Invariant Algebras and Major Indices for Classical Weyl Groups [1]

Riccardo Biagioli and Fabrizio Caselli

1 Introduction

Let W be a classical Weyl group, i.e. W is either the symmetric group A_{n-1}, the hyperoctahedral group B_n, or the even-signed permutation group D_n. Consider the natural diagonal and tensor actions of W and W^t, respectively, on the polynomial ring $\mathbf{C}[x_1, \ldots, x_n]^{\otimes t}$ and denote by DIA and TIA the corresponding invariant algebras. Let $\mathcal{Z}_W(\bar{q})$ be the quotient of the Hilbert series of DIA and TIA.

A well known result due to MacMahon [20] asserts that the major index is equidistributed with the length function on the symmetric group. The Euler-Mahonian distribution of descent number and major index was extensively studied (see e.g. [7, 12, 13, 16]) and its generating function is known as Carlitz's identity. Although its nature is combinatorial, the major index has also important algebraic properties. It is known that, if $W = A_{n-1}$, then $\mathcal{Z}_{A_{n-1}}(\bar{q})$ is a polynomial with non-negative integer coefficients, which admits an explicit simple formula in terms of the major index [14]. Moreover, Garsia and Stanton provide a descent basis for the coinvariant algebra of type A whose elements are monomials of degree uqual to the major index of the indexing permutation [15]. The problem of generalizing these results to the hyperoctahedral group has been open for many years. Several authors have defined analogues of the major index for B_n (see, e.g., [8, 9, 10, 19, 21]) but none of these is Mahonian, (i.e. equidistributed with length). Finally in a recent paper [1] Adin and Roichman introduced the flag-major index ($fmaj$) on the hyperoctahedral group. They show that it is Mahonian and find a formula for $\mathcal{Z}_{B_n}(\bar{q})$ by means of this new statistic. In [2] the previous two authors and Brenti give a generalization to B_n of Carlitz's identity. The flag-major index has been further studied in [3]; it plays a crucial role in representation theory, more precisely in the decomposition of the coinvariant algebra into irreducible modules. In [4] the first of the present authors defines the D-flag major index ($fmaj_D$) for the even-signed permutation group, and proves that it is Mahonian. Moreover, he defines a pair of Euler-Mahonian statistics that allows a generalization of Carlitz's identity to D_n. Neither similar formula for $\mathcal{Z}_{D_n}(\bar{q})$ nor other algebraic properties have been found so far.

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The purpose of this work is to introduce a new "major" statistic on D_n from which one can generalize all the combinatorial and algebraic properties known for type A and B. In particular we would like to find an explicit formula for $\mathcal{Z}_{D_n}(\bar{q})$ using this new statistic. Toward this end, we define two new Mahonian statistics ned_D and $Dmaj$. The latter $Dmaj$, defined in a combinatorial way, has the analogous algebraic meaning for D_n, as the major index for S_n, and $fmaj$ for B_n; namely, it allows us to find an explicit formula for $\mathcal{Z}_{D_n}(\bar{q})$ which implies, in particular, that this series (as in types A and B) is actually a polynomial with non-negative integer coefficients. To prove the results we introduce suitable even and odd t-partite partitions. These are related with the t-partite partitions introduced by Gordon in [17], and further studied by Garsia and Gessel in [14] where applications to the permutation enumerations are shown. Using similar ideas, we define the Mahonian statistic ned_B on B_n and we find a new and simpler proof of the Adin-Roichman formula for $\mathcal{Z}_{B_n}(\bar{q})$. Finally, we define a new descent number $Ddes$ on D_n so that the pair $(Ddes, Dmaj)$ gives a generalization to D_n of Carlitz's identity. In a upcoming paper [5] we show that $Dmaj$ and $Ddes$ play an important role in the decomposition in irreducible submodules of the coinvariant algebra of type D.

The organization of the paper is as follows. In §2 we introduce some preliminaries and notation. In particular we present some combinatorial properties of classical Weyl groups and we define their actions on the polynomial rings. In §3 we define several new combinatorial tools that are needed in the rest of our work and we prove some of their fundamental properties: we introduce the concept of parity of a partition and the new statistics ned_B, ned_D and $Dmaj$. In §4 we study some combinatorial properties of $Dmaj$. We prove that it is equidistributed with length and that, together with $Ddes$, satisfies the Carlitz's identity for D_n. Moreover we define another descent statistic on D_n, $fdes_D$, solving a problem stated in [4]. Section 5 is devoted to the proof of our main result, i.e. we find an explicit formula for the polynomial $\mathcal{Z}_{D_n}(\bar{q})$ using $Dmaj$. In §6 we show how the ideas developed for D_n can be applied to give a new and simpler proof of the analogous formula for $\mathcal{Z}_{B_n}(\bar{q})$.

2 Notation, Definitions and Preliminaries

2.1 Classical Weyl Groups

In this section we give some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} := \{1,2,3,\dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers and $\mathbf{C}$ be the field of complex numbers; for $a \in \mathbf{N}$ we let $[a] := \{1, 2, \ldots, a\}$ (where $[0] := \emptyset$). Given $a, b \in \mathbf{N}$ we let $[a, b) := \{i \in \mathbf{N} \ : \ min(a, b) \leq i < max(a, b)\}$ and similarly for $n, m \in \mathbf{Z}$, $n \leq m$, $[n, m] := \{n, n+1, \ldots, m\}$. Given $n, m \in \mathbf{Z}$, by $n \equiv m$ we mean $n \equiv m$ (mod 2). For a set A we denote its cardinality by $|A|$ and the set of all its subsets by 2^A. If $A \subseteq [n]$ its complementary set $[n] \setminus A$ will be denoted by $\mathcal{C}_n(A)$. Given two sets A and B we denote by $A \triangle B$ their symmetric difference $(A \cup B) \setminus (A \cap B)$.

We always consider the linear order on $\mathbf{Z}$

$$-1 \prec -2 \prec \cdots \prec -n \prec \cdots \prec 0 \prec 1 \prec 2 \prec \cdots \prec n \prec \cdots$$

instead of the usual ordering. Given a sequence $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbf{Z}^n$ we let $Neg(\sigma) := \{i \in [n] : \sigma_i < 0\}$ and $neg(\sigma) := |Neg(\sigma)|$. We say that $(i, j) \in [n] \times [n]$ is an *inversion* of σ if $i < j$ and $\sigma_i \succ \sigma_j$ and that $i \in [n-1]$ is a *descent* of σ if $\sigma_i \succ \sigma_{i+1}$. We denote by $Inv(\sigma)$ and $Des(\sigma)$ the set of inversions and the set of descents of σ and by $inv(\sigma)$ and $des(\sigma)$ their cardinalities, respectively. Moreover we define, for all $i \in [n-1]$

$$\varepsilon_i(\sigma) := \begin{cases} 1, & \text{if } \sigma_i > \sigma_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We also set $\varepsilon_n(\sigma) := 0$. Finally we let

$$maj(\sigma) := \sum_{i \in Des(\sigma)} i$$

and call it the *major index* of σ.

Given a set A we let $S(A)$ be the set of all bijections $\tau : A \to A$, and $S_n := S([n])$. If $\sigma \in S_n$ then we write $\sigma = \sigma_1 \ldots \sigma_n$ to mean that $\sigma(i) = \sigma_i$, for $i = 1, \ldots, n$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau := \sigma \circ \tau$ (composition of functions) so that, for example, $1423 \cdot 2134 = 4123$.

Given a variable q and a commutative ring R we denote by $R[q]$ (respectively, $R[[q]]$) the ring of polynomials (respectively, formal power series) in q with coefficients in R. For $i \in \mathbf{N}$ we let, as is customary, $[i]_q := 1 + q + q^2 + \ldots + q^{i-1}$ (so $[0]_q = 0$).

For $n \in \mathbf{P}$ we let

$$A_n(t, q) := \sum_{\sigma \in S_n} t^{des(\sigma)} q^{maj(\sigma)},$$

and $A_0(t, q) := 1$. For example, $A_3(t, q) = 1 + 2tq^2 + 2tq + t^2q^3$. The following result is due to Carlitz, and we refer the reader to [7] for its proof, (see also [3] for a refinement).

Theorem 2.1 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r\geq 0}[r+1]_q^n t^r = \frac{A_n(t,q)}{\prod_{i=0}^n (1-tq^i)}$$

in $\mathbf{Z}[q][[t]]$.

Let B_n be the group of all bijections β of the set $[-n,n]\setminus\{0\}$ onto itself such that

$$\beta(-i) = -\beta(i)$$

for all $i \in [-n,n]\setminus\{0\}$, with composition as the group operation. If $\beta \in B_n$ then, following [6], we write $\beta = [\beta_1,\ldots,\beta_n]$ to mean that $\beta(i) = \beta_i$, for $i = 1,\ldots,n$, and call this the *window* notation of β. The group B_n is often called the group of all *signed permutations* on $[n]$ or the *hyperoctahedral group* of rank n.
We find it convenient to introduce this *pair* notation: for each $\sigma \in S_n$ and $H \subseteq [n]$, we let $(\sigma,H) := [\beta_1,\ldots,\beta_n]$ be the signed permutation defined as follows:

$$\beta_i := \begin{cases} -\sigma_i, & \text{if } i \in H, \\ \sigma_i, & \text{if } i \notin H. \end{cases}$$

Note that in this notation we have

$$(\sigma,H)^{-1} = (\sigma^{-1},\sigma(H)) \tag{2}$$

and

$$(\sigma,H)(\tau,K) = (\sigma\tau, K\triangle\tau^{-1}(H)) \tag{3}$$

For example, if $(\sigma,H) = (43512,\{1,2,5\}) = [-4,-3,5,1,-2] \in B_5$ then $(\sigma,H)^{-1} = (45213,\{2,3,4\}) = [4,-5,-2,-1,3]$ and if $(\tau,K) = (21345,\{2,5\})$ then $(\sigma,H)(\tau,K) = (34512,\{1\})$. It is well known (see, e.g., [6]) that B_n is a Coxeter group with respect to the generating set $S := \{s_0,s_1,\ldots,s_{n-1}\}$ where

$$s_0 := [-1,2,3,\ldots,n]$$

and

$$s_i := [1,2,\ldots,i-1,i+1,i,i+2,\ldots,n] \tag{4}$$

for $i = 1,\ldots,n-1$. This gives rise to another natural statistic on B_n, the *length* (similarly definable for any Coxeter group),

$$\ell(\beta) := min\{r \in \mathbf{N} : \beta = s_{i_1}\ldots s_{i_r} \text{ for some } i_1,\ldots,i_r \in [0,n-1]\}.$$

There is a well known direct combinatorial way to compute this statistic (see, e.g., [6]), namely, for $\beta \in B_n$,

$$\ell(\beta) = inv(\beta) - \sum_{j \in Neg(\beta)} \beta(j).$$

Following [1] and [2] we define the *flag-major index* of $\beta \in B_n$ by

$$fmaj(\beta) := 2maj(\beta) + neg(\beta) \tag{5}$$

and the *flag descent number* of β by

$$fdes(\beta) := 2des(\beta) + \eta(\beta), \tag{6}$$

where

$$\eta(\beta) := \begin{cases} 1, & \text{if } \beta(1) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For example, if $\beta = [-4, -3, 5, 1, -2] \in B_5$ then $fmaj(\beta) = 2 \cdot 8 + 3 = 19$ and $fdes(\beta) = 2 \cdot 3 + 1 = 7$.
It is known that $fmaj$ is equidistributed with length on B_n, (see [1, Theorem 2] where $fmaj$ is denoted $flag - major$).
The pair of statistics $(fdes, fmaj)$ gives a generalization of Carlitz's identity (Theorem 2.1) to B_n. More precisely we have the following theorem due to Adin, Brenti and Roichman [2] (see also [3] for a refinement).

