

Stability of Unit Hopf Vector Fields on Quotients of Spheres

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Abstract. – The volume of a unit vector field V of a Riemannian manifold (M, g) is the volume of its image $V(M)$ in the unit tangent bundle endowed with the Sasaki metric. Unit Hopf vector fields, that is, unit vector fields that are tangent to the fiber of a Hopf fibration $\mathbb{S}^n \rightarrow \mathbb{C}P^{\frac{n-1}{2}}$ (n odd) are well known to be critical for the volume functional on the round n -dimensional sphere $\mathbb{S}^n(r)$ for every radius $r > 1$. Regarding the Hessian, it turns out that its positivity actually depends on the radius. Indeed, in [2], it is proven that for $n \geq 5$ there is a critical radius $r_c = \frac{1}{\sqrt{n-4}}$ such that Hopf vector fields are stable if and only if $r \leq r_c$. In this paper we consider the question of the existence of a critical radius for space forms $M^n(c)$ (n odd) of positive curvature c . These space forms are isometric quotients $\mathbb{S}^n(r)/\Gamma$ of round spheres and naturally carry a unit Hopf vector field which is critical for the volume functional. We prove that $r_c = +\infty$, unless Γ is trivial. So, in contrast with the situation for the sphere, the Hopf field is stable on $\mathbb{S}^n(r)/\Gamma$, $\Gamma \neq \{Id\}$, whatever the radius.

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1 Introduction and Main Results

In [10], H. Gluck and W. Ziller raised the question of finding unit vector fields of \mathbb{S}^{2m+1} which are the “best organised” ones. Precisely, they first defined a volume functional Vol on the space of unit vector fields and then asked for finding the infimum of Vol and possibly its minimizers. This volume functional is the natural one: it maps a unit vector field V onto the volume of the image submanifold in the unit tangent bundle $V(\mathbb{S}^{2m+1}) \subset T^1\mathbb{S}^{2m+1}$ endowed with the Sasaki metric (the metric which canonically extends the metric of the base space to the tangent space). For $m = 1$, Gluck and Ziller showed that the infimum is reached by *Hopf fields*, *i. e.* unit vector fields tangent to the fibers of a Hopf fibration $\mathbb{S}^{2m+1} \rightarrow \mathbb{C}P^m$. For $m > 1$, the

question revealed to be more muddled than initially expected and gave rise to a sizeable amount of work (see [6] for a survey). In particular, Hopf fields are no longer minimizers of the volume on the unit sphere \mathbb{S}^{2m+1} , $m > 1$. In 1993, S. L. Pedersen conjectured that for $m > 1$ the infimum is reached by a singular field derived from a Pontrjagin cycle [18]. This conjecture is still open. In 2004, a non-trivial lower bound for the volume is obtained by F. Brito, P. Chacón and A. Naveira [4], on the other hand the regularity of minimizers is investigated by D. Johnson and P. Smith in a sequel of articles [15],[16] and [17]. It was suspected in [9] that the radius of the base sphere could play a role and this was confirmed in [2]. Precisely, whatever the radius of $\mathbb{S}^{2m+1}(r)$ Hopf vector fields are minimal but, as soon as $m > 1$, the non-negativity of the Hessian, that is their stability, depends on the radius. In fact, for each odd-dimensional sphere \mathbb{S}^{2m+1} , $m > 1$, there exists a critical radius $r_c = \frac{1}{\sqrt{2m-3}}$ such that Hopf fields are stable if and only if $r \leq r_c$. It was then noticed that a similar phenomenon occurs in a more general setting, namely for K -contact manifolds [14] (see also [19]). The goal of this article is to investigate the question of the existence of critical radius for quotients of spheres. Any space form with positive sectional curvature is a quotient of a $\mathbb{S}^{2m+1}(r)$ by a finite fixed point free isometry subgroup Γ of $O(2m+2)$. It turns out that there is still (at least) a Hopf field H_Γ on the quotient $\mathbb{S}^{2m+1}(r)/\Gamma$ which remains critical for the volume functional. But something *a priori* unexpected occurs: the field H_Γ is always stable, whatever the radius $r > 0$.

Theorem 1.— *Let $M = \mathbb{S}^{2m+1}(r)/\Gamma$ be a space form with $\Gamma \neq \{Id\}$, then the Hopf field H_Γ is stable on $\mathbb{S}^{2m+1}(r)/\Gamma$.*

As a consequence, there is no critical radius for quotients $\mathbb{S}^{2m+1}(r)/\Gamma$, $\Gamma \neq \{Id\}$. Technics used to prove this theorem differ from the ones of [2]. In that last article the crucial point to prove the non negativity of the Hessian for $r \leq r_c$ was to take advantage of the Hopf fibration to read any vector field as a Fourier serie along the fibers of the fibration. Here, our starting point is the fact that the non negativity of the Hessian of the volume is implied by the non negativity of the Hessian of a simpler functional : the energy. Given a vector field V , its *energy* is the number

$$E(V) := \frac{1}{2} \int_{\mathbb{S}^{2m+1}(r)} (2m+1 + \|\nabla V\|^2) dvol,$$

the relevant term $B(V) := \int_{\mathbb{S}^{2m+1}(r)} \|\nabla V\|^2 dvol$ being called the *bending* of V . In constrast with the volume, the Hessian of the energy behaves homogeneously with the radius and no critical radius phenomenon can appear for it. We then observe that the stability of the Hopf vector field as a critical point of the energy is equivalent to a certain lower bound of the first eigenvalue of

an elliptic operator acting over vector fields orthogonal to the Hopf distribution. This lower bound is obtained by relating the elliptic operator with the rough Laplacian for functions over the sphere. The key point is that the geometry of any quotient $\mathbb{S}^{2m+1}(r)/\Gamma$, $\Gamma \neq \{Id\}$ forces the vanishing of the constant term of the homogeneous harmonic decomposition of a vector field. That is why vector fields that give unstable directions for the Hessian of volume on spheres do not descend to the quotient.

