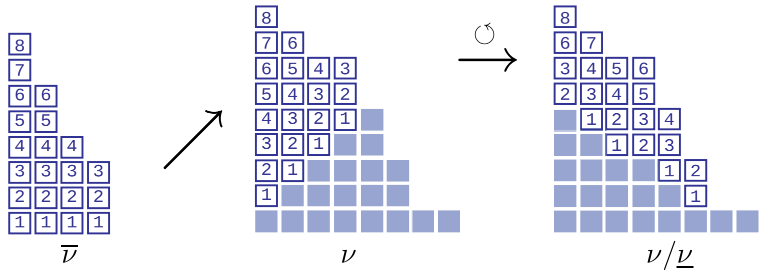

FIGURE 8. The LR-filling of $\nu/\underline{\nu}$ with reverse reading word $\Delta(\overline{\nu})$.

It immediately follows that

**Corollary 7.2.** *For any partition $\nu$ the difference*

$$s_{\overline{\nu}}s_{\underline{\nu}} - s_\nu$$

*is Schur positive. Thus, recalling that $s_0 = 1$, Conjecture 1.1 holds for pairs of the form $(0, \nu)$.*

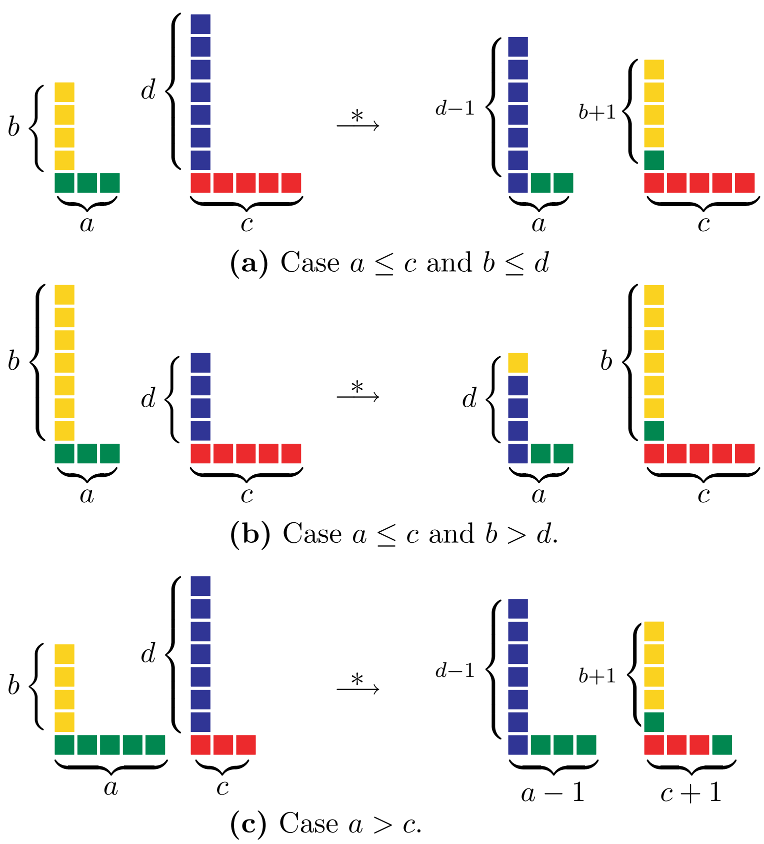

FIGURE 9. Three cases of $*$-operation on pairs of hooks.

We are now ready to prove Theorem 3.1 about other special instances of Conjecture 1.1, and of our extension Conjecture 2.9.