Theorem 2.2 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\beta \in B_n} t^{fdes(\beta)} q^{fmaj(\beta)}}{(1-t) \prod_{i=1}^{n} (1 - t^2 q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

We denote by D_n the subgroup of B_n consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

$$D_n := \{\gamma \in B_n : neg(\gamma) \equiv 0\}.$$

As for B_n we introduce a *pair* notation: for each $\sigma \in S_n$ and $K \subseteq [n-1]$ we let $(\sigma, K)_D := [\gamma_1, \ldots, \gamma_n]$ be the unique even-signed permutation γ such that $|\gamma_i| = \sigma_i$ for all $i \in [n]$ and $K \cup \{n\} \supseteq Neg(\gamma) \supseteq K$. More precisely

$$\gamma_i := \begin{cases} -\sigma_i, & \text{if } i \in K, \\ \sigma_i, & \text{if } i \notin K \cup \{n\}, \\ (-1)^{|K|}\sigma_n, & \text{if } i = n. \end{cases}$$

For example $(54312, \{1,3,4\})_D = [-5,4,-3,-1,-2] \in D_5$. We will usually omit the index D in the pair notation of D_n when there is no risk of confusion with the pair notation of B_n.
It is well known (see, e.g., [6]) that D_n is a Coxeter group with respect to the generating set $S := \{s_0^D, s_1, \ldots, s_{n-1}\}$ where

$$s_0^D := [-2, -1, 3, \ldots, n]$$

and s_i is defined as in (4), for $i \in [n-1]$. There is a well known direct combinatorial way to compute the length of $\gamma \in D_n$ (see, e.g., [6]), namely

$$\ell(\gamma) = inv(\gamma) - \sum_{j \in Neg(\gamma)} \gamma(j) - neg(\gamma).$$

Following [4], for every $\gamma \in D_n$ we define the *D-negative multiset*

$$DDes(\gamma) := Des(\gamma) \biguplus \{-\gamma(i) - 1 \ : \ i \in Neg(\gamma)\} \setminus \{0\},$$

$$ddes(\gamma) := |DDes(\gamma)|$$

and

$$dmaj(\gamma) := \sum_{i \in DDes(\gamma)} i.$$

For example, if $\gamma = [-5,4,-3,-1,-2] \in D_5$ then $DDes(\gamma) = \{1, 2^2, 3, 4\}$, $ddes(\gamma) = 5$ and $dmaj(\gamma) = 12$.
The pair of statistics $(ddes, dmaj)$ gives a generalization of Carlitz's identity to D_n. More precisely, we have the following theorem, (see [4, Theorem 3.4]).

Theorem 2.3 *Let $n \in \mathbf{P}$. Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\gamma \in D_n} t^{ddes(\gamma)} q^{dmaj(\gamma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

2.2 Group Actions on Polynomial Rings.

Let W be a classical Weyl group, i.e $W = S_n, B_n$ or D_n. There is a natural action of W on the polynomial ring $\mathbf{P}_n := \mathbf{C}[x_1, \ldots, x_n]$, $\varphi : W \to \mathrm{Aut}(\mathbf{P}_n)$ defined on the generators by

$$\varphi(w) : x_i \mapsto \frac{w(i)}{\mid w(i) \mid} x_{|w(i)|},$$

for all $w \in W$ and extended uniquely to an algebra homomorphism. This action gives rise to two actions on the tensor power $\mathbf{P}_n^{\otimes t} := \mathbf{P}_n \otimes \cdots \otimes \mathbf{P}_n$ (t-times): the natural *tensor action* φ_T of $W^t := W \times \cdots \times W$ (t-times), and the *diagonal action* of W on $\mathbf{P}_n^{\otimes t}$, $\varphi_D := \varphi_T \circ d$ defined using the diagonal embedding $d : W \hookrightarrow W^t$, $w \mapsto (w, \ldots, w)$.

The *tensor invariant algebra*

$$\mathrm{TIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_T(\bar{w})\bar{p} = \bar{p} \ \text{ for all } \bar{w} \in W^t\}$$

is a subalgebra of the *diagonal invariant algebra*

$$\mathrm{DIA} := \{\bar{p} \in \mathbf{P}_n^{\otimes t} : \varphi_D(w)\bar{p} = \bar{p} \ \text{ for all } w \in W\}.$$

These two algebras are naturally multigraded and hence we can consider the corresponding Hilbert series

$$F_D(\bar{q}) := \sum_{n_1, \ldots, n_t} \dim_{\mathbf{C}}(\mathrm{DIA}_{n_1, \ldots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

$$F_T(\bar{q}) := \sum_{n_1, \ldots, n_t} \dim_{\mathbf{C}}(\mathrm{TIA}_{n_1, \ldots, n_t}) q_1^{n_1} \cdots q_t^{n_t},$$

where $\mathrm{DIA}_{n_1, \ldots, n_t}$ and $\mathrm{TIA}_{n_1, \ldots, n_t}$ are the homogeneous components of multi-degree $(n_1, \ldots, n_t)$ in DIA and TIA respectively and $\bar{q} = (q_1, \ldots, q_t)$.

We denote the quotient series by

$$\mathcal{Z}_W(\bar{q}) := \frac{F_D(\bar{q})}{F_T(\bar{q})} \in \mathbf{Z}[[\bar{q}]].$$

3 New Statistics on B_n and D_n

In this section we introduce some new combinatorial objects and we prove some preliminary results that are used in the proof of our main result (Theorem 5.12).

3.1 Bijections and Parity Sets

Definition. We define a map $\varphi_n : 2^{[n]} \to 2^{[n]}$, for every $n \in \mathbf{N}$, in the following inductive way: for $n \geq 1$,

$$\varphi_n(H) := \begin{cases} \mathcal{C}_n \varphi_{n-1}(H), & \text{if } H \subseteq [n-1], \\ \varphi_{n-1}(H \setminus \{n\}), & \text{if } H \not\subseteq [n-1], \end{cases}$$

and $\varphi_0(\emptyset) := \emptyset$.

Lemma 3.1 *The map $\varphi_n : 2^{[n]} \to 2^{[n]}$, is a bijection for every $n \in \mathbf{N}$.*

Proof. It suffices to show that φ_n is injective. We proceed by induction on n. If $n = 0$, it is trivial so suppose $n > 0$. Let $H_1 \neq H_2 \in 2^{[n]}$ be such that $\varphi(H_1) = \varphi(H_2)$. Then necessarily we have that either $n \in H_1 \cap H_2$ or $n \notin H_1 \cup H_2$. In both cases we can easily conclude by the definition of φ_n and our induction hypothesis. ■

For example, let $n = 4$ and $H = \{2\}$, then,

$$\begin{aligned} \varphi_4(\{2\}) &= \mathcal{C}_4\varphi_3(\{2\}) = \mathcal{C}_4\mathcal{C}_3\varphi_2(\{2\}) = \mathcal{C}_4\mathcal{C}_3\varphi_1(\emptyset) = \mathcal{C}_4\mathcal{C}_3\mathcal{C}_1\varphi_0(\emptyset) \\ &= \mathcal{C}_4\mathcal{C}_3(\{1\}) = \mathcal{C}_4(\{2,3\}) = \{1,4\}. \end{aligned}$$

There is also a direct way to compute φ_n.

Lemma 3.2 *Let $n \in \mathbf{N}$ and $H \subseteq [n]$. Then*

$$\varphi_n(H) = \{i \in [n] : \ |[i,n] \setminus H| \equiv 1\}.$$

Proof. We proceed by induction on n. If $n = 0$ it is trivial, so suppose $n \geq 1$. If $n \in H$ we have

$$\begin{aligned} \varphi_n(H) &= \varphi_{n-1}(H \setminus \{n\}) \\ &= \{i \in [n-1] : |[i, n-1] \setminus (H \setminus \{n\})| \equiv 1\} \\ &= \{i \in [n] : |[i,n] \setminus H| \equiv 1\}. \end{aligned}$$

The case $n \notin H$ is similar and is left to the reader. ■

Our goal is to understand the action of a permutation σ on $\varphi_n(H)$. For this it is useful to introduce the following concept. From now on denote with $\mathcal{B}_{n,1}$ the set of all integer partitions with at most n parts.

Definition. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, 0, \ldots) \in \mathcal{B}_{n,1}$. We define the *parity set* of λ to be

$$H(\lambda) := \{i \in [n] \ : \ \lambda_i - \lambda_{i+1} \equiv 0\}.$$

Definition. Let $\sigma \in S_n$ and $H \subseteq [n]$. Let $\lambda \in \mathcal{B}_{n,1}$ be such that $H = H(\lambda)$. Then we define

$$H^{\sigma} := H(\mu),$$

where μ is any partition in $\mathcal{B}_{n,1}$ such that $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$.

Note that the definition of H^{σ} does not depend on λ and μ but only on H and σ. Observe that the following statements are equivalent:

i) $(H(\lambda))^{\sigma} = H(\mu)$;

ii) $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$.

For example, suppose $n = 4$ and $\sigma = 4312$. Let $\lambda_i = p_i + \cdots + p_n$ and $\mu_i = r_i + \cdots + r_n$, for $i = 1, \ldots, n$. The condition $\lambda_i + \mu_{\sigma(i)} \equiv 0$ for all $i \in [n]$ is equivalent to the following system of congruences:

$$\begin{cases} p_1 + p_2 + p_3 + p_4 \equiv r_4 \\ p_2 + p_3 + p_4 \equiv r_3 + r_4 \\ p_3 + p_4 \equiv r_1 + r_2 + r_3 + r_4 \\ p_4 \equiv r_2 + r_3 + r_4. \end{cases}$$

If $H = \{1, 3\}$ is the parity set of λ then p_1, p_3 are even, and p_2, p_4 are odd. All these conditions force r_3, r_4 to be even and r_1, r_2 to be odd, hence $H^{\sigma} = \{3, 4\}$.

It is also possible to give an explicit direct description of H^{σ}.

Lemma 3.3 *Let $n \in \mathbf{N}, H \subseteq [n]$ and $\sigma \in S_n$. Then*

$$H^{\sigma} = \{i \in [n] : |[\sigma^{-1}(i), \sigma^{-1}(i+1)) \setminus H| \equiv 0\},$$

where $\sigma^{-1}(n+1) := n+1$.

Proof. Let λ be a partition with parity set H and set $p_i := \lambda_i - \lambda_{i+1}$. Let μ be a partition such that $\lambda_i + \mu_{\sigma(i)} \equiv 0$ and set $r_i := \mu_i - \mu_{i+1}$. Then, by definition, $i \in H^{\sigma}$ if and only if r_i is even. But

$$\begin{aligned} r_i &= \mu_i - \mu_{i+1} \\ &\equiv \lambda_{\sigma^{-1}(i)} - \lambda_{\sigma^{-1}(i+1)} \\ &\equiv \sum_{j \in [\sigma^{-1}(i), \sigma^{-1}(i+1))} p_j \end{aligned}$$

and the result follows. ∎

We can now prove the main technical result of this section.

Lemma 3.4 *Let $n \in \mathbf{N}$. Then for all $H \subseteq [n]$ and $\sigma \in S_n$ we have*

$$\sigma\varphi_n(H) = \varphi_n(H^\sigma).$$

Proof. From Lemma 3.2 we have that

$$i \in \sigma\varphi_n(H) \iff |[\sigma^{-1}(i), n] \setminus H| \equiv 1,$$

and

$$i \in \varphi_n(H^\sigma) \iff |[i, n] \setminus H^\sigma| \equiv 1.$$

The latter condition is equivalent to the following statement: the number of the following congruences

$$\begin{aligned} |[\sigma^{-1}(i), \sigma^{-1}(i+1)) \setminus H| &\equiv 0 \\ |[\sigma^{-1}(i+1), \sigma^{-1}(i+2)) \setminus H| &\equiv 0 \\ &\vdots \\ |[\sigma^{-1}(n), \sigma^{-1}(n+1)) \setminus H| &\equiv 0 \end{aligned}$$

which are not satisfied is congruent to 1. Hence the sum of the members in the left-hand side is congruent to 1. But

$$\sum_{j=i}^{n} |[\sigma^{-1}(j), \sigma^{-1}(j+1)) \setminus H| \equiv |[\sigma^{-1}(i), n+1) \setminus H|$$

and we are done. ■

Note that Lemma 3.4 implies that $(\sigma, H) \mapsto H^\sigma$ is a left action of S_n on $2^{[n]}$.