In §4, we inspect the situation for quotients of Berger spheres. A Berger sphere $(\mathbb{S}^{2m+1}, g_\mu)$ is a sphere in which the usual metric has been modified of a factor μ in the Hopf direction (see §4 for a precise definition). In a Berger sphere, the stability of both the volume and the energy at the Hopf fields depends on μ , precisely there exists μ_{vol} (resp. μ_e) such that the Hopf field is stable for the volume (resp. the energy) if and only if $\mu \leq \mu_{vol}$ (resp. $\mu \leq \mu_e$) [8]. The numbers μ_{vol} and μ_e are both equal to 1 for \mathbb{S}^3 and decrease toward zero when the dimension of the sphere increases. Once again the situation simplifies for quotients due to the fact that certain unstable directions on $(\mathbb{S}^{2m+1}, g_\mu)$ do not descend to $(\mathbb{S}^{2m+1}/\Gamma, g_\mu)$.

Theorem 2 and 3.— *Let $0 < \mu \leq 1$ then the Hopf field H_Γ^μ is stable for both energy and volume on $(\mathbb{S}^{2m+1}/\Gamma, g_\mu)$, $\Gamma \neq \{Id\}$, $m \geq 1$. Moreover any non identically zero vector field orthogonal to H_Γ^μ provides an unstable direction for both energy and volume if μ is large enough.*

This result is sharp in the following sense: it cannot be improved for Berger projective spaces $(\mathbb{R}P^{4m-1}, g_\mu)$, $m \geq 1$.

Proposition 4.— *Let $\Gamma = \mathbb{Z}_2$. If $\mu > 1$, the Hopf vector field of H_Γ^μ is unstable for both energy and volume on $(\mathbb{R}P^{4m-1}, g_\mu)$, $m \geq 1$.*

2 Hopf Fields on Spherical Space Forms

2.1 Spherical Space forms

For the sake of completeness, we recall in this subsection the basic facts about spherical space forms that will be needed in the sequel of the article. Our reference is [24].

A well-known result of W. Killing and H. Hopf states that any complete connected manifold M^n with constant positive sectional curvature is isometric to a quotient $\mathbb{S}^n(r)/\Gamma$ of a sphere of radius $r > 0$ by a finite group

$\Gamma < O(n+1)$ of fixed point free isometries¹. The free action of Γ just means that only the identity element of Γ has +1 as eigenvalue. If G is a subgroup of $O(n+1)$ conjugate to Γ (*i. e.* $\exists h \in O(n+1)$ such that $\Gamma = hGh^{-1}$) then h induces an isometry $\bar{h} : \mathbb{S}^n(r)/\Gamma \rightarrow \mathbb{S}^n(r)/G$ by conjugating classes. Conversely, if $\bar{h} : \mathbb{S}^n(r)/\Gamma \rightarrow \mathbb{S}^n(r)/G$ is an isometry, standard arguments of Riemannian coverings theory show that Γ and G are conjugate.

The classification problem for complete connected Riemannian manifolds of constant positive curvature was solved by G. Vincent in the late 1940s [22]. Its main conceptual technique was to view Γ as the image of an abstract group G by a fixed point free real orthogonal representation $\sigma : G \rightarrow O(n+1)$: every complete connected Riemannian manifold M^n of positive constant sectional curvature is isometric to a quotient

$$\mathbb{S}^n(r)/\sigma(G) = \mathbb{S}^n(r)/(\sigma_1 \oplus \dots \oplus \sigma_s)(G)$$

where $\sigma = \bigoplus_{i=1}^s \sigma_i$, $\sigma_i : G \rightarrow O(W_i)$ are irreducible and fixed point free and $\sum_{i=1}^s \dim W_i = n+1$. We then need to determine groups which admit fixed point free irreducible representations and then to classify irreducible representations of such groups. This last point is achieved with the help of the Frobenius-Schur theorem by classifying complex irreducible representations $\pi : G \rightarrow \text{End}(V \otimes \mathbb{C})$. Every real representation $\sigma : G \rightarrow \text{End}(V)$ induces a complex representation $\sigma^{\mathbb{C}} : G \rightarrow \text{End}(V \otimes \mathbb{C})$ by \mathbb{C} -linear extension. Conversely, if $\bar{\pi}$ is the conjugate representation of π , then $\pi \oplus \bar{\pi}$ is equivalent to a representation $\sigma^{\mathbb{C}}$ with σ real.² Indeed, if $\pi(g) = X + iY$ with X, Y real matrices and if

$$A = \begin{bmatrix} Id & -Id \\ -Id & Id \end{bmatrix},$$

$$(\pi \oplus \bar{\pi})(g) = \begin{bmatrix} X + iY & 0 \\ 0 & X - iY \end{bmatrix} \in \text{End}((V \otimes \mathbb{C}) \oplus (V \otimes \mathbb{C})),$$

then

$$A \circ (\pi \oplus \bar{\pi})(g) \circ A^{-1} = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = \sigma^{\mathbb{C}}(g) \in \text{Gl}((V \oplus V) \otimes \mathbb{C})$$

where

$$\sigma(g) = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} = \begin{bmatrix} \text{Re}(\pi(g)) & -\text{Im}(\pi(g)) \\ \text{Im}(\pi(g)) & \text{Re}(\pi(g)) \end{bmatrix} \in \text{Gl}(V \oplus V).$$

