Title:
Equidistribution of negative statistics and quotients of Coxeter groups of type B and D

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Abstract:
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Equidistribution of negative statistics and quotients of Coxeter groups of type B and D

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Abstract

We generalize some identities and q-identities previously known for the symmetric group to Coxeter groups of type B and D. The extended results include theorems of Foata and Schützenberger, Gessel, and Roselle on various distributions of statistics, like of inversion number, major index, and descent number. In order to show our results we provide explicit caracterizations of the systems of minimal coset representatives of Coxeter groups of type B and D.

1 Introduction

A well known theorem of MacMahon [17] shows that the *length function* and the *major index* are equidistributed over the symmetric group S_n. We recall that the length of a permutation $\sigma \in S_n$ is given by the *number of inversions*, denoted $\mathrm{inv}(\sigma) := |\{(i,j) \mid i < j,\ \sigma(i) > \sigma(j)\}|$, and the major index of σ is the sum of all its *descents*. More precisely,

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i,$$

where $\mathrm{Des}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$. Foata gave a bijective proof of this equidistribution theorem in [9]. He studied further his bijection and together with Schützenberger derived the two following results [13]. The first one is a refinement of MacMahon's theorem, asserting the equidistribution of major index and number of inversions over descent classes.

Theorem 1.1 (Foata-Schützenberger)**.** *Let* $M = \{m_1, \ldots, m_t\}_< \subseteq \{1, \ldots, n-1\}$. *Then*

$$\sum_{\{\sigma \in S_n \mid \mathrm{Des}(\sigma^{-1}) = M\}} q^{\mathrm{maj}(\sigma)} = \sum_{\{\sigma \in S_n \mid \mathrm{Des}(\sigma^{-1}) = M\}} q^{\mathrm{inv}(\sigma)}.$$

*Dedicated to the memory of my friend and colleague Giulio Minervini.

The second one concerns the symmetry of the distribution of the major index and the inversion number over the symmetric group.

Theorem 1.2 (Foata-Schützenberger)**.** *The pairs of statistics* $(\mathrm{maj}, \mathrm{inv})$ *and* $(\mathrm{inv}, \mathrm{maj})$ *have the same distribution on* S_n*, namely*

$$S_n(t,q) := \sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n} t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

Theorem 1.1 has been extensively studied and generalized in many ways in the last three decades. Nevertheless, it still receives a lot of attention as shown by two recent papers of Hivert, Novelli, and Thibon [16], and of Adin, Brenti, and Roichman [3], where a multivariate generalization and an extension to the hyperoctahedral group of it are provided. In the latter paper, the problem of finding an analogue of this Foata-Schützenberger theorem for the Coxeter groups of type D is proposed [3, Problem 5.6].

In this paper we answer this question. Actually, we show that the *negative major indices* "nmaj", introduced in [2] on Coxeter groups of type B, and "dmaj", defined in [6] on Coxeter groups of type D, give generalizations of the first and second Foata-Schützenberger identities to B_n and D_n. In our analysis we derive nice relations among *quotients*, or sets of minimal coset representatives, of B_n and D_n that are interesting in their own. Explicit maps between these quotients are shown, and used to compute some generating functions.

Finally, we use our results, and the *negative descent numbers*, to give generalizations to B_n and D_n of two classical q-identities. The first one, due to Roselle [19] (see also Rawlings [18, (2.4)]), is the generating function of the inversion number and major index over the symmetric group: for undefined notation see next section.

Theorem 1.3 (Roselle)**.**

$$\sum_{n \geq 0} S_n(t,q) \frac{u^n}{(t;t)_n (q;q)_n} = \frac{1}{(u;t,q)_{\infty,\infty}},$$

where $S_0(t,q) = 1$. The second one is the trivariate distribution of inversion number, major index, and *number of descents*, due to Gessel [15, Theorem 8.4], (see also [14]).

Theorem 1.4 (Gessel)**.**

$$\sum_{n \geq 0} \frac{u^n}{[n]_q!} \frac{\sum_{\sigma \in S_n} t^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{des}(\sigma)}}{(t;q)_{n+1}} = \sum_{k \geq 0} p^k e[u]_q e[tu]_q \cdots e[t^k u]_q.$$

2 Preliminaries and notation

In this section we give some definitions, notation and results that will be used in the rest of this work. For $n \in \mathbb{N}$ we let $[n] := \{1, 2, \ldots, n\}$ (where $[0] := \emptyset$). Given $n, m \in \mathbb{Z},\ n \leq m$,

we let $[n,m] := \{n, n+1, \ldots, m\}$. We let $\mathbb{P} := \{1,2,3,\ldots\}$. The cardinality of a set A will be denoted by $|A|$ and we let $\binom{[n]}{2} := \{S \subseteq [n] \mid |S| = 2\}$. Given a set A, we denote $A_< := \{a_1, a_2, \ldots\}$ where $a_1 < a_2 < \ldots$.

For our study we need notation for q-analogs of the factorial, binomial coefficient, and multinomial coefficient. These are defined by the following expressions

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1}; \qquad [n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q;$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{[n]_q!}{[m]_q![n-m]_q!}; \qquad \begin{bmatrix} n \\ m_1,\ m_2, \ldots, m_t \end{bmatrix}_q := \frac{[n]_q!}{[m_1]_q![m_2]_q! \cdots [m_t]_q!}.$$

As usual we let

$$\begin{aligned} (a;q)_0 &:= 1 \\ (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}) \\ (a;q)_\infty &:= \prod_{n\geq 1}(1-aq^{n-1}). \end{aligned}$$

Moreover, for $r, s \in \mathbb{N}$ we let

$$(a;t,q)_{r,s} := \begin{cases} 1 & \text{if } r \text{ or } s \text{ are zero} \\ \displaystyle\prod_{1\leq i\leq r}\prod_{1\leq j\leq s}(1-at^{i-1}q^{j-1}) & \text{if } r,s \geq 1 \end{cases},$$

and

$$(a;t,q)_{\infty,\infty} := \prod_{i\geq 1}\prod_{j\geq 1}(1-at^{i-1}q^{j-1}).$$

Finally,

$$e[u]_q := \sum_{n\geq 0}\frac{u^n}{[n]_q!},$$

is the q-analogue of the exponential function. The following q-binomial theorem is well known (see e.g. [4])

Theorem 2.1.

$$(-xq;q)_n = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m+1}{2}}x^m.$$

2.1 Coxeter groups of type B and D

We denote by B_n the group of all bijections β of the set $[-n,n]\setminus\{0\}$ onto itself such that

$$\beta(-i) = -\beta(i)$$

for all $i \in [-n,n]\setminus\{0\}$, with composition as the group operation. This group is usually known as the group of *signed permutations* on $[n]$, or as the *hyperoctahedral group* of rank

n. If $\beta \in B_n$ then we write $\beta = [\beta(1), \ldots, \beta(n)]$ and we call this the *window* notation of β. As set of generators for B_n we take $S_B := \{s_1^B, \ldots, s_{n-1}^B, s_0^B\}$ where for $i \in [n-1]$

$$s_i^B := [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ and } s_0^B := [-1, 2, \ldots, n].$$

It is well known that (B_n, S_B) is a Coxeter system of type B (see e.g., [8, §8.1]).