Definition. Let $p : 2^{[n]} \to 2^{[n-1]}$ be the following projection of sets

$$H \mapsto \begin{cases} H, & \text{if } n \notin H, \\ \mathcal{C}_n(H), & \text{if } n \in H. \end{cases} \tag{7}$$

Definition. Let $\sigma \in S_n$, $H \subseteq [n]$ and $\lambda \in \mathcal{B}_{n,1}$ be such that $H(\lambda) = H$. We define

$$\overline{H^\sigma} := H(\mu) \tag{8}$$

where $\mu \in \mathcal{B}_{n,1}$ is such that $\lambda_i + \mu_{\sigma(i)} \equiv 1$ for all $i \in [n]$.

The proof of the following technical lemma is left to the reader.

Lemma 3.5 *Let $\sigma \in S_n$ and $H \subseteq [n]$. Then*

$$\overline{H^\sigma} = H^\sigma \triangle \{n\} = (H \triangle \{n\})^\sigma.$$

Lemma 3.6 *Let $\sigma \in S_n$ and $K \subseteq [n-1]$. Then*

$$\varphi_{n-1}(K^\sigma \setminus \{n\}) = p\left(\sigma \varphi_{n-1}(K)\right).$$

Proof. Suppose $n \notin K^\sigma$. Then, by Lemma 3.4, we have that

$$\varphi_{n-1}(K^\sigma) = \mathcal{C}_n \varphi_n(K^\sigma) = \mathcal{C}_n \sigma \varphi_n(K) = \mathcal{C}_n \sigma \mathcal{C}_n \varphi_{n-1}(K) = \sigma \varphi_{n-1}(K).$$

If $n \in K^\sigma$ we have similarly that

$$\varphi_{n-1}(K^\sigma \setminus \{n\}) = \varphi_n(K^\sigma) = \sigma \varphi_n(K) = \sigma \mathcal{C}_n \varphi_{n-1}(K) = \mathcal{C}_n \sigma \varphi_{n-1}(K)$$

and the result follows. ∎

3.2 Generalization to the Multivariable Case

In this section we generalize the definitions and results given in §3.1 to the multivariable case.

Let $n, t \in \mathbf{N}, \sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Let $\lambda^{(1)}, \ldots, \lambda^{(t)} \in \mathcal{B}_{n,1}$ be such that the parity set $H(\lambda^{(i)}) = H_i$ for all $i \in [t]$. Then we define

$$(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} := H(\mu),$$

where the partition $\mu \in \mathcal{B}_{n,1}$ is such that for all $j \in [n]$, $\lambda^{(1)}_j + \lambda^{(2)}_{\sigma_1(j)} + \cdots + \lambda^{(t)}_{\sigma_{t-1} \cdots \sigma_1(j)} + \mu_{\sigma_t \cdots \sigma_1(j)} \equiv 0$. Note that, as for the one-dimensional case, the definition of $(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)}$ does not depend on the $\lambda^{(i)}$'s and μ but only on the H_i's and σ_i's.

Observe that the following conditions are equivalent:

i) $(H(\lambda^{(1)}), \ldots, H(\lambda^{(t)}))^{(\sigma_1, \ldots, \sigma_t)} = H(\mu)$;

ii) $\lambda^{(1)}_j + \lambda^{(2)}_{\sigma_1(j)} + \cdots + \lambda^{(t)}_{\sigma_{t-1} \cdots \sigma_1(j)} + \mu_{\sigma_t \cdots \sigma_1(j)} \equiv 0$.

The following two technical lemmas are needed for the proof of the main result (Theorem 5.12).

Lemma 3.7 *Let $n, t \in \mathbf{N}$, $\sigma_i \in S_n$ and $H_i \subseteq [n]$ for all $i \in [t]$. Then*

$$(H_1, \ldots, H_t)^{(\sigma_1, \ldots, \sigma_t)} = \mathcal{C}_n^{t+1}\left(H_1^{\sigma_t \cdots \sigma_1} \triangle H_2^{\sigma_t \cdots \sigma_2} \triangle \cdots \triangle H_t^{\sigma_t}\right).$$

Proof. We sketch the proof in the case $t = 2$, for $t > 2$ it is similar. We have to prove that

$$(H_1, H_2)^{(\sigma_1,\sigma_2)} = \mathcal{C}_n(H_1^{\sigma_2\sigma_1} \triangle H_2^{\sigma_2}).$$

Let $\lambda^{(1)}, \lambda^{(2)}, \mu \in \mathcal{B}_{n,1}$ be such that for all $i = 1, 2$, $H(\lambda^{(i)}) = H_i$, and for all $j \in [n]$

$$\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} \equiv \mu_{\sigma_2\sigma_1(j)}. \tag{9}$$

Let $p_j = \lambda_j^{(1)} - \lambda_{j+1}^{(1)}$, $r_j = \lambda_j^{(2)} - \lambda_{j+1}^{(2)}$ and $s_j = \mu_j - \mu_{j+1}$ for all $j \in [n]$. The condition (9) is equivalent to

$$s_j \equiv \sum_{i \in [\sigma_2^{-1}(j), \sigma_2^{-1}(j+1))} r_i + \sum_{i \in [\sigma_1^{-1}\sigma_2^{-1}(j), \sigma_1^{-1}\sigma_2^{-1}(j+1))} p_i,$$

and the thesis follows from Lemma 3.3. ■

The next result says that the bijection φ_n is "almost" distributive with respect to the symmetric difference of sets.

Lemma 3.8 *Let $n \in \mathbf{N}$. Then for all $H_1, \dots, H_t \subseteq [n]$ we have:*

$$\varphi_n(H_1) \triangle \cdots \triangle \varphi_n(H_t) = \varphi_n \mathcal{C}_n^{t+1} (H_1 \triangle \cdots \triangle H_t).$$

Proof. We proceed by induction on t. If $t = 1$ it is trivial, so suppose $t = 2$. In this case we have to prove that

$$\varphi_n(H_1) \triangle \varphi_n(H_2) = \varphi_n \mathcal{C}_n (H_1 \triangle H_2). \tag{10}$$

By Lemma 3.2 the set in the left-hand side is given by the $i \in [n]$ that verify exactly one of the following congruences

$$|[i, n] \setminus H_1| \equiv 1$$
$$|[i, n] \setminus H_2| \equiv 1.$$

Hence

$$\begin{aligned} \varphi_n(H_1) \triangle \varphi_n(H_2) &= \{i \in [n] : |[i, n] \setminus H_1| + |[i, n] \setminus H_2| \equiv 1\} \\ &= \{i \in [n] : |H_1 \triangle H_2 \cap [i, n]| \equiv 1\}. \end{aligned}$$

The set in the right-hand side is

$$\begin{aligned} \varphi_n \mathcal{C}_n(H_1 \triangle H_2) &= \{i \in [n] : |[i, n] \setminus \mathcal{C}_n(H_1 \triangle H_2)| \equiv 1\} \\ &= \{i \in [n] : |H_1 \triangle H_2 \cap [i, n]| \equiv 1\}. \end{aligned}$$

Now suppose $t > 2$. We have

$$\begin{aligned}\varphi_n(H_1)\triangle\cdots\triangle\varphi_n(H_t) &= \left(\varphi_n\mathcal{C}_n^t(H_1\triangle\cdots\triangle H_{t-1})\right)\triangle\varphi_n(H_t)\\ &= \varphi_n\mathcal{C}_n\left(\mathcal{C}_n^t(H_1\triangle\cdots\triangle H_{t-1})\triangle H_t\right)\\ &= \varphi_n\mathcal{C}_n^{t+1}(H_1\triangle\cdots\triangle H_t),\end{aligned}$$

where we have used the induction hypothesis and (10). ∎

We can now prove the following generalization of Lemma 3.4.

Corollary 3.9 *Let $n \in \mathbf{N}$. Then for all $H_1, \cdots, H_t \in [n]$ and $\sigma_1, \ldots, \sigma_t \in S_n$ we have*

$$\sigma_t\cdots\sigma_1\varphi_n(H_1)\triangle\sigma_t\cdots\sigma_2\varphi_n(H_2)\triangle\cdots\triangle\sigma_t\varphi_n(H_t) = \varphi_n\left((H_1,\ldots,H_t)^{(\sigma_1,\ldots,\sigma_t)}\right).$$

Proof. By Lemmas 3.7, 3.8 and 3.4 there follows that

$$\begin{aligned}\varphi_n\left((H_1,\ldots,H_t)^{(\sigma_1,\ldots,\sigma_t)}\right) &= \varphi_n(\mathcal{C}_n^{t+1}\,(H_1^{\sigma_t\cdots\sigma_1}\triangle H_2^{\sigma_t\cdots\sigma_2}\triangle\cdots\triangle H_t^{\sigma_t}))\\ &= \varphi_n(H_1^{\sigma_t\cdots\sigma_1})\triangle\cdots\triangle\varphi_n(H_t^{\sigma_t})\\ &= \sigma_t\cdots\sigma_1\varphi_n(H_1)\triangle\cdots\triangle\sigma_t\varphi_n(H_t).\end{aligned}$$

∎

Let $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Moreover, let $\lambda^{(1)}, \ldots, \lambda^{(t)} \in \mathcal{B}_{n,1}$ be such that $H(\lambda^{(i)}) = H_i$ for all $i \in [n]$. Then we define

$$\overline{(H_1,\ldots,H_t)^{(\sigma_1,\ldots,\sigma_t)}} = H(\mu)$$

where $\mu \in \mathcal{B}_{n,1}$ is such that $\lambda_j^{(1)} + \lambda_{\sigma_1(j)}^{(2)} + \cdots + \lambda_{\sigma_{t-1}\cdots\sigma_1(j)}^{(t)} + \mu_{\sigma_t\cdots\sigma_1(j)} \equiv 1$ for all $j \in [n]$.

The following two results are natural generalizations of Lemmas 3.5 and 3.6 and again we leave the proof of the former to the reader.