**Proof of part (1) of Theorem 3.1.** Let $\mu = (a, 1^b)$ and $\nu = (c, 1^d)$ be two hook shapes, and $(\lambda, \rho)$ be equal to $(\mu, \nu)^*$. There are essentially 3 different cases for the effect of the $*$-operation on such a pair of hooks, depending on the relative values of $a$, $b$, $c$ and $d$. These are illustrated in Figure 9. In each case, for $s_\theta$ that appears both in the expansion of $s_\mu s_\nu$ and $s_\lambda s_\rho$, our objective is to construct an injection between LR-fillings of $\theta/\nu$ of type $\mu$ and LR-fillings of $\theta/\rho$ of type $\lambda$. Under the above hypothesis, it is easy to check that $s_\theta$ can appear in the product $s_\mu s_\nu$, with nonzero coefficient, only if $\theta$ has at most two parts larger than 2. Thus, in general, $\theta$ has the form $\theta = (r, s, 2^t, 1^u)$. Moreover, it is also clear that $r \geq \max(a, c)$, and $t+u+2 \geq \max(b+1, d+1)$, since otherwise it would be impossible to get a nonzero result using the Littlewood-Richardson Rule.

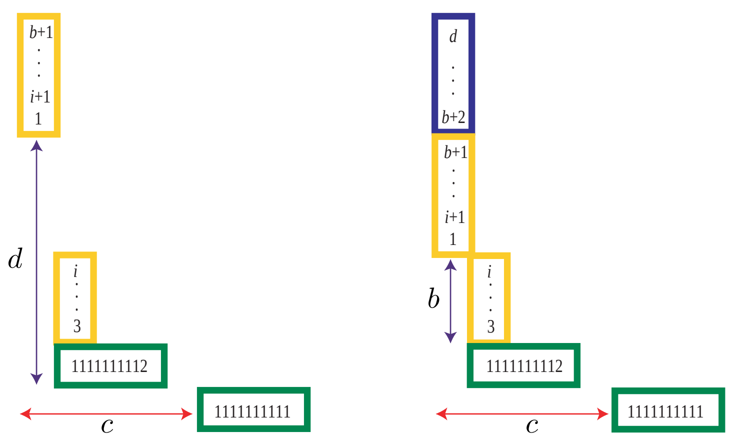


FIGURE 10. From a LR-filling of $\theta/\nu$ to a LR-filling of $\theta/\rho$. Case **(a)**

**(a)** ($a \leq c$ and $b \leq d$.) If $b = d$ or $b+1 = d$, then $(\mu, \nu)$ is a fixed point and the result is obvious. We can thus suppose that $d \geq b+2$. This situation is illustrated in Figure 9 case **(a)**, and we have $\lambda = (a, 1^{d-1})$ and $\rho = (c, 1^{b+1})$. Thus, the skew shape $\theta/\rho$ only differs from that of $\theta/\nu$ in the first column. There are now $d-(b+1)$ new boxes to be filled, which are all at the top end of the first column of $\theta$. Moreover since $d \geq b+2$, the first two columns of both $\theta/\rho$ and $\theta/\nu$ are vertical strips. From a LR-filling of $\theta/\nu$, of type $\mu$, we construct a filling of $\theta/\rho$ as follows. We simply slide down by $d-(b+1)$ positions the entries appearing in the first column, and then add new entries $b+2, \ldots, d$ in the $d-(b+1)$ cells at the top of this first column. All the other entries of the original filling are kept as they were, and the results stays semistandard since there are no interaction between the first and second columns. The resulting LR-filling is clearly of type $\rho$. An example of this procedure is given in Figure 10.

**(b)** ($a \leq c$ and $b > d$.) In this case $\lambda = (a, 1^d)$ and $\rho = (c, 1^b)$. Hence, contrary to case **(a)**, the first column of $\theta/\rho$ now has less boxes to be filled than the first column of $\theta/\nu$. To obtain a LR-filling of $\theta/\rho$ starting from one for $\theta/\nu$, we rather proceed as follows. Push up by $b-d$ positions the $d+1$ first entries of the first column up to those in the $(d+2)^{\text{th}}$ row. Then delete all the entries labeled from $d+2$ to $b+1$, while leaving unchanged the part of the filling in the remaining columns. This clearly produces a LR-filling of $\theta/\rho$. This is illustrated in Figure 11.