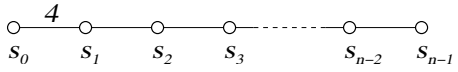

Figure 1: The Dynkin diagram of B_n

To give an explicit combinatorial description of the length function ℓ_B of B_n with respect to S_B, we need the following statistics. For $\beta \in B_n$ we let

$$\begin{aligned} \mathrm{inv}(\beta) &:= |\{(i,j) \mid i < j,\ \beta(i) > \beta(j)\}|, \\ \mathrm{N}_1(\beta) &:= |\{i \in [n] \mid \beta(i) < 0\}|, \text{ and} \\ \mathrm{N}_2(\beta) &:= \left|\left\{\{i,j\} \in \binom{[n]}{2} \mid \beta(i) + \beta(j) < 0\right\}\right|. \end{aligned}$$

Note that, if $\beta \in B_n$,

$$\mathrm{N}_1(\beta) + \mathrm{N}_2(\beta) = -\sum_{\{i \in [n] \mid \beta(i) < 0\}} \beta(i). \tag{1}$$

For example if $\beta = [-3, 1, -6, 2, -4, -5] \in B_6$ then $\mathrm{inv}(\beta) = 9$, $\mathrm{N}_1(\beta) = 4$, and $\mathrm{N}_2(\beta) = 14$.

The following characterizations of the length function, and of the right descent set of $\beta \in B_n$ are well known [8].

Proposition 2.2. *Let $\beta \in B_n$. Then*

$$\begin{aligned} \ell_B(\beta) &= \mathrm{inv}(\beta) + \mathrm{N}_1(\beta) + \mathrm{N}_2(\beta), \quad \textit{and} \\ \mathrm{Des}_B(\beta) &= \{i \in [0, n-1] \mid \beta(i) > \beta(i+1)\}, \end{aligned}$$

where $\beta(0) := 0$.

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\gamma \in B_n \mid \mathrm{N}_1(\gamma) \equiv 0 \pmod 2\}.$$

It is usually called the *even-signed permutation group*. As a set of generators for D_n we take $S_D := \{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$ where for $i \in [n-1]$

$$s_i^D := s_i^B \text{ and } s_0^D := [-2, -1, 3, \ldots, n].$$

There is a well known direct combinatorial way to compute the length, and the right descent set of $\gamma \in D_n$, (see, e.g., [8, §8.2]).

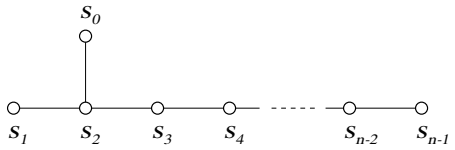

Figure 2: The Dynkin diagram of D_n

Proposition 2.3. *Let $\gamma \in D_n$. Then*

$$\begin{aligned} \ell_D(\gamma) &= \mathrm{inv}(\gamma) + \mathrm{N}_2(\gamma), \text{ and} \\ \mathrm{Des}_D(\gamma) &= \{i \in [0, n-1] \mid \gamma(i) > \gamma(i+1)\}, \end{aligned}$$

where $\gamma(0) := -\gamma(2)$.

2.2 Negative statistics

In [2], Adin, Brenti and Roichman introduced the following statistics on B_n. For $\beta \in B_n$ let

$$\mathrm{NDes}(\beta) := \mathrm{Des}(\beta) \biguplus \{-\beta(i) \mid \beta(i) < 0\},$$

where $\biguplus$ denotes disjoint union, and define

$$\mathrm{nmaj}(\beta) := \sum_{i \in \mathrm{NDes}(\beta)} i, \quad \text{and} \quad \mathrm{ndes}(\beta) := |\,\mathrm{NDes}(\beta)|.$$

Note that $\mathrm{NDes}(\beta)$ is a multiset. It follows from (1) that

$$\mathrm{nmaj}(\beta) = \mathrm{maj}(\beta) + \mathrm{N}_1(\beta) + \mathrm{N}_2(\beta), \quad \text{and} \tag{2}$$

$$\mathrm{ndes}(\beta) = \mathrm{des}(\beta) + \mathrm{N}_1(\beta). \tag{3}$$

For the element $\beta = [-3, 1, -6, 2, -4, -5] \in B_6$, $\mathrm{nmaj}(\beta) = 29$, and $\mathrm{ndes}(\beta) = 7$.

In [6], a notion of descent multiset for $\gamma \in D_n$ is introduced

$$\mathrm{DDes}(\gamma) := \mathrm{Des}(\gamma) \biguplus \{-\gamma(i) - 1 \mid \gamma(i) < 0\} \setminus \{0\},$$

and the following statistics are defined

$$\mathrm{dmaj}(\gamma) := \sum_{i \in \mathrm{DDes}(\gamma)} i, \quad \text{and} \quad \mathrm{ddes}(\gamma) := |\,\mathrm{DDes}(\gamma)|.$$

It easily follows that

$$\mathrm{dmaj}(\gamma) = \mathrm{maj}(\gamma) + \mathrm{N}_2(\gamma), \quad \text{and} \tag{4}$$

$$\mathrm{ddes}(\gamma) = \mathrm{des}(\gamma) + \mathrm{N}_1(\gamma) + \epsilon(\gamma), \tag{5}$$

where

$$\epsilon(\gamma) := \begin{cases} -1 & \text{if } 1 \notin \gamma([n]) \\ 0 & \text{if } 1 \in \gamma([n]). \end{cases} \tag{6}$$

For example if $\gamma = [-4, 1, 3, -5, -2, -6] \in D_6$ then $\mathrm{dmaj}(\gamma) = 21$, and $\mathrm{ddes}(\gamma) = 5$.