Lemma 3.10 *Let $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots, H_t \subseteq [n]$. Then for all $i \in [t]$ we have:*

$$\begin{aligned}\overline{(H_1,\ldots,H_t)^{(\sigma_1,\ldots,\sigma_t)}} &= (H_1,\ldots,H_t)^{(\sigma_1,\ldots,\sigma_t)}\triangle\{n\}\\ &= (H_1,\ldots,H_{i-1},H_i\triangle\{n\},H_{i+1},\ldots,H_t)^{(\sigma_1,\ldots,\sigma_t)}.\end{aligned}$$

Lemma 3.11 *Let $\sigma_1, \ldots, \sigma_{t-1} \in S_n$ and $K_1, \ldots, K_{t-1} \subseteq [n-1]$. Then*

$$\varphi_{n-1}((K_1,\ldots,K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}\setminus\{n\}) = \pi\left(\sigma_{t-1}\cdots\sigma_1\varphi_{n-1}(K_1)\triangle\cdots\triangle\sigma_{t-1}\varphi_{n-1}(K_{t-1})\right).$$

Proof. If $n \notin (K_1, \ldots, K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}$ we have, by Corollary 3.9,

$$\begin{aligned}
\varphi_{n-1}((K_1, \ldots, K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}) &= \mathcal{C}_n\varphi_n((K_1, \ldots, K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}) \\
&= \mathcal{C}_n(\sigma_{t-1}\cdots\sigma_1\varphi_n(K_1)\triangle\cdots\triangle\sigma_{t-1}\varphi_n(K_{t-1})) \\
&= \mathcal{C}_n(\sigma_{t-1}\cdots\sigma_1\mathcal{C}_n\varphi_{n-1}(K_1)\triangle\cdots\triangle\sigma_{t-1}\mathcal{C}_n\varphi_{n-1}(K_{t-1})) \\
&= \mathcal{C}_n^t(\sigma_{t-1}\cdots\sigma_1\varphi_{n-1}(K_1)\triangle\cdots\triangle\sigma_{t-1}\varphi_{n-1}(K_{t-1})).
\end{aligned}$$

Similarly, if $n \in (K_1, \ldots, K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}$ we have that

$$\varphi_{n-1}((K_1, \ldots, K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}) = \mathcal{C}_n^{t+1}(\sigma_{t-1}\cdots\sigma_1\varphi_{n-1}(K_1)\triangle\cdots\triangle\sigma_{t-1}\varphi_{n-1}(K_{t-1}))$$

and the result follows. ∎

3.3 The Statistics *ned* and *Dmaj*

In this section we introduce the fundamental statistics *ned* and *Dmaj* and study some of their basic properties.

For every $\beta \in B_n$ we define $\bar{\beta} \in B_{n-1}$ by deleting the last entry of β and scaling the others as follows

$$\bar{\beta}(i) := \begin{cases} \beta(i), & \text{if } |\beta(i)| < |\beta(n)|, \\ \beta(i) - 1, & \text{if } \beta(i) > 0 \text{ and } |\beta(i)| > |\beta(n)|, \\ \beta(i) + 1, & \text{if } \beta(i) < 0 \text{ and } |\beta(i)| > |\beta(n)|. \end{cases}$$

For example, if $\beta = [-4, -3, 5, 1, -2] \in B_5$ then $\bar{\beta} = [-3, -2, 4, 1]$.

Definition. We let B_n^+ be the set of the signed permutations $\beta \in B_n$ such that $\beta(n) > 0$.

Lemma 3.12 *Let $\beta \in B_n^+$. Then*

$$maj(-\beta) = maj(\beta) + neg(\beta),$$

where $-\beta := [-\beta(1), \ldots, -\beta(n)]$.

Proof. We proceed by induction on n. For $n = 1$ it is true, so let $n > 1$. We have three cases to consider:

i) $\beta(n-1) > \beta(n)$

Then $maj(\beta) = maj(\bar{\beta}) + n - 1$,$maj(-\beta) = maj(-\bar{\beta}) + n - 1$ and $neg(\beta) = neg(\bar{\beta})$. Since $\bar{\beta} \in B^+_{n-1}$ by induction we have

$$maj(-\bar{\beta}) = maj(\bar{\beta}) + neg(\bar{\beta}) \tag{11}$$

and the thesis follows.

ii) $\beta(n) > \beta(n-1) > 0$

Then $maj(\beta) = maj(\bar{\beta})$, $maj(-\beta) = maj(-\bar{\beta})$ and $neg(\beta) = neg(\bar{\beta})$, and the result follows by (11), as above.

iii) $\beta(n-1) < 0$

Then we have $maj(-\beta) = maj(-\bar{\beta}) + n - 1$ and $maj(\beta) = maj(\bar{\beta})$. Since $-\beta \in B^+_{n-1}$ by induction there follows that $maj(\bar{\beta}) = maj(-\bar{\beta}) + neg(-\bar{\beta})$. Hence

$$maj(-\beta) = maj(\bar{\beta}) - neg(-\bar{\beta}) + n - 1 = maj(\beta) - neg(-\bar{\beta}) + n - 1,$$

and the result follows since $neg(-\bar{\beta}) = n - 1 - neg(\beta)$. ■

Corollary 3.13 *Let* $\beta \in B^+_n$. *Then*

$$fmaj(-\beta) = fmaj(\beta) + n.$$

Proof. This follows immediately from $neg(-\beta) = n - neg(\beta)$ and Lemma 3.12. ■

The verification of the following observation is left to the reader.

Lemma 3.14 *Let* $\sigma \in S_n$ *and* $H \subseteq [n]$. *Then*

$$\overline{(\sigma, \varphi_n(H))} := \begin{cases} -(\bar{\sigma}, \varphi_{n-1}(H)), & \text{if } n \notin H, \\ (\bar{\sigma}, \varphi_{n-1}(H \setminus \{n\})), & \text{if } n \in H. \end{cases}$$

Recall the definition of $\varepsilon_i(\sigma)$ given in (1). We are ready to introduce two new fundamental statistics for this work.

Definition. For $(\sigma, H) \in B_n$ we let

$$ned_B(\sigma, H) := \sum_{i \in H} 2i\varepsilon_i(\sigma) + \sum_{i \in \mathcal{C}_n(H)} i. \tag{12}$$

For $(\sigma, K)_D \in D_n$ we let

$$ned_D(\sigma, K) := \sum_{i \in K} 2i\varepsilon_i(\sigma) + \sum_{i \in \mathcal{C}_{n-1}(K)} i. \tag{13}$$

For example, if $\beta = [-2, 4, -3, -1] = (2431, \{1, 3, 4\}) \in B_4$ then $ned_B(\beta) = 2{\cdot}3+2 = 8$ and if $\gamma = [2, 4, -3, -1] = (2431, \{3\}) \in D_4$ then $ned_D(\gamma) = 2 \cdot 3 + 1 + 2 = 9$. The main property of ned_B is the following one.

Theorem 3.15 *For every* $(\sigma, H) \in B_n$

$$ned_B(\sigma, H) = fmaj(\sigma, \varphi_n(H)). \tag{14}$$

Proof. We proceed by induction on n, (14) being easy to check for $n = 1$. Let $n > 1$, $H \subseteq [n]$ and $\sigma \in S_n$. We have four cases to consider.

a) $n \notin H$, $n-1 \in Des(\sigma)$ and $n-1 \in H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= 2(n-1) + \sum_{i \in H} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H)} i + n \\ &= 3n - 2 + ned_B(\bar{\sigma}, H). \end{aligned}$$

Let' s compute the right-hand side of (14). We have $n-1, n \in \varphi_n(H)$ and $n-1 \notin \varphi_{n-1}(H)$. From this, Lemma 3.14 and Corollary 3.13, it follows that

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_n(H)}) + 2(n-1) + 1 \\ &= fmaj(-(\bar{\sigma}, \varphi_{n-1}(H))) + 2n - 1 \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H)) + 3n - 2, \end{aligned}$$

so (14) follows from our induction hypothesis.

b) $n \notin H$ and either $n-1 \notin Des(\sigma)$ or $n-1 \notin H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= \sum_{i \in H} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H)} i + n \\ &= ned_B(\bar{\sigma}, H) + n. \end{aligned}$$

Consider now the right-hand side of (14). We have two possibilities.
If $n-1 \notin H$ then $n-1 \notin \varphi_n(H)$, $n \in \varphi_n(H)$ and $n-1 \in \varphi_{n-1}(H)$. By Lemma 3.14 and Corollary 3.13 we obtain

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_n(H)}) + 2(n-1) + 1 \\ &= fmaj(-(\bar{\sigma}, \varphi_{n-1}(H))) + 2n - 1 \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H)) + n. \end{aligned}$$

If $n-1 \notin Des(\sigma)$ and $n-1 \in H$ then $n-1, n \in \varphi_n(H)$ and $n-1 \notin \varphi_{n-1}(H)$. By Lemma 3.14 and Corollary 3.13 we have that

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_n(H)}) + 1 \\ &= fmaj(-(\bar{\sigma}, \varphi_{n-1}(H))) + 1 \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H)) + (n-1) + 1, \end{aligned}$$

and (14) follows.

c) $n \in H$, $n-1 \in Des(\sigma)$ and $n-1 \in H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= \sum_{i \in H \setminus \{n\}} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H\setminus\{n\})} i + 2n - 2 \\ &= ned_B(\bar{\sigma}, H \setminus \{n\}) + 2n - 2. \end{aligned}$$

On the other hand, from $n-1, n \notin \varphi_n(H)$ and Lemma 3.14 we have that

$$\begin{aligned} fmaj(\sigma, \varphi_n(H)) &= fmaj(\overline{\sigma, \varphi_{n-1}(H)}) + 2(n-1) \\ &= fmaj(\bar{\sigma}, \varphi_{n-1}(H \setminus \{n\})) + 2n - 2, \end{aligned}$$

and (14) again follows.

d) $n \in H$ and either $n-1 \notin Des(\sigma)$ or $n-1 \notin H$

Then

$$\begin{aligned} ned_B(\sigma, H) &= \sum_{i \in H \setminus \{n\}} 2i\varepsilon_i(\bar{\sigma}) + \sum_{\mathcal{C}_{n-1}(H\setminus\{n\})} i \\ &= ned_B(\bar{\sigma}, H \setminus \{n\}). \end{aligned}$$

But $n \notin \varphi_n(H)$ hence by Lemma 3.14 it follows that

$$fmaj(\sigma, \varphi_n(H)) = fmaj(\overline{\sigma, \varphi_n(H)}) = fmaj(\bar{\sigma}, \varphi_{n-1}(H \setminus \{n\})),$$

and this concludes the proof. ■

The next corollary says that ned_B is a Mahonian statistic on B_n.

Corollary 3.16 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\beta \in B_n} q^{ned_B(\beta)} = \sum_{\beta \in B_n} q^{fmaj(\beta)}.$$

The following statistic is fundamental for this work and its definition is naturally suggested by Theorem 3.15. We will see in §4 that it is Mahonian and in §5 that it has the same algebraic role for D_n, as maj for S_n and $fmaj$ for B_n, in the quotient of the Hilbert series of DIA and TIA defined in §2.

Definition. Let $\gamma \in D_n$, we define

$$Dmaj(\gamma) := fmaj([\gamma_1, \ldots, \gamma_{n-1}, |\gamma_n|]).$$

For example, if $\gamma = [-2, 3, -1, -5, -4]$, then $Dmaj(\gamma) = fmaj([-2, 3, -1, -5, 4]) = 2 \cdot 2 + 3 = 7$. Note that $Dmaj((\sigma, K)_D) = fmaj((\sigma, K))$.

The next result plays an important role in the computation of the Hilbert series and its proof follows immediately from Theorem 3.15.

Corollary 3.17 *Let* $(\sigma, K) \in D_n$. *Then*

$$ned_D(\sigma, K) = Dmaj(\sigma, \varphi_{n-1}(K)).$$

4 Combinatorial Properties of *Dmaj*

In this section we study the statistic $Dmaj$ introduced in §3 and we show that it is equidistributed with length on D_n. Moreover we introduce a new "descent number", $Ddes$, on D_n and we show that the pair $(Ddes, Dmaj)$ solves the generalization of Carlitz's identity to D_n. Finally, we give another generalization of Carlitz's identity to D_n, thus solving a question posed in [4].

4.1 Equidistribution

Following [1] we let, for all $i \in [n-1]$, $\tau_i := s_i s_{i-1} \cdots s_0 \in B_n$. The family $\{\tau_i\}_i$ is a set of generators for B_n and for any $\beta \in B_n$ there exist unique integers $r_0, \ldots, r_{n-1}$, with $0 \leq r_i \leq 2i+1$ for $i = 0, \ldots, n-1$, such that

$$\beta = \tau_{n-1}^{r_{n-1}} \cdots \tau_2^{r_2} \tau_1^{r_1} \tau_0^{r_0}. \tag{15}$$

The following is a further characterization of the flag-major index, (see [1]).