Frobenius-Schur theorem states that for every irreducible real representation $\sigma : G \rightarrow \text{Gl}(W)$ there exists an irreducible complex representation π

¹All our manifolds are assume to be without boundary

²Two representations $\phi, \psi : G \rightarrow \text{End}(V \otimes \mathbb{C})$ are *equivalent* if there exists a \mathbb{C} -linear isomorphism $A \in \text{Gl}(V \otimes \mathbb{C})$ such that: $\forall g \in G; A\phi(g)A^{-1} = \psi(g)$.

such that:

- either $\sigma^{\mathbb{C}}$ is equivalent to $\pi \oplus \widehat{\pi}$ with $\pi : G \rightarrow \text{End}(V \otimes \mathbb{C})$, $V \oplus V = W$,
- either $\sigma^{\mathbb{C}}$ is equivalent to π with $\pi : G \rightarrow \text{End}(W \otimes \mathbb{C})$, then we commit an abuse of language and say that π is equivalent to a real representation.

The classification of fixe point free irreducible complex representations π shows that they are not equivalent to a real representation except if $G = \{Id\}$ or \mathbb{Z}_2 . In other words, if M^n is complete, connected and with positive sectional curvature then:

- either M^n is isometric to $\mathbb{S}^n(r)$ or $\mathbb{R}P^n(r)$,
- either n is odd, M^n is isometric to $\mathbb{S}^n(r)/\Gamma$ with $\Gamma < U(\frac{n+1}{2})$.

Indeed, $\Gamma = (\sigma_1 \oplus \dots \oplus \sigma_s)(G)$ with $\sigma_i : G \rightarrow O(W_i)$ and if $G \neq \{Id\}$ and $G \neq \mathbb{Z}_2$ then, from the Frobenius-Schur theorem, for all $g \in G$ and for all $i \in \{1, \dots, s\}$, $\sigma_i(g)$ is of the form

$$\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \in U(V_i \otimes \mathbb{C}) \subset O(V_i \oplus V_i),$$

where $W_i = V_i \oplus V_i$. In particular, if M^{2m+1} is complete, connected and with positive sectional curvature then M^{2m+1} is orientable.

2.2 Hopf fields

Definition.— Let M^{2m+1} be a complete connected manifold with positive constant sectional curvature. We say that a unit vector field $V : M \rightarrow T^1M$ is a unit *Hopf field* on M^{2m+1} if there exists an isometry from M^{2m+1} to some $\mathbb{S}^{2m+1}(r)/\Gamma$ such that the image of V lifts to a unit Hopf vector field of \mathbb{S}^{2m+1} .

Proposition 1.— *Every complete connected manifold M^{2m+1} with positive constant sectional curvature admits (at least) one unit Hopf vector field.*

Proof of Proposition 1.— The result is obvious if M^{2m+1} is isometric to $\mathbb{S}^{2m+1}(r)$. If M^{2m+1} is isometric to $\mathbb{R}P^{2m+1}$, then the Hopf field $H(x) = Jx$ on $\mathbb{S}^{2m+1}(r) \subset (\mathbb{R}^{2m+2}, J) \approx \mathbb{C}^{m+1}$ defined a Hopf field on the quotient $\mathbb{R}P^{2m+1}(r)$ since $H(-x) = -H(x)$. In the other cases, from the results mentioned above, M^{2m+1} is isometric to $\mathbb{S}^{2m+1}(r)/\Gamma$ where $\Gamma < U(m+1)$. The field $H(x) = Jx$ of $\mathbb{S}^{2m+1}(r) \subset (\mathbb{R}^{2m+2}, J)$ induces a well-defined vector field on the quotient $\mathbb{S}^{2m+1}(r)/\Gamma$ if and only if

$$\forall g \in \Gamma, \forall x \in \mathbb{S}^{2m+1}(r) : d\pi_x(H(x)) = d\pi_{g(x)}(H(g(x)))$$

where $\pi : \mathbb{S}^{2m+1}(r) \rightarrow \mathbb{S}^{2m+1}(r)/\Gamma$ is the Riemannian covering. Since $\pi \circ g = \pi$ we have $d\pi_x(H(x)) = d\pi_{g(x)}(g(H(x)))$. It ensues that

$$d\pi_{g(x)}(g(H(x))) = d\pi_{g(x)}(H(g(x))).$$

Since $d\pi_x : T_x\mathbb{S}^{2m+1}(r) \rightarrow T_{\pi(x)}(\mathbb{S}^{2m+1}(r)/\Gamma)$ is an isomorphism we deduce that

$H(g(x)) = g(H(x))$ i. e. $gJ = Jg$ for every $g \in \Gamma$ and this last condition is fulfilled since $\Gamma < U(m+1)$. \square

Notation. We denote by H_Γ the unit Hopf vector field on $\mathbb{S}^{2m+1}(r)/\Gamma$ induced by the unit Hopf vector field $H(x) = Jx$ of $\mathbb{S}^{2m+1}(r) \subset (\mathbb{R}^{2m+2}, J)$. Given an isometry $f : \mathbb{S}^{2m+1}(r)/\Gamma \rightarrow M^{2m+1}$ the image f_*H_Γ is a unit Hopf field of M^{2m+1} . By a slight abuse of notation, we will still denote by H_Γ any vector field of M^{2m+1} of the form f_*H_Γ for some isometry f .