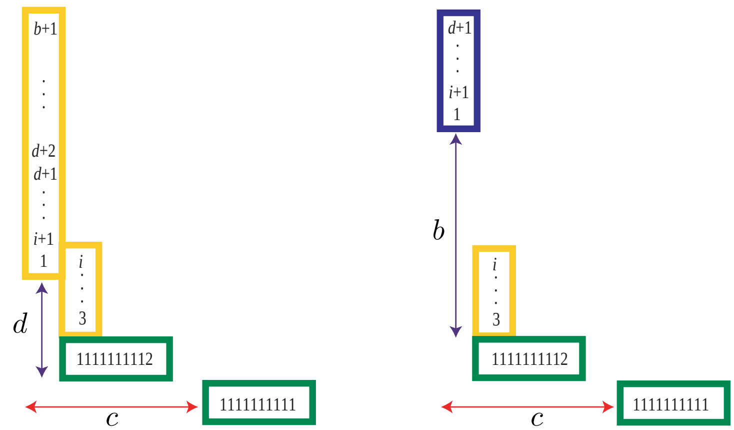


FIGURE 11. From a LR-filling of $\theta/\nu$ to a LR-filling of $\theta/\rho$. Case **(b)**

**(c)** $(a \leq c.)$ In this case $\lambda = (a-1, 1^{d-1})$ and $\rho = (c+1, 1^{b+1})$, so that the first row of $\theta/\rho$ has one less box than that of $\theta/\nu$, so that $\lambda_1(\mu,\nu) = \mu_1(\mu,\nu) - 1$. Note that in every LR-filling involved, the entries in the first row can be only 1's. We get a LR-filling of $\theta/\rho$ from one of $\theta/\nu$ by the same procedures used for cases **(a)** or **(b)**, depending on the relative values of $b$ and $d$, after which we remove a copy of 1 from the first row. □

**Proof of part (2) of Theorem 3.1.** To show that Conjecture 2.9 holds for pairs of skew shapes $(\mu/\alpha, \nu/\beta)$, where all partitions involved are hooks, with $\alpha = \beta$, we proceed as follows. If $\alpha = \beta = 0$ then the result follows from part **(1)**. So let $\alpha = \beta \neq 0$. We first recall (see Figure 9) that for a pair of hooks $\mu = (a, 1^b)$ and $\nu = (c, 1^d)$ the possible results of the $*$-operation

$$(\mu,\nu)^* = ((\mathfrak{a}, 1^{\mathfrak{b}}), (\mathfrak{c}, 1^{\mathfrak{d}})) \tag{7.1}$$

are

**(a)** $\begin{array}{ll} \mathfrak{a} = a, & \mathfrak{b} = d-1 \\ \mathfrak{c} = c, & \mathfrak{d} = b+1 \end{array}$ **(b)** $\begin{array}{ll} \mathfrak{a} = a, & \mathfrak{b} = d \\ \mathfrak{c} = c, & \mathfrak{d} = b \end{array}$ **(c)** $\begin{array}{ll} \mathfrak{a} = a-1, & \mathfrak{b} = d-1 \\ \mathfrak{c} = c+1, & \mathfrak{d} = b+1 \end{array}$

We also recall that the pair $(\alpha,\beta) = ((k, 1^m), (k, 1^m))$ is a fixed point for $*$-operation, and that the skew Schur functions $s_{\mu/\alpha}$ and $s_{\nu/\beta}$ are simply

$$s_{\mu/\alpha} = h_{a-k}\, e_{b-m}, \qquad \text{and} \qquad s_{\nu/\beta} = h_{c-k}\, e_{d-m}.$$