Proposition 4.1 *Let* $\beta \in B_n$. *Then*

$$fmaj(\beta) = \sum_{i=0}^{n-1} r_i.$$

Now recall the definition of B_n^+ given in §3.3. Let $\beta = \tau_{n-1}^{r_{n-1}} \cdots \tau_1^{r_1} \tau_0^{r_0} \in B_n$. Since $\tau_{n-1}^{r_{n-1}}(n) > 0$ if and only if $r_{n-1} = 0, \ldots, n-1$, it follows that $\beta \in B_n^+$ if and only if $r_{n-1} = 0, \ldots, n-1$. Now consider $\beta_1 = \tau_{n-2}^{r_{n-2}} \cdots \tau_1^{r_1} \tau_0^{r_0} \in B_{n-1}$. If $\beta_1 \in B_{n-1}^+$ then $r_{n-2} \leq n-2$. Otherwise, if $\beta_1 \notin B_{n-1}^+$, then $\beta_1 = \tau_{n-2}^{n-1} \beta_2$ with $\beta_2 \in B_{n-1}^+$. So we have the following decomposition of B_n^+

$$B_n^+ = \bigcup_c \bigcup_\xi (\{\tau_{n-1}^c \xi\} \biguplus \{\tau_{n-1}^c \tau_{n-2}^{n-1} \xi\}),$$

where $c = 0, \ldots, n-1$ and $\xi \in B_{n-1}^+$.

Following [4] we let for all $i \in [n-1]$,

$$t_i := s_i s_{i-1} \cdots s_0^D.$$

The family $\{t_i\}_i$ is a set of generators for D_n and for every $\gamma \in D_n$ there exists a unique representation

$$\gamma = t_0^{h_{n-1}} t_{n-1}^{k_{n-1}} t_0^{h_{n-2}} t_{n-2}^{k_{n-2}} \cdots t_0^{h_1} t_1^{k_1}$$

with $0 \leq h_r \leq 1$, $0 \leq k_r \leq 2r-1$ and $k_r \in \{2r-1, r-1\}$ if $h_r = 1$ for all $r = 1, \ldots, n-1$, (see [4, Proposition 4.1]).

For any $\gamma \in D_n$, the *D-flag major index* of γ is defined by (see [4])

$$fmaj_D := \sum_{i=1}^{n-1} k_i + \sum_{i=1}^{n-1} h_i.$$

This statistic is equidistributed with length on D_n, ([4, Proposition 4.2]).

Now we are ready to state and prove the main result of this section, namely that $Dmaj$ is equidistributed with length on D_n.

Proposition 4.2 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma \in D_n} q^{Dmaj(\gamma)} = \sum_{\gamma \in D_n} q^{\ell(\gamma)}.$$

Proof. We define a map $\Psi : D_n \to B_n^+$ as follows:

$$\Psi(\prod_{i=1}^{n-1} t_0^{h_{n-i}} t_{n-i}^{k_{n-i}}) := \prod_{i=1}^{n-1} \Psi(t_0^{h_{n-i}} t_{n-i}^{k_{n-i}}),$$

where

$$\begin{aligned}
\Psi(t_0 t_{n-i}^{2n-2i-1}) &:= \tau_{n-i}^{n-i} \tau_{n-i-1}^{n-i}; \\
\Psi(t_0 t_{n-i}^{n-i-1}) &:= \tau_{n-i-1}^{n-i}; \\
\Psi(t_{n-i}^{k_{n-i}}) &:= \tau_{n-i}^{k_{k-i}}, \qquad \text{if } k_{n-1} \leq n-i; \\
\Psi(t_{n-i}^{k_{n-i}}) &:= \tau_{n-i}^{k_{n-i}-n+i} \tau_{n-i-1}^{n-i}, \quad \text{if } k_{n-1} > n-i.
\end{aligned}$$

It is easy to see that the map Ψ is a bijection that sends $fmaj_D$ to $fmaj$. The thesis follows from the equidistribution of the D-flag major index and the definition of $Dmaj$. ■

4.2 Carlitz's Identity

In this section we give two different generalizations of Carlitz's identity to D_n.

For $\beta \in B_n$, by Corollary 3.13 we know that

$$fmaj(-\beta) = fmaj(\beta) + n. \tag{16}$$

¿From definition (6), it is not hard to prove that for $\beta \in B_n^+$

$$fdes(-\beta) = fdes(\beta) + 1. \tag{17}$$

Definition. For $\gamma \in D_n$ we define the *D-descent number* by

$$Ddes(\gamma) := fdes([\gamma_1, \ldots, \gamma_{n-1}, |\gamma_n|]).$$

For example, if $\gamma = [-2, -1, 4, 5, -6, -3] \in D_6$ then $Ddes(\gamma) = fdes([-2, -1, 4, 5, -6, 3]) = 2 \cdot 2 + 1 = 5$.
Now we are ready to prove the main result of this section

Theorem 4.3 *Let* $n \in \mathbf{P}$. *Then*

$$\sum_{r \geq 0} [r+1]_q^n t^r = \frac{\sum_{\gamma \in D_n} t^{Ddes(\gamma)} q^{Dmaj(\gamma)}}{(1-t)(1-tq^n)\prod_{i=1}^{n-1}(1-t^2q^{2i})}$$

in $\mathbf{Z}[q][[t]]$.

Proof. From (16) and (17) we have that

$$\begin{aligned} \sum_{\beta \in B_n} t^{fdes(\beta)} q^{fmaj(\beta)} &= \sum_{\beta \in B_n^+} t^{fdes(\beta)} q^{fmaj(\beta)} + t^{fdes(-\beta)} q^{fmaj(-\beta)} \\ &= \sum_{\beta \in B_n^+} t^{fdes(\beta)} q^{fmaj(\beta)} + t^{fdes(\beta)+1} q^{fmaj(\beta)+n} \\ &= (1+tq^n) \sum_{\beta \in B_n^+} t^{fdes(\beta)} q^{fmaj(\beta)} \\ &= (1+tq^n) \sum_{\gamma \in D_n} t^{Ddes(\gamma)} q^{Dmaj(\gamma)}. \end{aligned}$$

Now the result follows easily from Theorem 2.2. ■

Finally, we answer a question posed in [4]. To this end we define the following D-flag descent number on D_n,

$$fdes_D(\gamma) := fdes(\Psi(\gamma)),$$

where Ψ has been defined in the proof of Proposition 4.2. By Theorem 4.3 and the definition of Ψ it is easy to see that the two pairs of statistics $(fdes_D, fmaj_D)$ and $(Ddes, Dmaj)$ are equidistributed in D_n. This and Theorem 2.3 imply the next corollary.

Corollary 4.4 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma \in D_n} t^{ddes(\gamma)} q^{dmaj_D(\gamma)} = \sum_{\gamma \in D_n} t^{fdes_D(\gamma)} q^{fmaj_D(\gamma)} = \sum_{\gamma \in D_n} t^{Ddes(\gamma)} q^{Dmaj(\gamma)}.$$

Finally, the case $t = 1$ and Proposition 4.2 imply the following result.

Corollary 4.5 *Let $n \in \mathbf{P}$. Then*

$$\sum_{\gamma \in D_n} q^{\ell(\gamma)} = \sum_{\gamma \in D_n} q^{dmaj(\gamma)} = \sum_{\gamma \in D_n} q^{fmaj_D(\gamma)} = \sum_{\gamma \in D_n} q^{Dmaj(\gamma)}.$$

5 The Main Results

In this section we use the combinatorial tools developed in §3 to prove the main result of this work, namely, we find a closed formula for $\mathcal{Z}_{D_n}(\bar{q})$ in terms of the statistic $Dmaj$. This formula implies that the series is actually a polynomial with non-negative integer coefficients.

5.1 t-Partite Partitions

In this section we recall the language of t-partite partitions which was originally defined by Gordon [17] as well as some results of Garsia and Gessel [14] that we use in the rest of this work.

Let $\mathcal{F}_n$ be the set of all functions $f : [n] \to \mathbf{N}$. For $f \in \mathcal{F}_n$ we let

$$|f| := \sum_{i=1}^{n} f(i),$$

and we denote $\mathcal{F}_{n,t} := (\mathcal{F}_n)^t$. Moreover, for $f = (f_1, \ldots, f_t) \in \mathcal{F}_{n,t}$, we define

$$\alpha_j(f) := \sum_{i=1}^{t} f_i(j),$$

and we let $\mathcal{F}^e_{n,t} := \{f \in \mathcal{F}_{n,t} \ : \ \alpha_j(f) \equiv 0 \ \text{ for all } j \in [n]\}$ and $\mathcal{F}^o_{n,t} := \{f \in \mathcal{F}_{n,t} \ : \ \alpha_j(f) \equiv 1 \ \text{ for all } j \in [n]\}$.

A *t-partite partition with n parts* is a sequence $f = (f_1, \ldots, f_t) \in \mathcal{F}_{n,t}$,

$$f = \begin{pmatrix} f_1(1) & f_1(2) & \ldots & f_1(n) \\ f_2(1) & f_2(2) & \ldots & f_2(n) \\ \vdots & \vdots & & \vdots \\ f_t(1) & f_t(2) & \ldots & f_t(n) \end{pmatrix}$$

satisfying the following condition:
for $i_o \in [t]$ and $j \in [n]$, if $f_i(j) = f_i(j+1)$ for all $i < i_0$, then $f_{i_0}(j) \geq f_{i_0}(j+1)$.
Note, in particular, that for $i_0 = 1$ this implies that

$$f_1(1) \geq f_1(2) \geq \ldots \geq f_1(n) \geq 0,$$

so f_1 is a partition with at most n parts.
We denote the set of all the t-partite partitions with n parts by $\mathcal{B}_{n,t}$.
For example, if $n = 5$ and $t = 2$, then $f = (f_1, f_2)$ with $f_1 = (4,4,4,3,3)$ and $f_2 = (3,3,2,5,4)$ is a bipartite partition with 5 parts.

Given a permutation $\sigma = \sigma_1 \cdots \sigma_n$ we say that the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is *σ-compatible* if $\lambda_i - \lambda_{i+1} \geq \varepsilon_i(\sigma)$ for all $i \in [n-1]$, where $\varepsilon_i(\sigma)$ is defined in (1). Clearly, a partition λ is σ-compatible if and only if it is of the form

$$\lambda_i = p_i + p_{i+1} + \cdots + p_n$$

with $p_i \geq \varepsilon_i(\sigma)$ for all i. We let $\mathcal{P}(\sigma)$ be the set of all σ-compatible partitions.
For example, if $\sigma = 15342$ then $\lambda = (6,6,4,4,3) \in \mathcal{P}(\sigma)$.
The following two theorems are due to Garsia and Gessel (see [14, Theorems 2.1 and 2.2]):

Theorem 5.1 *The map Ω,*

$$(\sigma, \lambda, \mu) \longmapsto \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \mu_{\sigma_1} & \mu_{\sigma_2} & \ldots & \mu_{\sigma_n} \end{pmatrix},$$

is a bijection between $\mathcal{B}_{n,2}$ and the set $\mathcal{P}_{n,2}$ of the triplets (σ, λ, μ), where
i) $\sigma \in S_n$;
ii) $\lambda \in \mathcal{P}(\sigma)$;
iii) $\mu \in \mathcal{P}(\sigma^{-1})$.