Harmonicity and minimality. A unit vector field V on a Riemannian manifold (M, g) can be considered as a map $V : M \rightarrow T^1M$ of M into its unit tangent bundle. If we equip T^1M with the Sasaki metric g^S , V is called a *harmonic* unit vector field if the map is harmonic and a *minimal* unit vector field if the submanifold $V(M)$ is minimal in (T^1M, g^S) . (see.

Lemma 1.– *Unit Hopf fields are minimal and harmonic.*

Proof of Lemma 1.– This is obvious since unit Hopf fields of $\mathbb{S}^{2m+1}(r)$ are minimal and harmonic (see [7] for instance) and $\pi : \mathbb{S}^{2m+1}(r) \rightarrow \mathbb{S}^{2m+1}(r)/\Gamma$ is a Riemannian covering. \square

3 Bending over Sections of the Orthogonal Distribution of the Hopf Field

The key point in the proof of our main result is to obtain the following estimate for the bending of vector fields orthogonal to H_Γ .

Proposition 2. – *Let V be a smooth vector field of $\mathbb{S}^{2m+1}(r)/\Gamma$, $\Gamma \neq \{Id\}$, such that at every point $\langle H_\Gamma, V \rangle = 0$, then:*

$$\int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|\nabla V\|^2 dvol \geq \frac{2m}{r^2} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|V\|^2 dvol.$$

General strategy of the proof.– Let Δ be the rough Laplacian on sections of $T\mathbb{S}^{2m+1}(r)$, that is $\Delta = \nabla^*\nabla$ where ∇^* is the L_2 -adjoint of ∇ . Given a Hopf field H , we denote by $\mathcal{E} = \Gamma(H^\perp)$ the space of sections of the orthogonal distribution to H and we define an operator $L : \mathcal{E} \rightarrow \mathcal{E}$ by:

$$L(V) := \Delta V - \langle \Delta V, H \rangle H$$

Since L is an elliptic operator, the set of its eigenvalues is countably infinite with $+\infty$ as limit point (see for instance [21] p. 39, proposition 5.1 and p. 69,

theorem 8.3 for a proof of that classical result). Moreover:

$$\begin{aligned}
\int_{\mathbb{S}^{2m+1}(r)} \langle L(V), V \rangle dvol &= \int_{\mathbb{S}^{2m+1}(r)} (\langle \Delta V, V \rangle - \langle \Delta V, H \rangle \langle V, H \rangle) dvol \\
&= \int_{\mathbb{S}^{2m+1}(r)} \langle \Delta(V), V \rangle dvol \\
&= \int_{\mathbb{S}^{2m+1}(r)} \|\nabla V\|^2 dvol \geq 0
\end{aligned}$$

since $\langle V, H \rangle = 0$. Therefore Proposition 2 reduces to an estimation of the lowest eigenvalue of L . The precise computation of eigenvalues of L proves to be difficult, we bypass this by introducing another operator $\bar{L} : \mathcal{E} \rightarrow \mathcal{E}$ the eigenvalues of which are easier to control. In the sequel, we first define \bar{L} and study its link with L (Lemma 2), we then state a property of unit vector fields that descend on a proper quotient of spheres (Lemma 3), that property is crucial to obtain an estimate (Proposition 3) which will imply Proposition 2.

Operator \bar{L} .— Precisely let

$$\bar{L}(V) = \pi \circ \bar{\Delta}V$$

where π is the canonical projection onto the orthogonal distribution H^\perp of H :

$$\pi : T\mathbb{S}^{2m+1}(r) \rightarrow H^\perp$$

and $\bar{\Delta}$ is the rough Laplacian on sections of the trivial bundle

$$T\mathbb{R}^{2m+2} |_{\mathbb{S}^{2m+1}(r)} = \mathbb{S}^{2m+1}(r) \times \mathbb{R}^{2m+2} \rightarrow \mathbb{S}^{2m+1}(r).$$

In particular, if (E_1, \dots, E_{2m+1}) is a local orthonormal frame of $T\mathbb{S}^{2m+1}(r)$, $\bar{\nabla}$ is the standard connection of $\mathbb{R}^{2m+2} \supset \mathbb{S}^{2m+1}(r)$ and V is a vector field of the sphere considered as a map from $\mathbb{S}^{2m+1}(r)$ to \mathbb{R}^{2m+2} , then

$$\bar{\Delta}V = - \sum_{i=1}^{2m+1} (\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V - \bar{\nabla}_{\nabla_{E_i} E_i} V).$$

Lemma 2. — *If V is a section of $H^\perp \rightarrow \mathbb{S}^{2m+1}(r)$ then $L(V) = \bar{L}(V) - \frac{1}{r^2}V$.*