It follows that the statement that we have to prove simply translates into

$$\underbrace{(h_{\mathfrak{a}-k}\, e_{\mathfrak{b}-m})}_{s_\lambda} \underbrace{(h_{\mathfrak{c}-k}\, e_{\mathfrak{d}-m})}_{s_\rho} - \underbrace{(h_{a-k}\, e_{b-m})}_{s_{\mu/\alpha}} \underbrace{(h_{c-k}\, e_{d-m})}_{s_{\nu/\beta}} \tag{7.2}$$

being Schur positive. In case **(b)**, this is simply vacuously true. For case **(a)**, (7.2) equals

$$h_{a-k}h_{c-k}(e_{d-m-1}e_{b-m+1} - e_{d-m}e_{b-m}). \tag{7.3}$$

But a special case of the (dual) Jacobi-Trudi formula states that

$$s_{2^k,1^{\ell-k}} = \det \begin{pmatrix} e_\ell & e_{\ell+1} \\ e_{k-1} & e_k \end{pmatrix}, \tag{7.4}$$

when $\ell \geq k \geq 1$. Observe here that the determinant in (7.4) is equal to the determinant obtained by exchanging $k$ and $\ell$. To get the positivity of (7.3), we use (7.4) with $\ell = d-m-1$ and $k = b-m+1$ when $d > b+1$, and $\ell = b-m+1$ and $k = d-m-1$ otherwise. Hence (7.3) is simply equal to $h_{a-k}h_{c-k}s_{2^k,1^{\ell-k}}$, which is Schur positive by Pieri's formula (see [6, Corollary 7.15.3]). Finally, for case **(c)**, (7.2) equals

$$\begin{aligned} h_{a-k-1}h_{c-k+1}e_{d-m-1}e_{b-m+1} - h_{a-k}h_{c-k}e_{d-m}e_{b-m} = \\ h_{a-k-1}h_{c-k+1}(e_{d-m-1}e_{b-m+1} - e_{d-m}e_{b-m}) + \\ e_{d-m}e_{b-m}(h_{a-k-1}h_{c-k+1} - h_{a-k}h_{c-k}) \end{aligned}$$

which is readily seen to be positive, by a similar argument. □

**Proof of part (3) of Theorem 3.1.** For $\nu/\beta$ a ribbon, set $(\lambda, \rho) = (0, \nu/\beta)^*$. We will show that $s_\lambda s_\rho$ can be expanded as a (positive integer coefficient) sum of skew Schur functions indexed by ribbons, with $s_{\nu/\beta}$ appearing with nonzero coefficient. The slightly more general case of weak ribbons is entirely similar. We first need to recall that a $N$-cell ribbon $R = \nu/\beta$ is entirely described by the sequence of row lengths of $R$, reding down from the top. This results in a *composition* $c(R) = (c_1, \dots, c_n)$ of $N$. In term of this composition description, it is classical that the product of two ribbon Schur functions can be expressed as a sum of two ribbon Schur functions:

$$s_{c_1,\dots,c_n} s_{d_1,\dots,d_k} = s_{c_1,\dots,c_n,d_1,\dots,d_k} + s_{c_1,\dots,c_n+d_1,\dots,d_k} \tag{7.5}$$

The two resulting ribbons are illustrated in Figure 12. Let us split the ribbon $R = \nu/\beta$ in

FIGURE 12. The two ribbons in the right hand side of (7.5).

two sub-ribbons $R'$ and $R''$, according to the inequalities $y > x$ and $y \leq x$, respectively, for cells $(x, y)$ of $R$. This is illustrated in the left hand side of Figure 13. From Lemma 2.3 it

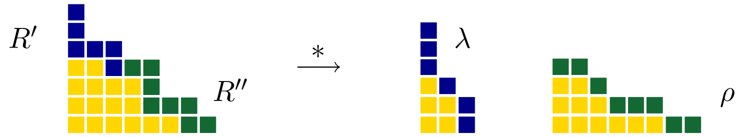


FIGURE 13. The ribbons $R'$ and $R''$.