The statistics nmaj and dmaj are usually called *negative major indices*; ndes and ddes *negative descent numbers* for B_n and D_n, respectively. The negative major indices are Mahonian statistics, namely they are equidistributed with the length over the group,

$$\sum_{\beta \in B_n} q^{\mathrm{nmaj}(\beta)} = \sum_{\beta \in B_n} q^{\ell_B(\gamma)}, \quad \text{and} \quad \sum_{\gamma \in D_n} q^{\mathrm{dmaj}(\gamma)} = \sum_{\gamma \in D_n} q^{\ell_D(\gamma)}.$$

The pairs $(\mathrm{ndes}, \mathrm{nmaj})$ and $(\mathrm{ddes}, \mathrm{dmaj})$ give generalizations to B_n and D_n of a famous identity of Carlitz, see [2, Theorem 3.2], and [6, Theorem 3.4].

2.3 Quotients of Coxeter groups

To show some of the next results we will need of the following decomposition that comes from the general theory of Coxeter group. We refer the reader to [8] for any undefined notation.

Let (W, S) be a Coxeter system, for $J \subseteq S$ we let W_J be the *parabolic subgroup* of W generated by J, and

$$W^J := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J\},$$

the set of minimal left coset representatives of W_J, or the *(right) quotient*. The quotient W^J is a poset according to the Bruhat order. The following is well known (see [8, §2.4]).

Proposition 2.4. *Let (W, S) be a Coxeter system, and let $J \subseteq S$. Then:*

i) Every $w \in W$ has a unique factorization $w = w^J w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.

ii) For this factorization $\ell(w) = \ell(w^J) + \ell(w_J)$.

As a first application of this decomposition to the groups B_n (and D_n), let us consider the parabolic subgroup generated by $J := S_B \setminus \{s_0^B\}$. In this case, by looking at the Dynkin diagram in Figure 1, we obtain that $B_J = S_n$. Moreover it is not hard to see that

$$B^J := B_n^J = \{u \in B_n \mid u(1) < u(2) < \ldots < u(n)\}. \tag{7}$$

Hence from Proposition 2.4 we get

$$B_n = \biguplus_{\sigma \in S_n} \{u\sigma \mid u \in B^J\}. \tag{8}$$

Note that in the case D_n, for $J := S_D \setminus \{s_0^D\}$, a similar decomposition holds,

$$D_n = \biguplus_{\sigma \in S_n} \{u\sigma \mid u \in D^J\},$$

where once again $D_J = S_n$, and $D^J = \{u \in D_n \mid u(1) < u(2) < \ldots < u(n)\}$.

Remark 2.5. The construction or right quotient can be mirrored, by considering *left descents*. Let $J \subseteq S$. A *left quotient* of W is defined by

$${}^J W := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in J\}.$$

Proposition 2.4 holds for left quotients too, but the factorization in i) becomes $w = w_J \cdot {}^J w$, with ${}^J w \in {}^J W$. Left and right quotients are isomorphic posets, by means of the inversion map. In the next section, we will work with subsets of B_n and D_n that are left quotients. They are called descent classes for reasons that will be immediately clear.

3 Combinatorial description of descent classes

Let us fix a subset of descents $M := \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$. The set

$$B(M) := \{\beta \in B_n \mid \mathrm{Des}_B(\beta^{-1}) \subseteq M\}, \tag{9}$$

is usually called a *B-descent class*. Note that this set is nothing but a left quotient of B_n. More precisely, it is the one corresponding to the subset $J = S \setminus \tilde{M}$, where $\tilde{M} := \{s_i \mid i \in M\}$. The following result can be found in [3, Lemma 4.1].

Lemma 3.1. *Let $\beta \in B_n$, and $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$. Let $m_{t+1} := n$. Then $\mathrm{Des}_B(\beta^{-1}) \subseteq M$ if and only if there exist (unique) integers $r_1, \ldots, r_t$ satisfying $m_i \leq r_i \leq m_{i+1}$ for all i, and such that β is a shuffle of the following increasing sequences:*

$$\begin{array}{c} [1, 2, \ldots, m_1]\,, \\ [-r_1, -r_1+1, \ldots, -(m_1+1)]\,, \\ [r_1+1, r_1+2, \ldots, m_2]\,, \\ \vdots \\ [-r_t, -r_t+1, \ldots, -(m_t+1)]\,, \\ [r_t+1, r_t+2, \ldots, n]\,. \end{array} \tag{10}$$

Some of these sequences may be empty, if $r_i = m_i$ or $r_i = m_{i+1}$ for some i, or if $m_i = 0$.

The following one is an explicit description of D-descent classes

$$D(M) := \{\gamma \in D_n \mid \mathrm{Des}_D(\gamma^{-1}) \subseteq M\}. \tag{11}$$

Lemma 3.2. *Let $\gamma \in D_n$, and $M = \{m_1, \ldots, m_t\}_< \subseteq [0, n-1]$. Let $m_{t+1} := n$. Then $\mathrm{Des}_D(\gamma^{-1}) \subseteq M$ if and only if there exist (unique) integers $r_1, \ldots, r_t$ satisfying $m_i \leq r_i \leq m_{i+1}$ for all i, and such that γ is a shuffle of the following increasing sequences. There are three cases, and six possible "blocks" of sequences.*

1) If $0 \in M$: ($m_1 = 0$)

$$\begin{array}{l} [-r_1, -r_1+1, \ldots, -2, -1]\,, \\ [r_1+1, r_1+2, \ldots, m_2]\,, \\ \qquad\vdots \\ [-r_t, -r_t+1, \ldots, -(m_t+1)]\,, \\ [r_t+1, r_t+2, \ldots, n]\,, \end{array} \tag{12}$$

with $\displaystyle\sum_{i=1}^{t}(r_i - m_i) \equiv 0 \pmod 2$.