Proof of Lemma 2. — Let (E_1, \dots, E_{2m}, H) be a local orthonormal basis of $\mathbb{S}^{2m+1}(r)$ such that, at the point x where the computation is done, $\nabla_{E_i} E_i(x) = 0$ for all $i \in \{1, \dots, 2m\}$. We have:

$$\begin{aligned}
\nabla_{E_i} V &= \bar{\nabla}_{E_i} V - \langle \bar{\nabla}_{E_i} V, N \rangle N \\
&= \bar{\nabla}_{E_i} V + \langle \bar{\nabla}_{E_i} N, V \rangle N \\
&= \bar{\nabla}_{E_i} V + \frac{1}{r} \langle V, E_i \rangle N
\end{aligned}$$

so

$$\nabla_{E_i} \nabla_{E_i} V = \bar{\nabla}_{E_i} \nabla_{E_i} V - \langle \bar{\nabla}_{E_i} \nabla_{E_i} V, N \rangle N.$$

But

$$\begin{aligned} \bar{\nabla}_{E_i} \nabla_{E_i} V &= \bar{\nabla}_{E_i} (\bar{\nabla}_{E_i} V + \frac{1}{r} \langle V, E_i \rangle N) \\ &= \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V + \frac{1}{r} (\langle \bar{\nabla}_{E_i} V, E_i \rangle N + \langle V, \bar{\nabla}_{E_i} E_i \rangle N + \langle V, E_i \rangle \bar{\nabla}_{E_i} N) \end{aligned}$$

Since $\bar{\nabla}_{E_i} E_i = \nabla_{E_i} E_i + \langle \bar{\nabla}_{E_i} E_i, N \rangle N = -\frac{1}{r} N$, we have

$$\nabla_{E_i} \nabla_{E_i} V = \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V - \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V, N \rangle N + \frac{1}{r^2} \langle V, E_i \rangle E_i.$$

Therefore

$$(\nabla_{E_i} \nabla_{E_i} V)^{H^\perp} = (\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V)^{H^\perp} + \frac{1}{r^2} \langle V, E_i \rangle E_i$$

where $(X)^{H^\perp}$ denotes the H^\perp -component of X . It remains to deal with $\nabla_H(\nabla_H V)$, we have:

$$\begin{aligned} \nabla_H V &= \bar{\nabla}_H V - \langle \bar{\nabla}_H V, N \rangle N \\ &= \bar{\nabla}_H V + \frac{1}{r} \langle V, H \rangle N \\ &= \bar{\nabla}_H V \end{aligned}$$

and

$$\begin{aligned} \nabla_H \nabla_H V &= \bar{\nabla}_H \bar{\nabla}_H V - \langle \bar{\nabla}_H \bar{\nabla}_H V, N \rangle N \\ &= \bar{\nabla}_H \bar{\nabla}_H V + \langle \bar{\nabla}_H V, \bar{\nabla}_H N \rangle N \\ &= \bar{\nabla}_H \bar{\nabla}_H V + \langle V, N \rangle N \\ &= \bar{\nabla}_H \bar{\nabla}_H V. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} L(V) &= - \sum_{i=1}^{2m} (\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V + \frac{1}{r^2} \langle V, E_i \rangle E_i)^{H^\perp} - (\bar{\nabla}_H \bar{\nabla}_H V)^{H^\perp} \\ &= - \sum_{i=1}^{2m+1} (\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V)^{H^\perp} - \frac{1}{r^2} V = \bar{L}(V) - \frac{1}{r^2} V \end{aligned}$$

□

Let (e_1, \dots, e_{2m+2}) be the standard basis of \mathbb{R}^{2m+2} and $V = \sum_{i=1}^{2m+2} V_i e_i$, then a direct computation shows that

$$\bar{\Delta} V = \sum_{i=1}^{2m+2} (\Delta V_i) e_i$$

where Δ is the Laplacian on functions of the sphere. The link between \bar{L} and Δ is crucial to estimate the first eigenvalue of L but we need an extra argument to show that this eigenvalue is not too small, *i. e.* greater or equal to $2m$. This will be a consequence of the following lemma.

Lemma 3.— *Let $\Gamma \neq \{Id\}$ be a finite fixed point free isometry subgroup of $O(2m+2)$ and let $V : \mathbb{S}^{2m+1}(r) \rightarrow \mathbb{R}^{2m+2}$ be a continuous map equivariant under the action of Γ (*i.e.* $\forall g \in \Gamma, V \circ g = g \circ V$) then $\int_{\mathbb{S}^{2m+1}(r)} V dvol = 0$.*

Proof of Lemma 3.— We denote by N the cardinality of Γ and by $1, g_1, \dots, g_{N-1}$ the elements of Γ . Let $X \subset \mathbb{S}^{2m+1}$ be a fundamental domain for the Γ -action: $X \cup g_1 X \cup \dots \cup g_{N-1} X = \mathbb{S}^{2m+1}$. We have:

$$\begin{aligned} \int_{\mathbb{S}^{2m+1}(r)} V dvol &= \int_X V dvol + \int_{g_1 X} V dvol + \dots + \int_{g_{N-1} X} V dvol \\ &= \int_X V dvol + \int_X V \circ g_1 dvol + \dots + \int_X V \circ g_{N-1} dvol \\ &= \int_X V dvol + \int_X g_1 \circ V dvol + \dots + \int_X g_{N-1} \circ V dvol \\ &= \int_X (1 + g_1 + \dots + g_{N-1}) \circ V dvol. \end{aligned}$$