Theorem 5.2 *Let $W = S_n$. Then*

$$\mathcal{Z}_{S_n}(q_1, q_2) = \frac{\sum_{f \in \mathcal{B}_{n,2}} q_1^{|f_1|} q_2^{|f_2|}}{\sum_{g,h \in \mathcal{B}_{n,1}} q_1^{|g|} q_2^{|h|}} = \sum_{\sigma \in S_n} q_1^{maj(\sigma)} q_2^{maj(\sigma^{-1})}.$$

We let

$$\mathcal{B}^e_{n,2} := \{f \in \mathcal{B}_{n,2} \ : \ \alpha_j(f) \equiv 0 \ \text{ for all } j \in [n]\}$$

and

$$\mathcal{B}^o_{n,2} := \{f \in \mathcal{B}_{n,2} \ : \ \alpha_j(f) \equiv 1 \ \text{ for all } j \in [n]\}$$

be the sets of all the *even* and *odd* bipartite partitions with n parts, respectively. Moreover, we let

$$\mathcal{P}^e_{n,2} := \{(\sigma, \lambda, \mu) \in \mathcal{P}_{n,2} \ : \ \lambda_i + \mu_{\sigma(i)} \equiv 0 \ \text{ for all } i \in [n]\}$$

and

$$\mathcal{P}^o_{n,2} := \{(\sigma, \lambda, \mu) \in \mathcal{P}_{n,2} \ : \ \lambda_i + \mu_{\sigma(i)} \equiv 1 \ \text{ for all } i \in [n]\}.$$

It is clear that, by restriction, the map Ω of Theorem 5.1 gives rise to two bijections $\mathcal{B}^e_{n,2} \leftrightarrow \mathcal{P}^e_{n,2}$ and $\mathcal{B}^o_{n,2} \leftrightarrow \mathcal{P}^o_{n,2}$.

Theorems 5.1 and 5.2 can be extended to the general case ($t > 2$) as follows, (see [14, Remark 2.2]).

Theorem 5.3 *There exists a bijection between $\mathcal{B}_{n,t}$ and the set $\mathcal{P}_{n,t}$ of the $2t$-tuples*

$$(\sigma_1, \ldots, \sigma_t, \lambda^{(1)}, \ldots, \lambda^{(t)})$$

where $\sigma_i \in S_n$, $\lambda^{(i)} \in \mathcal{P}(\sigma_i)$ for all $i \in [t]$ and $\sigma_t \cdots \sigma_2\sigma_1 = id$. This bijection is given by

$$\Omega(\sigma_1, \ldots \sigma_t, \lambda^{(1)}, \ldots, \lambda^{(t)}) := \begin{pmatrix} \lambda^{(1)}_1 & \lambda^{(1)}_2 & \cdots & \lambda^{(1)}_n \\ \lambda^{(2)}_{\sigma_1(1)} & \lambda^{(2)}_{\sigma_1(2)} & \cdots & \lambda^{(2)}_{\sigma_1(n)} \\ \vdots & \vdots & & \vdots \\ \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(1)} & \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(2)} & \cdots & \lambda^{(t)}_{\sigma_{t-1}\cdots\sigma_1(n)} \end{pmatrix}.$$

We define $\mathcal{B}^e_{n,t}$, $\mathcal{B}^o_{n,t}$, $\mathcal{P}^e_{n,t}$ and $\mathcal{P}^o_{n,t}$ analogously to the case $t = 2$. Note again that the correspondence Ω restricts to bijections $\mathcal{B}^e_{n,t} \leftrightarrow \mathcal{P}^e_{n,t}$ and $\mathcal{B}^o_{n,t} \leftrightarrow \mathcal{P}^o_{n,t}$.

Theorem 5.4 *Let $W = S_n$ and $t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{S_n}(\bar{q}) = \sum_{\sigma_1,\ldots,\sigma_t} \prod_{i=1}^{t} q_i^{maj(\sigma_i)},$$

where the sum is over all t-tuples $(\sigma_1, \ldots, \sigma_t)$ of permutation in S_n such that $\sigma_t\sigma_{t-1}\cdots\sigma_1 = id$.

The following is the corresponding result of Theorem 5.4 for B_n and it is due to Adin and Roichman [1].

Theorem 5.5 *Let $n, t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\beta_1,\ldots,\beta_t} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},$$

where the sum is over all the signed permutation $\beta_1, \ldots, \beta_t \in B_n$ such that $\beta_t \cdots \beta_1 = id$.

5.2 A Basis for TIA and DIA for D_n

Let $W = D_n$. The tensor invariant algebra TIA is $(\mathbf{P}_n^{D_n})^{\otimes t}$, and $\mathbf{P}_n^{D_n}$ is freely generated (as an algebra) by the $n-1$ elementary symmetric functions $e_j(x_1^2, \ldots, x_n^2)$ for $j \in [n-1]$ and the monomial $x_1 \cdots x_n$ (see, e.g., [18, §3]). Hence

$$F_T(\bar{q}) = \prod_{i=1}^{t} \Big(\frac{1}{(1-q_i^n)} \prod_{j=1}^{n-1} \frac{1}{(1-q_i^{2j})} \Big).$$

A linear basis for $\mathbf{P}_n^{\otimes t}$ consists of all tensor monomials

$$\bar{x}^f := \bigotimes_{i=1}^{t} \prod_{j=1}^{n} x_j^{f_i(j)}$$

where $f = (f_1, \ldots, f_t) \in \mathcal{F}_{n,t}$. The canonical projection $\pi : \mathbf{P}_n^{\otimes t} \to$ DIA is defined by

$$\pi(\bar{x}^f) := \sum_{\gamma \in D_n} \varphi_D(\gamma)(\bar{x}^f)$$

so that

$$\text{DIA} =< \{\pi(\bar{x}^f) : \ f \in \mathcal{F}_{n,t}\} > .$$

Lemma 5.6 *For* $f \in \mathcal{F}_{n,t}$,

$$\pi(\bar{x}^f) \neq 0 \Longleftrightarrow f \in \mathcal{F}_{n,t}^e \cup \mathcal{F}_{n,t}^o,$$

where $\mathcal{F}_{n,t}^e$ *and* $\mathcal{F}_{n,t}^o$ *are defined in §5.1.*

Proof. Let $\delta_i = [-1, 2, 3, \ldots, -i, \ldots, n]$ for $i \in [2, n]$. Note that

$$\varphi_D(\delta_i)(\bar{x}^f) = (-1)^{\alpha_1(f)+\alpha_i(f)} \bar{x}^f.$$

Therefore, if C is any coset in D_n of the subgroup $T_i = \{id, \delta_i\}$, then

$$\sum_{\gamma \in C} \varphi_D(\gamma)(\bar{x}^f) \neq 0$$

if and only if

$$\alpha_1(f) + \alpha_i(f) \equiv 0.$$

Hence we conclude that if $\pi(\bar{x}^f) \neq 0$ then $f \in \mathcal{F}_{n,t}^e \cup \mathcal{F}_{n,t}^o$.
For the converse, let H be the subgroup of all the *generalized identity permutations* $h = (id, B) \in D_n$, (i.e. $|h(i)| = i$ for all $i \in [n]$). We have that $D_n = S_n \ltimes H$, hence

every $\gamma \in D_n$ has a unique representation $\gamma = \sigma \cdot h$ with $\sigma \in S_n$ and $h \in H$. For any $f \in \mathcal{F}^e_{n,t} \cup \mathcal{F}^o_{n,t}$ and for any $h \in H$ we have $\varphi_D(h)(\bar{x}^f) = \bar{x}^f$ hence

$$\sum_{\gamma \in \sigma H} \varphi_D(\gamma)(\bar{x}^f) = |H| \varphi_D(\sigma)(\bar{x}^f),$$

for any $\sigma \in S_n$, and the thesis follows. ■

Clearly $\mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}$ is a complete system of representatives for the orbits of all $f \in \mathcal{F}^e_{n,t} \cup \mathcal{F}^o_{n,t}$, under the action of the symmetric group. Hence we have

Proposition 5.7 *The set*

$$\{\pi(\bar{x}^f) : f \in \mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}\}$$

is a homogeneous basis for DIA.

Corollary 5.8 *The Hilbert series for* DIA *is*

$$F_D(\bar{q}) = \sum_{f \in \mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}} q_1^{|f_1|} \cdots q_t^{|f_t|}.$$

5.3 The Polynomial $\mathcal{Z}_{D_n}(q_1, q_2)$

We start showing the case $t = 2$. The following definition is fundamental.

Definition. We define an involution $\alpha : D_n \to D_n$ by

$$(\sigma, K) \quad \mapsto \quad (\sigma^{-1}, p(\sigma(K))), \tag{18}$$

where p is the projection defined in (7).

For example, $\alpha(4213, \{1,3\}) = (3241, p(\{1,4\}) = (3241, \{2,3\})$.

We are now ready to state and prove the following

Theorem 5.9 *Let* $n \in \mathbf{N}$. *Then*

$$\mathcal{Z}_{D_n}(q_1, q_2) = \sum_{\gamma \in D_n} q_1^{Dmaj(\gamma)} q_2^{Dmaj(\alpha(\gamma))}.$$

Proof. By Corollary 5.8 and the note below Theorem 5.2 we have that

$$\begin{aligned} F_D(q_1, q_2) &= \sum_{f \in \mathcal{B}^e_{n,2} \cup \mathcal{B}^o_{n,2}} q_1^{|f_1|} q_2^{|f_2|} \\ &= \sum_{(\sigma,\lambda,\mu) \in \mathcal{P}^e_{n,2}} q_1^{|\lambda|} q_2^{|\mu|} + \sum_{(\sigma,\lambda,\mu) \in \mathcal{P}^o_{n,2}} q_1^{|\lambda|} q_2^{|\mu|}. \end{aligned} \tag{19}$$

By the definitions of $H(\lambda)$ and H^σ given in §3.1, the first part in (19) can be rewritten as

$$\sum_{(\sigma,\lambda,\mu)\in\mathcal{P}^e_{n,2}} q_1^{|\lambda|}q_2^{|\mu|} = \sum_{\sigma\in S_n}\sum_{p_i,r_i} q_1^{\sum jp_j} q_2^{\sum jr_j}$$

where the last sum runs through all $p_i, r_i \in \mathbf{N}$, $i\in[n]$, such that $p_i \geq \varepsilon_i(\sigma)$, $r_i \geq \varepsilon_i(\sigma^{-1})$ and $(H(\lambda))^\sigma = H(\mu)$, with $\lambda_i = p_i + \cdots + p_n$ and $\mu_i = r_i + \cdots + r_n$. Now we split the previous sum according to the parity set of λ. Note that p_i is even if and only if $i\in H(\lambda)$, and similarly for μ. Hence we obtain

$$\begin{aligned}\sum_{\mathcal{P}^e_{n,2}} q_1^{|\lambda|}q_2^{|\mu|} &= \sum_{\sigma\in S_n}\sum_{H\subseteq[n]}\left(\prod_{i\in H} q_1^{2i\varepsilon_i(\sigma)}\prod_{i\in\mathcal{C}_n(H)} q_1^i \prod_{i\in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(H^\sigma)} q_2^i \sum_{\pi_i,\rho_i\in\mathbf{N}} q_1^{2\sum j\pi_j} q_2^{2\sum j\rho_j}\right)\\ &= \prod_{i=1}^{2}\prod_{j=1}^{n}\frac{1}{(1-q_i^{2j})}\sum_{\sigma\in S_n}\sum_{H\subseteq[n]}\left(\prod_{i\in H} q_1^{2i\varepsilon_i(\sigma)}\prod_{i\in\mathcal{C}_n(H)} q_1^i \prod_{i\in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(H^\sigma)} q_2^i\right) \quad (20)\end{aligned}$$

where $p_i = 2\pi_i + 2\varepsilon_i(\sigma)$ for $i\in H$, $p_i = 2\pi_i + 1$ for $i\in\mathcal{C}_n(H)$, $r_i = 2\rho_i + 2\varepsilon_i(\sigma^{-1})$ for $i\in H^\sigma$ and $r_i = 2\rho_i + 1$ for $i\in\mathcal{C}_n(H^\sigma)$.
Analogously, recalling the definition of $\overline{H^\sigma}$ given in §3.1, we can evaluate the second part of (19), substituting H^σ with $\overline{H^\sigma}$, obtaining

$$\sum_{(\sigma,\lambda,\mu)\in\mathcal{P}^o_{n,2}} q_1^{|\lambda|}q_2^{|\mu|} = \prod_{i=1}^{2}\prod_{j=1}^{n}\frac{1}{(1-q_i^{2j})}\sum_{\sigma\in S_n}\sum_{H\subseteq[n]}\left(\prod_{i\in H} q_1^{2i\varepsilon_i(\sigma)}\prod_{i\in\mathcal{C}_n(H)} q_1^i \prod_{i\in \overline{H^\sigma}} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(\overline{H^\sigma})} q_2^i\right). \quad (21)$$