2) If $0, 1 \notin M$: (note $m_1 \geq 2$)

$$\begin{array}{ll} [1, 2, \ldots, m_1]\,, & [-1, 2, \ldots, m_1] \\ [-r_1, -r_1+1, \ldots, -(m_1+1)]\,, & [-r_1, -r_1+1, \ldots, -(m_1+1)] \\ [r_1+1, r_1+2, \ldots, m_2]\,, & [r_1+1, r_1+2, \ldots, m_2] \\ \qquad\vdots & \qquad\vdots \\ [-r_t, -r_t+1, \ldots, -(m_t+1)]\,, & [-r_t, -r_t+1, \ldots, -(m_t+1)] \\ [r_t+1, r_t+2, \ldots, n] & [r_t+1, r_t+2, \ldots, n] \end{array} \tag{13}$$

with

$$\sum_{i=1}^{t}(r_i - m_i) \equiv 0 \pmod 2; \quad \sum_{i=1}^{t}(r_i - m_i) \equiv 1 \pmod 2.$$

3) If $0 \notin M$ and $1 \in M$: (note $m_2 \geq 2$, and $r_1 \geq 2$)

$$\begin{array}{lll} [1] & & [-r_1, \ldots, -2, 1] \\ [2, 3, \ldots, m_2]\,, & [-1, 2, 3, \ldots, m_2] & [r_1+1, r_1+2, \ldots, m_2] \\ [-r_2, -r_2+1, \ldots, -(m_2+1)]\,, & [-r_2, -r_2+1, \ldots, -(m_2+1)]\,, & [-r_2, -r_2+1, \ldots, -(m_2+1)] \\ [r_2+1, r_2+2, \ldots, m_3]\,, & [r_2+1, r_2+2, \ldots, m_3]\,, & [r_2+1, r_2+2, \ldots, m_3] \\ \qquad\vdots & \qquad\vdots & \qquad\vdots \\ [-r_t, -r_t+1, \ldots, -(m_t+1)]\,, & [-r_t, -r_t+1, \ldots, -(m_t+1)]\,, & [-r_t, -r_t+1, \ldots, -(m_t+1)] \\ [r_t+1, r_t+2, \ldots, n] & [r_t+1, r_t+2, \ldots, n] & [r_t+1, r_t+2, \ldots, n] \end{array} \tag{14}$$

with

$$\sum_{i=2}^{t}(r_i - m_i) \equiv 0 \pmod 2; \quad \sum_{i=2}^{t}(r_i - m_i) \equiv 1 \pmod 2; \quad \sum_{i=1}^{t}(r_i - m_i) \equiv 0 \pmod 2.$$

Some of these sequences may be empty, if $r_i = m_i$ or $r_i = m_{i+1}$ for some i, or if $m_i = 0$.

Proof. The only difference with respect to the B_n case is for the $0,1$ descents. They depend on the relative positions of ± 1 and ± 2 in the window notation of γ. The following are the D-descent classes of all elements of B_2. We have that

$$\begin{aligned} D(\emptyset) &= \{[1,2],[-1,2]\} \\ D(\{0\}) &= \{[2,-1],[-2,-1]\} \\ D(\{1\}) &= \{[-2,1],[2,1]\} \\ D(\{0,1\}) &= \{[1,-2],[-1,-2]\}. \end{aligned}$$

From this, the parity conditions $\sum_{i=1}^{t}(r_i - m_i) \equiv 0$ or $\equiv 1 \pmod 2$, and Lemma 3.1 the result follows. □

Remark 3.3. Let us fix a subset of descents $M := \{m_1, m_2, \ldots, m_t\}$. Consider the decompositions of B_n and D_n given by Proposition 2.4 by using left quotients. Recall that $|B_n| = 2^n n!$ and that $|D_n| = 2^{n-1} n!$. By looking at the Dynkin diagrams in Figure 1 and Figure 2, it is easy to derive the following equalities.

- ♣ If $0 \in M$, then $|B(M)| = 2 \cdot |D(M)|$;
- ♣ If $0, 1 \notin M$, then $|B(M)| = |D(M)|$;
- ♣ If $0 \notin M$, and $1 \in M$, then $|B(M)| = m_2 \cdot |D(M)|$.

Now we make explicit these equalities by showing relations between D and B left quotients.

Proposition 3.4. *Let $M \subseteq [0, n-1]$, such that $0 \in M$. Then*

i) $B(M)$ splits into the disjoint union

$$B(M) = D(M) \uplus \bar{D}(M),$$

where $\bar{D}(M) := \{\bar{\gamma} = [-\gamma(1), \gamma(2), \ldots, \gamma(n)] \mid \gamma \in D(M)\} = \{\gamma \cdot s_0^B \mid \gamma \in D(M)\}$.

ii) Moreover

$$\sum_{\beta \in B(M)} q^{\ell_D(\beta)} = 2 \cdot \sum_{\gamma \in D(M)} q^{\ell_D(\gamma)}.$$

Proof. Let $\gamma \in D(M)$. By Lemma 3.2 γ is a shuffle of the sequences in (12), and so it can also be obtained as a shuffle of the sequences in (10). Hence $\gamma \in B(M)$. Now, let us change the sign to the first entry of γ, by getting $\bar{\gamma}$. We are changing the sign of $-r_i$, or of $r_i + 1$ for $i \in [t]$, in one of the sequences in (12). Note that this operation does not create a new B-descent for $\bar{\gamma}$. Hence $\bar{\gamma} \in B(M) \setminus D(M)$. More precisely, $\bar{\gamma}$ can be obtained by shuffling the same sequences that give γ where the twos involving r_i are replaced either by $[-r_i + 1, \ldots, -(m_i + 1)]$ and $[r_i, r_i + 1, \ldots, m_i]$, or by $[-r_i - 1, \ldots, -(m_i + 1)]$

and $[r_i+2,\dots,m_i]$, depending if it is the sign of $-r_i$, or of r_i+1, that changes. All those sequences belong to (10). So i) follows by Remark 3.3.

Now, it is easy to see that for all $\gamma \in D(M)$, one has $\ell_D(\gamma) = \ell_D(\bar{\gamma})$. To see that, suppose $\gamma(1) > 0$. Then

$$\mathrm{inv}(\bar{\gamma}) = \mathrm{inv}(\gamma) - (\gamma(1)-1) \text{ and } \mathrm{N}_2(\bar{\gamma}) = \mathrm{N}_2(\gamma) + (\gamma(1)-1),$$

and so the length ℓ_D is stable. If $\gamma(1) < 0$ a similar computations holds, hence ii) follows. □

Note that the two subsets $D(M)$ and $\bar{D}(M)$ are not isomorphic as posets, when they are considered as sub-posets of $(B(M), <_B)$, where $<_B$ denotes the B-Bruhat order . An example is given for $n=3$ and $M=\{0,2\}$.