Let x be any point of \mathbb{S}^{2m+1} , we denote by $W(x)$ the vector

$$(1 + g_1 + \dots + g_{N-1})(V(x)) \in \mathbb{R}^{2m+2}.$$

It is obvious that $W(x)$ is fixed by any element of Γ . Since Γ is a fixed point free isometry subgroup of $O(2m+2)$, necessarily $W(x)$ vanishes. Thus

$$\int_{\mathbb{S}^{2m+1}(r)} V dvol = 0. \quad \square$$

Proposition 3.— *Let $\Gamma \neq \{Id\}$ be a fixed point free isometry subgroup of $O(2m+2)$ acting isometrically on $\mathbb{S}^{2m+1}(r)$. If $V \in \Gamma(H_\Gamma^\perp)$ descends to the quotient $\mathbb{S}^{2m+1}(r)/\Gamma$ then*

$$\int_{\mathbb{S}^{2m+1}(r)/\Gamma} \langle \bar{\Delta} V, V \rangle dvol \geq \frac{2m+1}{r^2} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|V\|^2 dvol.$$

Remark. — In that proposition $\bar{\Delta}$ must be understood as an operator of the space \mathcal{E}_Γ of sections of the distribution H_Γ^\perp over $\mathbb{S}^{2m+1}(r)/\Gamma$. Note that \mathcal{E}_Γ can be identified with the space of vector fields of $\mathbb{S}^{2m+1}(r)$ that are both equivariant for the action of Γ and orthogonal to the distribution H . This

last space is stable by L and \bar{L} , we denote by L_Γ and \bar{L}_Γ the restriction of these operators on that space. Of course, we can also consider these new operators L_Γ and \bar{L}_Γ as operators on \mathcal{E}_Γ .

Proof of Proposition 3.– From the density of polynomial functions in $C^0(\mathbb{S}^n)$, it is enough to show the proposition in the case where the components $(V^1, V^2, \dots, V^{2m+2})$ of $V = \sum_{i=1}^{2m+2} V^i e_i$ are polynomial functions (we see V as an equivariant vector field of $\mathbb{S}^{2m+1}(r) \subset \mathbb{R}^{2m+2}$, thus defining a map $V : \mathbb{S}^{2m+1}(r) \rightarrow \mathbb{R}^{2m+2}$). Let $k \geq 0$, we set:

$$\mathcal{P}_{\leq k} := \{ \text{restrictions } P|_{\mathbb{S}^{2m+1}(r)} \text{ of polynomial functions } P : \mathbb{R}^{2m+2} \rightarrow \mathbb{R} \text{ of degree } \leq k \}$$

and

$$\mathcal{H}_k := \{ \text{restrictions } P|_{\mathbb{S}^{2m+1}(r)} \text{ of homogeneous harmonic polynomial functions } P : \mathbb{R}^{2m+2} \rightarrow \mathbb{R} \text{ of degree } k \}.$$

It is well-known (see [5] for instance) that homogeneous harmonic polynomial functions are eigenvector for the Laplacian on the sphere $\mathbb{S}^{2m+1}(r)$:

$$\forall P \in \mathcal{H}_k, \quad \Delta P = \lambda_k P \quad \text{with } \lambda_k = \frac{k(k+2m)}{r^2}.$$

Moreover $\mathcal{P}_{\leq k}$ admits an orthogonal decomposition $\mathcal{P}_{\leq k} = \bigoplus_{j=0}^k \mathcal{H}_j$, hence V has an orthogonal decomposition:

$$V = V_0 + V_1 + \dots + V_k$$

with

$$V_j = \begin{pmatrix} V_j^1 \\ \vdots \\ V_j^{2m+2} \end{pmatrix} \quad j \in \{0, \dots, k\} \quad V_j \in \overbrace{\mathcal{H}_j \times \dots \times \mathcal{H}_j}^{(2m+2) \text{ times}}$$

It ensues that

$$\bar{\Delta} V = \lambda_1 V_1 + \dots + \lambda_k V_k,$$

since $\lambda_0 = 0$. From Lemma 3, and since V descends to the quotient $\mathbb{S}^{2m+1}(r)/\Gamma$, we have:

$$\int_{\mathbb{S}^{2m+1}(r)} V \, dvol = 0 \in \mathbb{R}^{2m+2}.$$

For $j > 0$ we have $\lambda_j \neq 0$ and therefore:

$$\begin{aligned} \int_{\mathbb{S}^{2m+1}(r)} V_j \, dvol &= \frac{1}{\lambda_j} \int_{\mathbb{S}^{2m+1}(r)} \lambda_j V_j \, dvol \\ &= \frac{1}{\lambda_j} \int_{\mathbb{S}^{2m+1}(r)} \bar{\Delta} V_j \, dvol = 0, \end{aligned}$$

since $\mathbb{S}^{2m+1}(r)$ is compact. Thus the relation $\int_{\mathbb{S}^{2m+1}(r)} V \, dvol = 0$ implies that $\int_{\mathbb{S}^{2m+1}(r)} V_0 \, dvol = 0$ which in turn implies that $V_0 = 0$ since \mathcal{H}_0 is the space of constant functions. We have:

$$\begin{aligned} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \langle \bar{\Delta} V, V \rangle \, dvol &= \sum_{j=1}^k \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \lambda_j \|V_j\|^2 \, dvol \\ &\geq \lambda_1 \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \left(\sum_{j=1}^k \|V_j\|^2 \right) \, dvol \\ &\geq \lambda_1 \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|V\|^2 \, dvol. \end{aligned}$$