Hence by (19), (20) and (21) we have that

$$\begin{aligned}F_D(q_1,q_2)\prod_{i=1}^{2}\prod_{j=1}^{n}(1-q_i^{2j}) &= \sum_{\sigma}\sum_{H\subseteq[n]}\prod_{i\in H} q_1^{2i\varepsilon_i(\sigma)}\prod_{i\in\mathcal{C}_n(H)} q_1^i \cdot\\ &\quad\left(\prod_{i\in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(H^\sigma)} q_2^i + \prod_{i\in\overline{H^\sigma}} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(\overline{H^\sigma})} q_2^i\right)\\ &= \sum_{\sigma}\sum_{K\subseteq[n-1]}\sum_{H\in\{K,K\cup\{n\}\}}\prod_{i\in H} q_1^{2i\varepsilon_i(\sigma)}\prod_{i\in\mathcal{C}_n(H)} q_1^i\cdot\\ &\quad\left(\prod_{i\in H^\sigma} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(H^\sigma)} q_2^i + \prod_{i\in H^\sigma\triangle\{n\}} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(H^\sigma\triangle\{n\})} q_2^i\right)\\ &= \sum_{\sigma}\sum_{K\subseteq[n-1]}(1+q_1^n)(1+q_2^n)\prod_{i\in K\cup\{n\}} q_1^{2i\varepsilon_i(\sigma)}\prod_{i\in\mathcal{C}_n(K\cup\{n\})} q_1^i\cdot\\ &\quad\prod_{i\in K^\sigma\cup\{n\}} q_2^{2i\varepsilon_i(\sigma^{-1})}\prod_{i\in\mathcal{C}_n(K^\sigma\cup\{n\})} q_2^i\end{aligned}$$

where we have used the fact that $\varepsilon_n(\sigma) = 0$ for all $\sigma \in S_n$, and Lemma 3.5. Applying Corollary 3.17 and Lemma 3.6 it follows that

$$\begin{aligned}
\mathcal{Z}_{D_n}(q_1, q_2) &= \sum_{\sigma} \sum_{K \subseteq [n-1]} q_1^{ned_D(\sigma, K)} q_2^{ned_D(\sigma^{-1}, (K^\sigma \setminus \{n\}))} \\
&= \sum_{\sigma} \sum_{K \subseteq [n-1]} q_1^{Dmaj(\sigma, \varphi_{n-1}(K))} q_2^{Dmaj(\sigma^{-1}, \varphi_{n-1}(K^\sigma \setminus \{n\}))} \\
&= \sum_{\sigma} \sum_{K \subseteq [n-1]} q_1^{Dmaj(\sigma, \varphi_{n-1}(K))} q_2^{Dmaj(\sigma^{-1}, p\sigma\varphi_{n-1}(K))} \\
&= \sum_{\gamma \in D_n} q_1^{Dmaj(\gamma)} q_2^{Dmaj(\alpha(\gamma))},
\end{aligned}$$

as desired. ■

Example. Consider the case $n = 2$. One may easily check that $\alpha(\gamma) = \gamma$ for all $\gamma \in D_2$ and hence

$$\begin{aligned}
\mathcal{Z}_{D_2}(q_1, q_2) &= (q_1 q_2)^{Dmaj(1,2)} + (q_1 q_2)^{Dmaj(2,1)} + (q_1 q_2)^{Dmaj(-1,-2)} + (q_1 q_2)^{Dmaj(-2,-1)} \\
&= (1 + q_1 q_2)^2.
\end{aligned}$$

We denote by ι the inversion in D_n so that $\iota(\gamma) := \gamma^{-1}$. The next lemma says that it is possible to "substitute" α with ι in Theorem 5.9.

Lemma 5.10 *α and ι are conjugate in $S(D_n)$.*

Proof. It is well known that two elements of a symmetric group are conjugate if and only if they have the same cycle type. Since both α and ι are involutions it is enough to show that they have the same number of fixed points. For this it is sufficient to prove that $i_\sigma = a_\sigma$, for every $\sigma \in S_n$, where

$$i_\sigma := \left| \left\{ K \in 2^{[n-1]} : (\sigma, K)^{-1} = (\sigma, K) \right\} \right|$$

and

$$a_\sigma := \left| \left\{ K \in 2^{[n-1]} : \alpha(\sigma, K) = (\sigma, K) \right\} \right|.$$

It is clear that $i_\sigma = a_\sigma = 0$ if σ is not an involution in S_n. On the other hand if σ is an involution with some fixed point then we have $i_\sigma = a_\sigma = 2^{c_1(\sigma)+c_2(\sigma)-1}$ while if σ has no fixed point then $a_\sigma = i_\sigma = 2^{c_2(\sigma)}$, where $c_i(\sigma)$ is the number of cycles of length i of σ. ■

Corollary 5.11 *There exists a function $M : D_n \to \mathbf{N}$, equidistributed with length, such that*

$$\mathcal{Z}_{D_n}(q_1, q_2) = \sum_{\gamma \in D_n} q_1^{M(\gamma)} q_2^{M(\gamma^{-1})}.$$

Proof. By Lemma 5.10 we know that there exists $\psi \in S(D_n)$ such that $\alpha\psi = \psi\iota$. Then the function $M := Dmaj \circ \psi$ realizes the above formula for $\mathcal{Z}_{D_n}(q_1, q_2)$. It follows immediately from Proposition 4.2 that this M is equidistributed with length on D_n. ■

We will show a combinatorial description of the parameter M when n is odd in §5.5.

5.4 The Polynomial $\mathcal{Z}_{D_n}(\bar{q})$

In this section we prove the main result of this work, i.e. we provide an explicit simple formula for the polynomial $\mathcal{Z}_{D_n}(\bar{q})$ in terms of the $Dmaj$.
We denote by $\alpha : D_n^{t-1} \to D_n$ the map

$$((\sigma_1, K_1), \ldots, (\sigma_{t-1}, K_{t-1})) \mapsto \left((\sigma_{t-1} \cdots \sigma_1)^{-1}, p\left(\sigma_{t-1} \cdots \sigma_1(K_1) \triangle \cdots \triangle \sigma_{t-1}(K_{t-1})\right)\right).$$

For example,

$$\begin{aligned} \alpha\left((4231, \{1,3\}), (2143, \{3\})\right) &= (2413, p(3142(\{1,3\}) \triangle 2143(\{3\})) \\ &= (2413, p(\{3\})) = (2413, \{3\}). \end{aligned}$$

Note that this is consistent with the definition of α given in (18).

Theorem 5.12 *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_n}(\bar{q}) = \sum_{\gamma_1, \ldots, \gamma_t \in D_n} \prod_{i=1}^{t} q_i^{Dmaj(\gamma_i)},$$

where the sum runs through all $\gamma_1, \ldots, \gamma_t \in D_n$ such that $\gamma_t = \alpha(\gamma_1, \ldots, \gamma_{t-1})$.

Proof. The proof is similar to that of Theorem 5.9, and hence we will not go through all the details. By Corollary 5.8 we have that

$$F_D(\bar{q}) = \sum_{(f_1, \ldots, f_t) \in \mathcal{B}^e_{n,t} \cup \mathcal{B}^o_{n,t}} \prod_{i=1}^{t} q_i^{|f_i|}. \tag{22}$$

Let's consider the sum in (22) restricted to $\mathcal{B}^e_{n,t}$. By the note below Theorem 5.4 we have that

$$\sum_{(f_1, \ldots f_t) \in \mathcal{B}^e_{n,t}} \prod_{i=1}^{t} q_i^{|f_i|} = \sum_{(\sigma_1, \ldots, \sigma_t, \lambda^{(1)}, \ldots, \lambda^{(t)}) \in \mathcal{P}^e_{n,t}} \prod_{i=1}^{t} q_i^{|\lambda_i|} = \sum_{\sigma_t \cdots \sigma_1 = id} \sum_{p_j^{(i)}} \prod_{i=1}^{t} q_i^{\sum k p_k^{(i)}}, \tag{23}$$

where the last sum is over all $p_j^{(i)} \in \mathbf{N}$, for $i \in [n]$ and $j \in [t]$, such that $p_j^{(i)} \geq \varepsilon_j(\sigma_i)$ and $(H(\lambda^{(1)}), \ldots, H(\lambda^{(t-1)}))^{(\sigma_1,\ldots,\sigma_{t-1})} = H(\lambda^{(t)})$, with $\lambda_j^{(i)} = p_j^{(i)} + \cdots + p_n^{(i)}$, where the set $(H_1, \ldots, H_t)^{(\sigma_1,\ldots,\sigma_t)}$ has been defined in §3.7. We proceed in a similar way for the sum in (22) over $\mathcal{B}^o_{n,t}$. If we split these sums according to the parity sets of the $\lambda^{(i)}$'s for $i \in [t-1]$ we obtain, by Lemma 3.10, that

$$F_D(\bar{q}) \prod_{i=1}^{n} \prod_{j=1}^{t} (1 - q_j^{2i}) = \sum_{\sigma_1,\ldots,\sigma_t} \sum_{H_1,\ldots,H_t} \prod_{j=1}^{t-1} \prod_{h \in H_j} q_j^{2h\varepsilon_h(\sigma_j)} \prod_{h \in \mathcal{C}_n(H_j)} q_j^h \tag{24}$$
$$\left(\prod_{h \in H_t} q_t^{2h\varepsilon_h(\sigma_t)} \prod_{h \in \mathcal{C}_n(H_t)} q_t^h + \prod_{h \in H_t \triangle \{n\}} q_t^{2h\varepsilon_h(\sigma_t)} \prod_{h \in \mathcal{C}_n(H_t \triangle \{n\})} q_t^h \right),$$

where the sums run through all $\sigma_1, \ldots, \sigma_t \in S_n$ such that $\sigma_t = (\sigma_{t-1} \cdots \sigma_1)^{-1}$ and all $H_1, \ldots, H_t \subseteq [n]$ such that $H_t = (H_1, \ldots, H_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}$. Now using the fact that $\varepsilon_n(\sigma) = 0$ for all $\sigma \in S_n$, and Lemma 3.10 we obtain that

$$F_D(\bar{q}) \prod_{i=1}^{n} \prod_{j=1}^{t} (1 - q_j^{2i}) = \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{i=1}^{t} (1 + q_i^n) \prod_{k \in K_i \cup \{n\}} q_k^{2k\varepsilon_k(\sigma_i)} \prod_{k \in \mathcal{C}_n(K_i \cup \{n\})} q_k^i$$

where the second sum runs over all $K_1, \ldots, K_t \subseteq [n-1]$ such that $K_t = (K_1, \ldots, K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})}$ and hence, by Corollary 3.17, Lemma 3.11 and the definition of α we conclude that

$$\begin{aligned}
\mathcal{Z}_{D_n,t}(\bar{q}) &= \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{k \in K_i \cup \{n\}} q_k^{2k\varepsilon_k(\sigma_i)} \prod_{k \in \mathcal{C}_n(K_i \cup \{n\})} q_k^i \\
&= \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{i=1}^{t} q_i^{ned_D(\sigma_i, K_i \setminus \{n\})} \\
&= \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{i=1}^{t-1} q_i^{Dmaj(\sigma_i, \varphi_{n-1}(K_i)} q_t^{Dmaj(\sigma_t, \varphi_{n-1}\left((K_1,\ldots,K_{t-1})^{(\sigma_1,\ldots,\sigma_{t-1})} \setminus \{n\}\right)} \\
&= \sum_{\sigma_1,\ldots,\sigma_t} \sum_{K_1,\ldots,K_t} \prod_{i=1}^{t-1} q_i^{Dmaj(\sigma_i, \varphi_{n-1}(K_i)} q_t^{Dmaj(\sigma_t, p(\sigma_{t-1} \cdots \sigma_1 \varphi_{n-1}(K_1) \triangle \cdots \triangle \sigma_1 \varphi_{n-1}(K_{t-1}))} \\
&= \sum_{\gamma_1,\ldots\gamma_{t-1} \in D_n} \prod_{i=1}^{t-1} q_i^{Dmaj(\gamma_i)} q_t^{Dmaj(\alpha(\gamma_1,\ldots,\gamma_{t-1}))}.
\end{aligned}$$