Proposition 3.5. *Let $M \subseteq [0,n-1]$, such that $0,1 \notin M$. Then*

i) The map $\varphi : B(M) \longrightarrow D(M)$ defined by

$$\beta \mapsto \begin{cases} \beta, & \text{if } \beta \in D_n; \\ s_0^B \cdot \beta, & \text{otherwise,} \end{cases}$$

is a bijection.

ii) Moreover

$$\sum_{\beta \in B(M)} q^{\ell_D(\beta)} = \sum_{\gamma \in D(M)} q^{\ell_D(\gamma)}.$$

Proof. Let $\beta \in B(M)$, it is a shuffle of the sequences in (10). If $\beta \in D_n$, then it is also a shuffle of the sequences in the first block of (13). Hence $\beta \in D(M)$. Now suppose that $\beta \notin D_n$. Since $0 \notin \mathrm{Des}_B(\beta^{-1})$, then $1 \in \beta[n]$. By multiplying on the left by s_0^B, we change the sign of 1, and so the parity of β. Hence $s_0^B \cdot \beta \in D_n$. Actually, we obtain an element which is a shuffle of the sequences in second block of (13). From Remark 3.3 i) follows.

Since $\mathrm{N}_2(\gamma) = \mathrm{N}_2(\varphi(\gamma))$ and $\mathrm{inv}(\gamma) = \mathrm{inv}(\varphi(\gamma))$, one has $\ell_D(\gamma) = \ell_D(\varphi(\gamma))$, and so ii) follows. □

The map φ is not a poset isomorphism between $(B(M), <_B)$ and $(D(M), <_D)$, where $<_B$ and $<_D$ denote the corresponding Bruhat orders. When $n=3$, and $M=\{2\}$, $B(M)$ is a chain, while in $D(M)$ there are two not comparable elements.

Proposition 3.6. *Let $M \subseteq [0,n-1]$, such that $0 \notin M$, and $1 \in M$. Then*

i) $B(M)$ splits as the disjoint union of the following m_2 subsets

$$B(M) = D_1(M) \uplus D_{12}(M) \uplus \ldots \uplus D_{12\ldots m_2}(M).$$

Each $D_{1\ldots i}(M)$ is in bijection with $D(M)$, and it is recursively defined as follows:

1) $D_1(M)$ is obtained by shuffling the sequences defining $D(M)$ where -1 (if present) is replaced with 1.

2) For each $i \geq 2$, $D_{12\dots i}(M)$ is obtained by shuffling the sequences defining $D_{12\dots i-1}(M)$ where:

- ♣ *1 and $\pm i$ are swiched if they are in the same sequence;*
- ♣ *i is replaced by 1, and 1 is replaced by $-i$, otherwise. This case happens when 1 is at the beginning of a sequence of type $[1, -(i-1), \dots, -2]$, and i is the initial value of the sequence $[i, i+1, \dots, m_2]$.*

ii) Moreover

$$\sum_{\beta \in B(M)} q^{\ell_D(\beta)} = [m_2]_q \cdot \sum_{\gamma \in D(M)} q^{\ell_D(\gamma)}.$$

Before writing down the proof let us consider an example.

Example 3.7. Consider $n = 4$ and $M = \{1, 3\}$. Then $D(M)$ is given by the shuffles of the following blocks of increasing sequences (written in column).

$$D(M) = \left\{ \begin{array}{cccc} [1] & [-1,2,3] & [-2,1] & [-3,-2,1] \\ [2,3] & [-4] & [3] & [4] \\ [4] & & [-4] & \end{array} \right\}$$

Then $B(M)$ splits as disjoint union of the following three subsets:

$$D_1(M) = \left\{ \begin{array}{cccc} [1] & [1,2,3] & [-2,1] & [-3,-2,1] \\ [2,3] & [-4] & [3] & [4] \\ [4] & & [-4] & \end{array} \right\}$$

$$D_{12}(M) = \left\{ \begin{array}{cccc} [-2] & [2,1,3] & [1,-2] & [-3,1,-2] \\ [1,3] & [-4] & [3] & [4] \\ [4] & & [-4] & \end{array} \right\}$$

$$D_{123}(M) = \left\{ \begin{array}{cccc} [-2] & [2,3,1] & [-3,-2] & [1,-3,-2] \\ [3,1] & [-4] & [1] & [4] \\ [4] & & [-4] & \end{array} \right\}.$$

Proof. The transformations defining $D_{1\dots i}(M)$ involve only the first two sequences of the three blocks of (14). It is easy to see that $D_{1\dots i}(M) \subseteq B(M)$ for all $i \in [m_2]$, and that $D_{1\dots i}(M)$ and $D_{1\dots j}(M)$ are disjoint if $i \neq j$. Hence the decomposition in i) follows from Remark 3.3.

Since changing -1 into 1 in a signed permutation γ affects neither $\mathrm{inv}(\gamma)$ nor $\mathrm{N}_2(\gamma)$, it follows that

$$\sum_{\gamma\in D(M)} q^{\ell_D(\gamma)} = \sum_{\gamma\in D_1(M)} q^{\ell_D(\gamma)}.$$

Now let us show that for all $i \geq 2$

$$\sum_{\gamma\in D_{1\dots i}(M)} q^{\ell_D(\gamma)} = q \sum_{\gamma\in D_{1\dots i-1}(M)} q^{\ell_D(\gamma)}.$$

Let $\gamma \in D_{1\dots i-1}(M)$. Consider the block in (14) whose a particular shuffle gives γ.

If 1 and $\pm i$ are in the same sequence, it can be either of the form $[\dots, 1, i, \dots, m_2]$, or of the form $[-r_1, \dots, -i, 1 \dots, -2]$. Now consider the shuffle giving γ, where 1 has been switched with $\pm i$. We get a new element $\bar{\gamma} \in D_{1\dots i}(M)$. It is clear that $\bar{\gamma}$ has one more inversion with respect to γ, and so the D-length go up by 1. In fact, all other sequences in the block (whose shuffle gives γ) are made by elements that are either all bigger or all smaller of both 1 and $\pm i$. Hence the difference between $\mathrm{inv}(\gamma)$ and $\mathrm{inv}(\bar{\gamma})$ depends only on the relative positions of 1 and $\pm i$ within the same sequence.