easily follows that $\lambda = \overline{\nu}/\overline{\beta}$ is a vertical strip and that $\rho = \underline{\nu}/\underline{\beta}$ is a horizontal strip. More specifically, the column lengths of $\lambda$ are equal to the column lengths of $R'$. Whereas, the row lengths of $\rho$ are equal to the row lengths of $R''$. This is illustrated in the right hand side of Figure 13. It follows from the definition of skew Schur functions (see [3]) that $s_\lambda$ is simply a product of elementary symmetric functions, one for each column length of $R'$. Similarly, $s_\rho$ is a product of complete homogeneous symmetric functions, one for each row length of $R''$. Repeated applications of the multiplication rule for ribbons (7.5) clearly give non negative coefficient expansions

$$s_\lambda = s_{R'} + \sum_c a_c s_c \quad \text{and} \quad s_\rho = s_{R''} + \sum_c b_c s_c,$$

which include respectively the terms $s_{R'}$ and $s_{R''}$, each with coefficient 1. Yet another application of the multiplication rule makes it evident that $s_R$ appears in the ribbon expansion of the product of $s_\lambda$ and $s_\rho$. Thus the theorem is proved. □

## 8. Reduction to a finite set of pairs in bounded height case

In this section we show that the bounded height case of Conjecture 1.1 can be reduced to checking that it holds for a finite number of pairs, for any given height. In order to do this, and to state our result, we need some definitions. Let $\mu$ and $\theta$ be two partitions such that $\mu \subseteq \theta$ and consider the skew partition $\theta/\mu$. Given a partition $\mu$ containing a column of height $k$, we denote $\mu - 1^k$ the partition obtained by removing this column. In other words,

$$(\mu - 1^k)_j := \begin{cases} \mu_j - 1, & \text{if } j \leq k, \\ \mu_j, & \text{if } j > k. \end{cases}$$

We say that $\mu$ has a *k-full* column in $\theta$, if there is a $j$ such that the $j^{\text{th}}$ column of $\mu$ and $\theta$

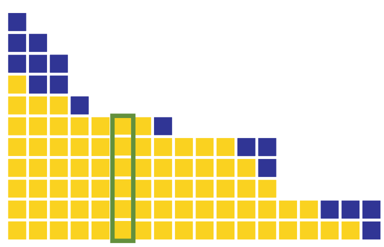

Figure 14. A 6-full column.

are both of height $k$. When this is the case, setting $\beta := \theta - 1^k$ and $\gamma := \mu - 1^k$, we observe that

$$\langle s_\theta, s_\mu s_\nu \rangle = \langle s_\beta, s_\gamma s_\nu \rangle, \tag{8.1}$$

since, using (6.1), this is clearly equivalent to $s_{\theta/\mu} = s_{\beta/\gamma}$ which holds trivially. When $\mu \subseteq \theta$, the fact that $\mu$ has a $k$-full column in $\theta$ is equivalent to

$$\theta_k \geq \mu_k > \theta_{k+1}, \tag{8.2}$$

assuming that $\theta_{k+1} = 0$ when $k$ is the number of parts of $\theta$.

**Proof of Theorem 3.2.** We proceed by induction on the number of columns of $\mu$, for the set of partitions $\mu$ with height bounded by $p$, and such that

$$\mu_1 > p\,(\nu_1 + p). \tag{8.3}$$

Once again, let $(\lambda, \rho) := (\mu, \nu)^*$. For any $\theta$, such that $s_\theta$ appears with nonzero coefficient in $s_\mu s_\nu$, we will show (8.3) implies that there exists a $k$, such that both $\mu$ and $\lambda$ have a $k$-full column in $\theta$, and such that

$$\lambda(\mu - 1^k, \nu) = \lambda - 1^k \qquad \text{and} \qquad \rho(\mu - 1^k, \nu) = \rho. \tag{8.4}$$

It will then follow that

$$\begin{aligned}
\langle s_\theta, s_\mu s_\nu \rangle &= \langle s_{\theta - 1^k}, s_{\mu - 1^k} s_\nu \rangle && \text{by (8.1)}\\
&\leq \langle s_{\theta - 1^k}, s_{\lambda(\mu - 1^k, \nu)} s_{\rho(\mu - 1^k, \nu)} \rangle && \text{by induction hypothesis}\\
&= \langle s_{\theta - 1^k}, s_{\lambda - 1^k} s_\rho \rangle && \text{by (8.4)}\\
&= \langle s_\theta, s_\lambda s_\rho \rangle && \text{by (8.1)}
\end{aligned}$$

which will prove the theorem.