Since $\lambda_1 = \frac{2m+1}{r^2}$ we obtain the proposition. \square

Proof of Proposition 2.— It follows easily from Proposition 1 since:

$$\begin{aligned} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \langle L(V), V \rangle \, dvol &= \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \left(\langle \bar{L}(V), V \rangle - \frac{1}{r^2} \|V\|^2 \right) \, dvol \\ &= \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \left(\langle \bar{\Delta}(V), V \rangle - \frac{1}{r^2} \|V\|^2 \right) \, dvol \\ &\geq \left(\lambda_1 - \frac{1}{r^2} \right) \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|V\|^2 \, dvol \\ &\geq \frac{2m}{r^2} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|V\|^2 \, dvol. \end{aligned}$$

\square

Proof of Theorem 1

Theorem 1.— *Let $M = \mathbb{S}^{2m+1}(r)/\Gamma$ be a space form with $\Gamma \neq \{Id\}$, then the Hopf field H_Γ is stable on $\mathbb{S}^{2m+1}(r)/\Gamma$ for both energy and volume.*

Proof of Theorem 1.— ENERGY.— Let $V \in \mathcal{E}_\Gamma$, from Lemma 10 of [9] the Hessian of the energy functional E at H_Γ has the following expression:

$$(Hess E)_{H_\Gamma}(V) = \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \left(-\frac{2m}{r^2} \|V\|^2 + \|\nabla V\|^2 \right) \, dvol.$$

From Proposition 2 we have:

$$\int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|\nabla V\|^2 dvol \geq \frac{2m}{r^2} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \|V\|^2 dvol,$$

thus $(Hess E)_{H_\Gamma}(V) \geq 0$ and the Hopf field H_Γ is stable for E on $\mathbb{S}^{2m+1}(r)/\Gamma$.

VOLUME.— Let $V \in \mathcal{E}_\Gamma$, the volume of V is given by

$$Vol(V) := \frac{1}{2} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \sqrt{\det L_V} dvol$$

where $L_V = Id + {}^T \nabla V \circ \nabla V$. From the expression of the Hessian of the volume functional Vol at H_Γ given in the Main Proposition of [2] we deduce:

$$(Hess Vol)_{H_\Gamma}(V) = \left(1 + \frac{1}{r^2}\right)^{m-2} \int_{\mathbb{S}^{2m+1}(r)/\Gamma} \left(-\frac{2m}{r^2} \|V\|^2 + \|\nabla V\|^2 + \frac{1}{r^2} \|\nabla_{H_\Gamma} V + \frac{1}{r} JV\|^2\right) dvol,$$

hence

$$(Hess Vol)_{H_\Gamma}(V) \geq \left(1 + \frac{1}{r^2}\right)^{m-2} (Hess E)_{H_\Gamma}(V).$$

Therefore the Hopf field H_Γ is stable for Vol on $\mathbb{S}^{2m+1}(r)/\Gamma$. \square

Remark.— A lot of works have been devoted to the study of volume and energy of vector fields in dimension three (see [11],[13], [23] for instance), in particular, for Riemannian quotients of $\mathbb{S}^3(r)$, our theorem can be deduced from the work of F. Brito [3].

4 Energy, Volume and Quotients of Berger Spheres

4.1 Hopf Fields on Berger Spheres

Recall that the Berger metric on $\mathbb{S}^{2m+1}(1) \subset (\mathbb{R}^{2m+2}, \langle \cdot, \cdot \rangle)$ is the one-parameter family of metrics $(g_\mu)_{\mu>0}$ defined by:

$$g_\mu|_{\mathcal{H}} = \mu \langle \cdot, \cdot \rangle, \quad g_\mu|_{H^\perp} = \langle \cdot, \cdot \rangle \quad \text{and} \quad g_\mu(\mathcal{H}, H^\perp) = 0$$

where \mathcal{H} denotes the distribution spanned by $H(p) = Jp$. The unit vector field $H^\mu(p) = \frac{1}{\sqrt{\mu}} Jp$ is called the Hopf field of the Berger sphere $(\mathbb{S}^{2m+1}, g_\mu)$. Let $\Gamma \neq \{Id\}$ be a finite subgroup of $U(m+1)$, the component of the isometry group of the Berger sphere $(\mathbb{S}^{2m+1}, g_\mu)$, $\mu \neq 1$, which contains the identity. We will always assume that Γ is finite and acts freely so that \mathbb{S}^{2m+1}/Γ is a manifold locally isometric to $(\mathbb{S}^{2m+1}, g_\mu)$ and that the Hopf field H^μ descends to a ‘‘Hopf vector field’’ H_Γ^μ on the quotient. In [8], it is shown that the Hopf field H^μ is harmonic and minimal on $(\mathbb{S}^{2m+1}, g_\mu)$, it ensues that H_Γ^μ is itself harmonic and minimal.

4.2 Stability for $0 < \mu \leq 1$

Theorem 2.— *Let $0 < \mu \leq 1$ then the Hopf field H_Γ^μ is stable for both energy and volume on $(\mathbb{S}^{2m+1}/\Gamma, g_\mu)$, $\Gamma \neq \{Id\}$, $m \geq 1$.*

Remark.— The case $m = 1$ of this theorem follows from the following result of [8]: Hopf fields minimise both energy and volume in (\mathbb{S}^3, g_μ) if $\mu \leq 1$.

Proof of Theorem 2.— The proof of this theorem is similar to the one of Theorem 1 but more involved. We just give the main steps.