Suppose that 1 and i are not in the same sequence. This means that 1 is at the beginning of the sequence $[1, -(i-1), \dots, -2]$ and i is at the beginning of the sequence $[i, i+1, \dots, m_2]$. So $\bar{\gamma} \in D_{1\dots i}(M)$, the element corresponding to γ after the switch, is obtained by shuffling a block that contains the following two sequences

$$[-i, -(i-1), \dots, -2] \quad \text{and} \quad [1, i+1, \dots, m_2]\,.$$

Once again all other sequences of the block are made by elements that are either all smaller or bigger of both 1 and i. The difference between the values of $\mathrm{inv}(\bar{\gamma})$ and $\mathrm{inv}(\gamma)$ depends only on the relative positions of 1 and i. Hence $\bar{\gamma}$ loses $i-2$ inversions with respect to γ (the ones given by the 1 at the beginning of the sequence), and $\mathrm{N}_2(\bar{\gamma}) = \mathrm{N}_2(\gamma) + (i-1)$ thanks to $-i$. So $\ell_D(\bar{\gamma}) = \ell_D(\gamma) + 1$. □

4 Equidistribution over descent classes

In this section we show generalizations of Theorem 1.1 to Coxeter groups of type B and D. We need the following classical result; see [14, Theorem 3.1], and [20, Example 2.2.5] for a proof.

Theorem 4.1. *Let $n \in \mathbb{P}$ and $M = \{m_1, m_2, \dots, m_t\}_< \subseteq [n-1]$. Then*

$$\sum_{\{\sigma\in S_n \mid \mathrm{Des}(\sigma^{-1})\subseteq M\}} q^{\mathrm{maj}(\sigma)} = \sum_{\{\sigma\in S_n \mid \mathrm{Des}(\sigma^{-1})\subseteq M\}} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} n \\ m_1,\ m_2-m_1, \dots, n-m_t \end{bmatrix}_q.$$

Theorem 4.2. *Let $n \in \mathbb{P}$ and $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$. Then*

$$\begin{aligned}\sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) \subseteq M\}} q^{\mathrm{nmaj}(\beta)} &= \sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) \subseteq M\}} q^{\ell_B(\beta)} = \sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) \subseteq M\}} q^{\mathrm{fmaj}(\beta)} \\ &= \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{i=m_1+1}^{n} (1 + q^i). \end{aligned} \tag{15}$$

Proof. Let us denote by $\mathrm{Sh}(r_1, \ldots, r_t)$ the set of signed permutations obtained as shuffles of the sequences in (10), with prescribed $r_1, \ldots, r_t$. From Theorem 4.1 it follows that

$$\sum_{\beta \in \mathrm{Sh}(r_1, \ldots, r_t)} q^{\mathrm{maj}(\beta)} = \sum_{\beta \in \mathrm{Sh}(r_1, \ldots, r_t)} q^{\mathrm{inv}(\beta)} = \begin{bmatrix} n \\ m_1,\ r_1 - m_1, \ldots, r_t - m_t,\ n - r_t \end{bmatrix}_q. \tag{16}$$

In fact inversion number and major index of a shuffle depend only on the order of the elements in the shuffled sequences. From this, and the definitions of $\mathrm{nmaj}(\beta) = \mathrm{maj}(\beta) + \mathrm{N}_1(\beta) + \mathrm{N}_2(\beta)$ and of $\ell_B(\beta) = \mathrm{inv}(\beta) + \mathrm{N}_1(\beta) + \mathrm{N}_2(\beta)$, the first equality in (15) follows. The second equality and the sum have been computed in [3]. The symbol fmaj denote the *flag-major index* introduced by Adin and Roichman in [1]. □

By the Principle of Inclusion-Exclusion we obtain

Corollary 4.3.

$$\sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) = M\}} q^{\mathrm{nmaj}(\beta)} = \sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) = M\}} q^{\ell_B(\beta)} = \sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) = M\}} q^{\mathrm{fmaj}_B(\beta)}.$$

The following lemma will be useful in the computation of our main result Theorem 4.5.

Lemma 4.4. *Let $n \in \mathbb{P}$ and $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$. Then*

$$\sum_{\{\beta \in B_n | \mathrm{Des}_B(\beta^{-1}) \subseteq M\}} q^{\ell_D(\beta)} = \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{i=m_1}^{n-1} (1 + q^i).$$

Proof. Let $\beta \in B(M)$. Recall that $\ell_D(\beta) = \ell_B(\beta) - \mathrm{N}_1(\beta)$, and that $\ell_B(\beta) = \mathrm{inv}(\beta) + \sum_{\beta(i)<0} |\beta(i)|$. Note that $\beta(i) < 0$ if and only if there exists a j such that $m_j + 1 \leq |\beta(i)| \leq r_j$. Therefore

$$\begin{aligned}\sum_{\beta(i)<0} |\beta(i)| &= \sum_{i=1}^{t} (m_i + 1) + \ldots + r_i \\ &= \sum_{i=1}^{t} \left[(r_i - m_i) m_i + \frac{(r_i - m_i)(r_i - m_i + 1)}{2} \right] \\ &= \sum_{i=1}^{t} \frac{1}{2} (r_i - m_i)(r_i + m_i + 1). \end{aligned}$$

Moreover $\mathrm{N}_1(\beta) = \sum_{i=1}^{t}(r_i - m_i)$, and so

$$\begin{aligned}\ell_D(\beta) &= \mathrm{inv}(\beta) + \sum_{i=1}^{t} \frac{1}{2}(r_i - m_i)(r_i + m_i + 1) - (r_i - m_i) \\ &= \mathrm{inv}(\beta) + \sum_{i=1}^{t} \binom{r_i - m_i + 1}{2} + (r_i - m_i)(m_i - 1)\end{aligned}$$