To show that there is a $k$ with the properties announced above, we proceed as follows. Observe that at least one of the differences $\mu_j - \mu_{j+1}$, where $1 \leq j \leq p$, is strictly larger then $\nu_1 + p$, since otherwise

$$\begin{aligned}
\mu_1 &= \sum_{j=1}^{p} \mu_j - \mu_{j+1}\\
&\leq p\,(\nu_1 + p)
\end{aligned}$$

which would contradict (8.3). We can thus choose $k$ to be the smallest integer, between 1 and $p$, such that

$$\mu_k > \mu_{k+1} + \nu_1 + p. \tag{8.5}$$

For $\theta$ as above, we must clearly have $\mu_i \leq \theta_i \leq \mu_i + \nu_1$. Thus

$$\begin{aligned}
\theta_k &\geq \mu_k\\
&> \mu_{k+1} + \nu_1 + p\\
&> \theta_{k+1},
\end{aligned}$$

so that $\mu$ has a $k$-full column in $\theta$, by criteria (8.2). Moreover, for $1 \leq i \leq k$, it is clear that $\mu_i > \nu_1 + p$, and thus (1.2) simplifies to

$$\lambda_i = \mu_i - i, \qquad \text{for} \qquad 1 \leq i \leq k.$$

It follows that

$$\begin{aligned}
\lambda_k &= \mu_k - k\\
&> \mu_{k+1} + \nu_1 + (p - k)\\
&\geq \theta_{k+1},
\end{aligned}$$

so that $\lambda$ also has a $k$-full column in $\theta$. The last verification that we need to do is that (8.4) holds. Now, it is clear that the first $k$ lines of $\gamma := \mu - 1^k$ are all too large for the first part of (2.1) to apply. In fact, considering the way $k$ has been chosen, we see that for $1 \leq i \leq k$

$$\begin{aligned} \gamma_i - i &= \mu_i - (i+1) \\ &> \mu_1 - (i-1)\,(\nu_1 + p) - (i+1) \\ &> (p - i + 1)\,(\nu_1 + p) - (i+1) \\ &\geq \nu_1 - 1. \end{aligned}$$

This makes it obvious that (8.4) holds, thus finishing our proof. □

## 9. Final remarks

We believe that to get a better understanding of the $*$-operation, a refined study of its effect on tableaux and semistandard tableaux will be crucial. For instance this should lead to a proof of "monomial" versions of Conjectures 1.1 and 2.9. More precisely, recall that the expansion of any Schur function in the basis of monomial symmetric functions involves only positive integers. It would thus follow from the conjectures that the expansion of the difference of products considered have positive integer coefficients when expanded in term of monomial symmetric function. In particular, using definition (5.2), one should have

$$f_{(\lambda,\rho)} \geq f_{(\mu,\nu)}. \tag{9.1}$$

whenever $(\lambda, \rho) = (\mu, \nu)^*$. An independent proof of these facts would clearly lend support to the conjectures.

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(F. Bergeron) Département de Mathématiques, Université du Québec à Montréal, Montréal, Québec, H3C 3P8, CANADA

(R. Biagioli) LaCIM, Université du Québec à Montréal, Montréal, Québec, H3C 3P8, CANADA

(M. Rosas) Department of Mathematics and Statistics, York University, N520 Ross Building, 4700 Keele St. Toronto, Ontario, CANADA

*E-mail address*: bergeron.francois@uqam.ca, biagioli@math.uqam.ca, rosas@mathstat.yorku.ca