■

5.5 The case n odd

If n is odd the formula appearing in Theorem 5.12 can be slightly improved. In particular we define one more statistic, $Dmaj^o$, that allows us to obtain a formula for $\mathcal{Z}_{D_n}(\bar{q})$ similar to the corresponding ones for S_n and B_n appearing in Theorem 5.4 and Theorem 5.5. Consider the set $S_n \times 2^{[n-1]}$ with the binary operation

$$(\sigma, H) * (\tau, K) := \left(\sigma\tau, p(K \triangle \tau^{-1}(H))\right).$$

Proposition 5.13 *Let $n > 1$. Then $\Delta_n = \left(S_n \times 2^{[n-1]}, *\right)$ is a group.*

Proof. The operation is clearly well-defined, the identity element is $(e, \emptyset)$ and inversion is given by $(\sigma, H)^{-1} = (\sigma^{-1}, p\sigma(H))$. We check the associativity property

$$\begin{aligned}(\sigma, H) * ((\tau, K) * (\upsilon, L)) &= (\sigma, H) * \left(\tau\upsilon, p(L \triangle \upsilon^{-1}(K))\right) \\ &= \left(\sigma\tau\upsilon, p(p(L \triangle \upsilon^{-1}(K)) \triangle \upsilon^{-1}\tau^{-1}(H))\right) \\ &= \left(\sigma\tau\upsilon, p(L \triangle \upsilon^{-1}(K) \triangle \upsilon^{-1}\tau^{-1}(H))\right)\end{aligned}$$

and

$$\begin{aligned}((\sigma, H) * (\tau, K)) * (\upsilon, L) &= \left(\sigma\tau, p(K \triangle \tau^{-1}(H))\right) * (\upsilon, L) \\ &= \left(\sigma\tau\upsilon, p(L \triangle \upsilon^{-1} p(K \triangle \tau^{-1}(H)))\right) \\ &= \left(\sigma\tau\upsilon, p(L \triangle \upsilon^{-1}(K) \triangle \upsilon^{-1}\tau^{-1}(H))\right),\end{aligned}$$

where we have used the distributivity of υ^{-1} with respect to the symmetric difference and the fact that $p(p(H) \triangle K) = p(H \triangle K)$ for all $H, K \subseteq [n]$. ∎

Theorem 5.14 *Δ_n is isomorphic to D_n if and only if n is odd.*

Proof. It is not difficult to see that, if n is odd, the map $\Phi : D_n \to \Delta_n$ defined by

$$\gamma \mapsto (|\gamma|, p(Neg(\gamma)))$$

is an isomorphism, where $|\gamma| = (|\gamma_1|, \ldots, |\gamma_n|)$. Now suppose that n is even and let $\varphi : D_n \to \Delta_n$ be a group homomorphism. Let $(\sigma_i, K_i) = \varphi(s_i)$, for $i = 0, \ldots, n-1$, be the images of the Coxeter generators of D_n. Then the Coxeter relations for D_n force the permutations $\sigma_0, \ldots, \sigma_{n-1}$ to have the same sign and the sets $K_0, \ldots, K_{n-1}$ to have all the same parity. These conditions imply that the set $\{(\sigma_i, K_i) : i = 0, \ldots, n-1\}$ cannot generate Δ_n. ∎

Let $n \in \mathbf{N}$ be odd. Then we let

$$Dmaj^o := Dmaj \circ \Phi,$$

where we identify Δ_n with D_n through the pair notation and Φ is defined as in the proof of Theorem 5.14 above.
For example $Dmaj^o([3, -1, 5, 2, -4]) = Dmaj(31524, \{1, 3, 4\}) = 2 \cdot 5 + 3 = 13$.

Corollary 5.15 *Let $n \in \mathbf{N}$. Then*

$$\mathcal{Z}_{D_{2n+1}}(\bar{q}) = \sum_{\gamma_1,\ldots,\gamma_t} \prod_{i=1}^{t} q_i^{Dmaj^o(\gamma_i)}.$$

where the sum is over all $\gamma_1, \ldots, \gamma_t \in D_{2n+1}$ such that $\gamma_t \cdots \gamma_1 = id$.

Proof. It is an immediate consequence of the proof of Theorem 5.14 that

$$\alpha(\Phi(\gamma_1), \ldots, \Phi(\gamma_2)) = \Phi(\gamma_t \cdots \gamma_1)^{-1}$$

and the thesis follows from Theorem 5.12. ■

Theorem 5.14 implies that, if n is even, there is no $\Phi \in S(D_n)$ such that $\alpha(\Phi(\gamma_1), \ldots, \Phi(\gamma_2)) = \Phi(\gamma_t \cdots \gamma_1)^{-1}$ that would imply the corresponding result of Corollary 5.15. Nevertheless, we know that this result holds for $t = 2$ (Corollary 5.11) but we haven't been able to define a nice statistic, $Dmaj^e$, that works in this case, or to understand if it exists for $t > 2$. We therefore propose the following

Problem. Let $n \in \mathbf{N}$ be even. Is there a statistic $Dmaj^e : D_n \to \mathbf{N}$, necessarily equidistributed with length on D_n, such that

$$\mathcal{Z}_{D_n}(\bar{q}) = \sum_{\gamma_1,\ldots,\gamma_t} \prod_{i=1}^{t} q_i^{Dmaj^e(\gamma_i)}$$

with $\gamma_t \cdots \gamma_1 = id$?

6 Applications to Weyl groups of type B

In this last section we show how the ideas developed for the Weyl groups of type D can be used to give a new and simpler proof of the closed formula for $\mathcal{Z}_{B_n}(\bar{q})$ appearing in Theorem 5.5 which was discovered by Adin and Roichman [1] using different methods.

6.1 A Basis for TIA and DIA for B_n.

Let $W = B_n$. The tensor invariant algebra TIA is clearly equal to $(\mathbf{P}_n^{B_n})^{\otimes t}$. It is well known, (see, e.g.,[18, §3]), that $\mathbf{P}_n^{B_n}$ is freely generated (as an algebra) by the n elementary symmetric functions in the squares of the indeterminates, $x_1^2, \ldots, x_n^2$,

$$e_j(x_1^2, \ldots, x_n^2) := \sum_{1 \le i_1 < \ldots < i_j \le n} x_{i_1}^2 \cdots x_{i_j}^2$$

for $j \in [n]$. Hence

$$F_T(\bar{q}) = \prod_{i=1}^{t}\prod_{j=1}^{n} \frac{1}{(1-q_i^{2j})}.$$

For all $\bar{x}^f \in \mathbf{P}_n^{\otimes t}$ let

$$\pi(\bar{x}^f) := \sum_{\beta \in B_n} \varphi_D(\beta)(\bar{x}^f)$$

be the corresponding invariant element in DIA. Following [1, Claim 5.1] we have that

$$\pi(\bar{x}^f) \neq 0 \Longleftrightarrow f \in \mathcal{F}_{n,t}^e,$$

where $\mathcal{F}_{n,t}^e$ is defined in §5.1. An easy consequence is the following.

Lemma 6.1 *The set*

$$\left\{\pi(\bar{x}^f) : f \in \mathcal{B}_{n,t}^e\right\}$$

is a homogeneous basis for DIA.

Corollary 6.2 *The Hilbert series for* DIA *is*

$$F_D(\bar{q}) = \sum_{f \in \mathcal{B}_{n,t}^e} q_1^{|f_1|} \cdots q_t^{|f_t|}.$$

Note that we choose a different parametrization of the basis of DIA with respect to [1, Corollary 5.4] and we will use this one to compute the generating function $F_D(\bar{q})$.

6.2 The polynomial $\mathcal{Z}_{B_n}(\bar{q})$

Theorem 6.3 *Let $n, t \in \mathbf{N}$. Then*

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\beta_1,\ldots,\beta_t \in B_n} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},$$

where the sum is over all the signed permutation $\beta_1, \ldots, \beta_t \in B_n$ such that $\beta_t \cdots \beta_1 = id$.

Proof. By Corollary 6.2, (23) and (24) in the proof of Theorem 5.12 we easily obtain that

$$\mathcal{Z}_{B_n}(\bar{q}) = \sum_{\sigma_1,\ldots,\sigma_t} \sum_{H_1,\ldots,H_t} \prod_{i=1}^{t} \left(\prod_{h \in H_i} q_i^{2h\varepsilon_h(\sigma_i)} \prod_{h \in \mathcal{C}_n(H_i)} q_i^h \right),$$

where the sums run through all $\sigma_1, \ldots, \sigma_t \in S_n$ and $H_1, \ldots H_t \subseteq [n]$ such that $\sigma_t = (\sigma_{t-1} \cdots \sigma_1)^{-1}$ and $H_t = (H_1, \ldots, H_{t-1})^{(\sigma_1, \ldots, \sigma_{t-1})}$. By Theorem 3.15 and Corollary 3.9, we conclude that

$$
\begin{aligned}
\mathcal{Z}_{B_n}(\bar{q}) &= \sum_{\sigma_1, \ldots, \sigma_t} \sum_{H_1, \ldots, H_t} \prod_{i=1}^{t} q_i^{ned_B(\sigma_i, (H_i))} \\
&= \sum_{\sigma_1, \ldots, \sigma_t} \sum_{H_1, \ldots, H_t} \prod_{i=1}^{t} q_i^{fmaj(\sigma_i, \varphi_n(H_i))} \\
&= \sum_{\sigma_1, \ldots, \sigma_t} \sum_{H_1, \ldots, H_{t-1}} \prod_{i=1}^{t-1} q_i^{fmaj(\sigma_i, \varphi_n(H_i))} q_t^{fmaj(\sigma_t, \sigma_{t-1} \ldots \sigma_1 \varphi_n(H_1) \triangle \cdots \triangle \sigma_{t-1} \varphi_n(H_{t-1}))} \\
&= \sum_{\beta_t \cdots \beta_1 = id} \prod_{i=1}^{t} q_i^{fmaj(\beta_i)},
\end{aligned}
$$

since

$$((\sigma_{t-1}, H_{t-1}) \cdots (\sigma_1, H_1))^{-1} = ((\sigma_{t-1} \cdots \sigma_1)^{-1}, \sigma_{t-1} \cdots \sigma_1(H_1) \triangle \cdots \triangle \sigma_{t-1}(H_{t-1})).$$

■

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Biagioli Riccardo
LaCIM, Université du Québec à Montréal,
Case Postale 8888, succursale Centre-ville, Montréal, Québec, H3C 3P8
Canada
e-mail: biagioli@math.uqam.ca

Caselli Fabrizio
Università di Roma "La Sapienza"
Dipartimento di Matematica
P.le A. Moro 5, 00185 Roma, Italy
e-mail: caselli@mat.uniroma1.it