Hence by (16)

$$\begin{aligned}\sum_{\beta\in B(M)} q^{\ell_D(\beta)} &= \sum_{r_1,\ldots,r_t} \sum_{\beta\in \mathrm{Sh}(r_1,\ldots,r_t)} q^{\mathrm{inv}(\beta)} q^{\sum_{i=1}^{t}\binom{r_i-m_i+1}{2}+(r_i-m_i)} \\ &= \sum_{r_1,\ldots,r_t} \begin{bmatrix} n \\ m_1,\ r_1 - m_1, \ldots, n - r_t \end{bmatrix}_q \cdot q^{\sum_{i=1}^{t}\binom{r_i-m_i+1}{2}+(r_i-m_i)(m_i-1)} \\ &= \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{i=1}^{t} \sum_{r_i=m_i}^{m_{i+1}} \begin{bmatrix} m_{i+1} - m_i \\ r_i - m_i \end{bmatrix}_q \cdot q^{\binom{r_i-m_i+1}{2}+(r_i-m_i)(m_i-1)} \\ &= \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{i=1}^{t} \prod_{j=m_i}^{m_{i+1}-1} (1 + q^j) \qquad (17) \\ &= \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{j=m_1}^{n-1} (1 + q^j)\end{aligned}$$

where the sum runs over $m_i \le r_i \le m_{i+1}$, and (17) is obtained by applying the q-binomial Theorem 2.1 with $x = q^{(m_i-1)}$. □

Theorem 4.5. *Let $n \in \mathbb{P}$ and $M = \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n-1]$. Then*

$$\begin{aligned}\sum_{\gamma\in D(M)} q^{\mathrm{dmaj}(\gamma)} &= \sum_{\gamma\in D(M)} q^{\ell_D(\gamma)} \\ &= \begin{cases} \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{i=1}^{n-1}(1 + q^i) & \text{if } 0 \in M; \\ \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \prod_{i=m_1}^{n-1}(1 + q^i) & \text{if } 0, 1 \notin M; \\ \begin{bmatrix} n \\ m_1,\ m_2 - m_1, \ldots, n - m_t \end{bmatrix}_q \cdot \dfrac{\prod_{i=1}^{n-1}(1 + q^i)}{[m_2]_q} & \text{if } 0 \notin M, \text{ and } 1 \in M. \end{cases}\end{aligned}$$

Proof. Once again the first equality follows from (16) and the definitions of dmaj and ℓ_D. The computation of the sum is now an easy application of Lemma 4.4, together with Propositions 3.4, 3.5, and 3.6. □

As corollary we obtain the desired generalization.

Corollary 4.6.

$$\sum_{\{\gamma\in D_n|\mathrm{Des}_D(\gamma^{-1})=M\}} q^{\mathrm{dmaj}(\gamma)} = \sum_{\{\gamma\in D_n|\mathrm{Des}_D(\gamma^{-1})=M\}} q^{\ell_D(\gamma)}.$$

Remark 4.7. If we replace Des_B with the usual descent set Des, Corollary 4.3 is still valid. It easily follows from Theorem 4.2 since $\mathrm{Des}(\beta^{-1}) \subseteq M$ if and only of $\mathrm{Des}_B(\beta^{-1}) \subseteq M \cup \{0\}$. Analogously, by replacing Des_D with Des, Corollary 4.6 holds for the Coxeter group of type D.
The two corollaries are not true if as descent set one chooses NDes for B_n and DDes for D_n.

5 Symmetry of the joint distribution

In this section we find generalizations of Foata-Schützenberger Theorem 1.2, Roselle Theorem 1.3, and Gessel Theorem 1.4.

The following is an easy computation.

Lemma 5.1. *Let $n \in \mathbb{P}$. Then*

$$\sum_{u\in B^J} p^{\mathrm{N}_1(u)} q^{\mathrm{N}_1(u)+\mathrm{N}_2(u)} = \sum_{S\subseteq[n]} p^{|S|} q^{\sum_{i\in S} i} = \prod_{i=1}^{n} (1+pq^i) = (-pq;q)_n.$$

Moreover

$$\sum_{u\in D^J} p^{\mathrm{N}_1(u)+\epsilon(u)} q^{\mathrm{N}_2(u)} = \sum_{S\subseteq[n-1]} p^{|S|} q^{\sum_{i\in S} i} = \prod_{i=1}^{n-1} (1+pq^i) = (-pq;q)_{n-1}.$$

Proposition 5.2. *The distribution of (nmaj, ℓ_B) over B_n is symmetric, namely*

$$B_n(t,q) := \sum_{\beta\in B_n} t^{\mathrm{nmaj}(\beta)} q^{\ell_B(\beta)} = \sum_{\beta\in B_n} t^{\ell_B(\beta)} q^{\mathrm{nmaj}(\beta)}$$

Proof. Let consider the decomposition (8) of B_n. Let $u \in B^J$ (or D^J) and $\sigma \in S_n$. Then the following equalities hold

$$\mathrm{maj}(u\sigma) = \mathrm{maj}(\sigma) \ \text{ and } \ \mathrm{inv}(u\sigma) = \mathrm{inv}(u).$$

Moreover

$$\mathrm{N}_1(u\sigma) = \mathrm{N}_1(u) \ \text{ and } \ \mathrm{N}_2(u\sigma) = \mathrm{N}_2(u).$$

Then from Theorem 1.2 it follows

$$\begin{aligned}
\sum_{\beta\in B_n} t^{\ell_B(\beta)}q^{\mathrm{nmaj}(\gamma)} &= \sum_{u\in B^J}\sum_{\sigma\in S_n} t^{\mathrm{inv}(u\sigma)+\mathrm{N}_1(u\sigma)+\mathrm{N}_2(u\sigma)}q^{\mathrm{maj}(u\sigma)+\mathrm{N}_1(u\sigma)+\mathrm{N}_2(u\sigma)}\\
&= \sum_{u\in B^J} t^{\mathrm{N}_1(u)+\mathrm{N}_2(u)}q^{\mathrm{N}_1(u)+\mathrm{N}_2(u)}\sum_{\sigma\in S_n} t^{\mathrm{inv}(\sigma)}q^{maj(\sigma)}\\
&= \sum_{u\in B^J} t^{\mathrm{N}_1(u)+\mathrm{N}_2(u)}q^{\mathrm{N}_1(u)+\mathrm{N}_2(u)}\sum_{\sigma\in S_n} t^{\mathrm{maj}(\sigma)}q^{\mathrm{inv}(\sigma)}\\
&= \sum_{u\in B^J}\sum_{\sigma\in S_n} t^{\mathrm{maj}(u\sigma)+\mathrm{N}_1(u\sigma)+\mathrm{N}_2(u\sigma)}q^{\mathrm{inv}(u\sigma)+\mathrm{N}_1(u\sigma)+\mathrm{N}_2(u\sigma)}\\
&= \sum_{\beta\in B_n} t^{\mathrm{nmaj}(\beta)}q^{\ell_B(\beta)}.
\end{aligned}$$

□

The analogous result holds for D_n. The proof is very similar to that of B_n and is left to the reader.

Proposition 5.3. *The pair of statistics* (dmaj, ℓ_D) *is symmetric, namely*

$$D_n(t,q) := \sum_{\gamma\in D_n} t^{\mathrm{dmaj}(\gamma)}q^{\ell_D(\gamma)} = \sum_{\gamma\in D_n} t^{\ell_D(\gamma)}q^{\mathrm{dmaj}(\gamma)}.$$

Note that, the flag-major index and the *D-major index* [7] do not share with nmaj and dmaj this symmetric distribution property.

The following identities are generalizations of Theorem 1.3 of Roselle to B_n and D_n. They easily follow from the proof of Proposition 5.2, Lemma 5.1, and from Theorem 1.3.

Proposition 5.4 (Roselle Identities for B_n and D_n)**.**

$$\begin{aligned}
\sum_{n\geq 0} B_n(t,q)\frac{u^n}{(t;t)_n(q;q)_n(-qt;qt)_n} &= \frac{1}{(u;t,q)_{\infty,\infty}}, \quad (B_0(t,q) := 0);\\
1+\sum_{n\geq 1} D_n(t,q)\frac{u^n}{(t;t)_n(q;q)_n(-qt;qt)_{n-1}} &= \frac{1}{(u;t,q)_{\infty,\infty}}.
\end{aligned}$$

Similarly the following identities, which generalize Gessel formula, follow from the proof of Proposition 5.2, Lemma 5.1, and Theorem 1.4.

Proposition 5.5 (Gessel Identities for B_n and D_n)**.**

$$\begin{aligned}
\sum_{n\geq 0}\frac{u^n}{[n]_q!}\frac{\sum_{\beta\in B_n} t^{\mathrm{nmaj}(\sigma)}q^{\ell_B(\beta)}p^{\mathrm{ndes}(\beta)}}{(-tqp;tq)_n(t;q)_{n+1}} &= \sum_{k\geq 0} p^k e[u]_q e[tu]_q\cdots e[t^k u]_q;\\
\frac{1}{1-t}+\sum_{n\geq 1}\frac{u^n}{[n]_q!}\frac{\sum_{\gamma\in D_n} t^{\mathrm{dmaj}(\gamma)}q^{\ell_D(\gamma)}p^{\mathrm{ddes}(\gamma)}}{(-tqp;tq)_{n-1}(t;q)_{n+1}} &= \sum_{k\geq 0} p^k e[u]_q e[tu]_q\cdots e[t^k u]_q.
\end{aligned}$$

6 Concluding remarks

As we mentioned along the paper, there exists another family of statistics, the *flag-statistics*, defined on Coxeter groups of type B, D (see [1] and [7]), and more generally on complex reflection groups [5]. Several generating functions involving flag-statistics have already been computed. In particular, we refer to the series of papers of Foata and Han [10, 11, 12], for a complete overview on the argument.

We remark that among the series computed, none involve a combination of flag-statistics and length. This is why we conclude the paper with the following interesting proposal.

Problem 6.1. What kind of identities, generalizing the ones of Roselle and Gessel, might be obtained by using flag-statistics ?

References

[1] R. M. Adin and Y. Roichman, The Flag Major Index and Group Actions on Polynomial Rings, Europ. J. Combinatorics, 22 (2001), 431-446.

[2] R. M. Adin, F. Brenti and Y. Roichman, Descent Numbers and Major Indices for the Hyperoctahedral Group, Adv. in Appl. Math., 27 (2001), 210-224.

[3] R. M. Adin, F. Brenti and Y. Roichman, Equi-distribution over descent classes of the Hyperoctahedral Group, J. Combin. Theory Ser. A, 113 (2006), 917-933.

[4] G.E. Andrews, The Theory of Partitions, The Encyclopedia of Mathematics and Its Applications Series, Addison-Wesley Pub. Co., NY, 300 pp. (1976).

[5] E. Bagno and R. Biagioli, Colored-Descent Representations of Complex Reflection Groups $G(r, p, n)$, Israel J. Math., 160 (2007), 317-348.

[6] R. Biagioli, Major and descent statistics for the even-signed permutation group, Adv. in Appl. Math., 31 (2003), 163-179.

[7] R. Biagioli and F. Caselli, Invariant algebras and major indices for classical Weyl groups, Proc. London Math. Soc., 88 (2004), 603-631.

[8] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Grad. Texts in Math. 231, Springer-Verlag, Berlin, 2005.

[9] D. Foata, On the Netto Inversion Number of a Sequence, Proc. Amer. Math. Soc., 19 (1968), 236-240.

[10] D. Foata and G.N. Han, Signed words and permutations, II: the Euler-Mahonian polynomials, Electronic J. Combinatorics, 11(2), R22, (2005).

[11] D. Foata and G.N. Han, Signed words and permutations, III: the MacMahon Verfahren, Sém. Lothar. Combin., vol. 54a, (2006), 20 pages (The Viennot Festschrift).

[12] D. Foata and G.N. Han, Signed words and permutations, V: a sextuple distribution, Ramanujan J., to appear.

[13] D. Foata and M.P. Schützenberger, Major index and inversion number of permutations, Math. Nachr., 83 (1978), 143-159.

[14] A. Garsia and I. Gessel, Permutation statistics and partitions, Adv. Math., 31 (1979), 288-305.

[15] I. Gessel, Generating functions and enumeration of sequences, M.I.T. doctoral thesis, 1977.

[16] F. Hivert, J.C. Novelli, and J.Y. Thibon, Multivariate generalizations of the Foata-Schützenberger equidistribution, Discr. Math. Theo. Comp. Sci., Proc. of Fourth Colloquium on Mathematics and Computer Science, Nancy, 2006.

[17] P.A. MacMahon, Combinatory Analysis, vol. 1, Cambridge Univ. Press, London, 1915.

[18] D. Rawlings, The combinatorics of certain products, Proc. Amer. Math. Soc., 89 (1983), 560-562.

[19] D.P. Roselle, Coefficients associated with the expansion of certain products, Proc. Amer. Math. Soc., 45 (1974), 144-150.

[20] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Stud. Adv. Math., no. 49, Cambridge Univ. Press, Cambridge, 1997.