STEP 1: NOTATIONS AND A FIRST LOWER BOUND.— We denote by ∇^μ the Levi-Civita connection of g_μ , it is related to the Levi-Civita connection ∇ of the standard metric of the sphere by the following relations (see. [8]):

$$\nabla_H^\mu X = \nabla_H X + (\mu - 1)\nabla_X H, \quad \nabla_X^\mu H = \mu\nabla_X H \quad \text{and} \quad \nabla_X^\mu Y = \nabla_X Y$$

for every X, Y in $\mathcal{E} = \Gamma(H^\perp)$. We denote indifferently by Δ^μ the rough Laplacian on functions or vector fields of $(\mathbb{S}^{2m+1}, g_\mu)$. If (E_1, \dots, E_{2m+1}) is a local g_μ -orthonormal frame and V a vector field we have:

$$\Delta^\mu V = - \sum_{i=1}^{2m+1} (\nabla_{E_i}^\mu \nabla_{E_i}^\mu V - \nabla_{\nabla_{E_i}^\mu E_i}^\mu V).$$

This Laplacian on functions is studied in [1] (see also [20]). From these papers it is easily deduced that the first non zero eigenvalue of Δ^μ is $\lambda_1^\mu = 2m + \frac{1}{\mu}$. We also denote by $\bar{\Delta}^\mu$ the rough Laplacian on the trivial bundle over $(\mathbb{S}^{2m+1}, g_\mu)$ with fiber \mathbb{R}^{2m+2} :

$$\bar{\Delta}^\mu V = - \sum_{i=1}^{2m+1} (\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} V - \bar{\nabla}_{\nabla_{E_i}^\mu E_i} V).$$

As before, $\bar{\nabla}$ denotes the usual connection of \mathbb{R}^{2m+2} . In particular, if (e_1, \dots, e_{2m+2}) is the standard frame of \mathbb{R}^{2m+2} and $V = \sum_i V_i e_i$ then:

$$\bar{\Delta}^\mu V = \sum_i^{2m+2} (\Delta^\mu V_i) e_i.$$

Arguments similar to those seen above show that

$$\int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\bar{\Delta}^\mu V, V) dv_{g_\mu} \geq (2m + \frac{1}{\mu}) \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu},$$

with $p > 1$.

STEP 2.– In that step we prove that

$$\int_{\mathbb{S}^{2m+1}/\Gamma} \|\nabla^\mu V\|_\mu^2 dv_{g_\mu} \geq C(\mu, m) \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu}$$

with

$$C(\mu, m) = \frac{2m^2\mu + (1-\mu)(2+\mu+2m-2m\mu)}{m\mu - \mu + 1}$$

if $0 < \mu \leq 1$. For every $V \in \mathcal{E}$ we put:

$$L^\mu(V) = \pi \circ \Delta^\mu V \quad \text{and} \quad \bar{L}^\mu(V) = \pi \circ \bar{\Delta}^\mu V$$

where π is the orthogonal projection $T\mathbb{S}^{2m+1} \rightarrow \mathcal{H}^\perp$. A computation yields to:

$$L^\mu(V) = \bar{L}^\mu(V) + \frac{2\mu^2 - 4\mu + 1}{\mu} V - \frac{2(\mu - 1)}{\mu} J\nabla_{H_\Gamma} V.$$

From this:

$$\begin{aligned} \int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(L^\mu(V), V) dv_{g_\mu} &= \int_{\mathbb{S}^{2m+1}/\Gamma} \left(g_\mu(\bar{L}^\mu(V), V) + \frac{2\mu^2 - 4\mu + 1}{\mu} \|V\|^2 \right) dv_{g_\mu} \\ &\quad + \frac{2(\mu - 1)}{\sqrt{\mu}} \int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\nabla_{H_\Gamma} V, JV) dv_{g_\mu} \quad (*) \end{aligned}$$

The presence of the last term introduces a technical difficulty. Using the relation between ∇ and ∇^μ we can write:

$$\begin{aligned} \frac{2(\mu - 1)}{\sqrt{\mu}} \int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\nabla_{H_\Gamma} V, JV) dv_{g_\mu} &= \frac{2(\mu - 1)}{\sqrt{\mu}} \int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\nabla_{H_\Gamma}^\mu V, JV) dv_{g_\mu} \\ &\quad - \frac{2(\mu - 1)^2}{\mu} \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu} \end{aligned}$$

And in [8], it is shown that:

$$\begin{aligned} \frac{-2(1-\mu)}{\sqrt{\mu}} \int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\nabla_{H_\Gamma}^\mu V, JV) dv_{g_\mu} &= \frac{1-\mu}{m\mu} \int_{\mathbb{S}^{2m+1}/\Gamma} (\|\nabla_{H_\Gamma}^\mu V\|_\mu^2 - \|\nabla^\mu V\|_\mu^2 \\ &\quad + \frac{1}{2} \|\pi \circ D^C V\|_{\mu, \mathcal{H}^\perp}^2 \\ &\quad + (2 + \mu - 2m\mu + 2m) \|V\|^2) dv_{g_\mu} \end{aligned}$$

where $D^C V$ denotes the operator $D_X^C V = \bar{\nabla}_{JX} V - J\bar{\nabla}_X V$ and the norm is taken on the orthogonal distribution H^\perp . Thus, in one hand we have (we assume $0 < \mu \leq 1$):

$$\frac{-2(1-\mu)}{\sqrt{\mu}} \int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\nabla_{H_\Gamma}^\mu V, JV) dv_{g_\mu} \geq$$

$$\frac{1-\mu}{m\mu} \int_{\mathbb{S}^{2m+1}/\Gamma} ((2+\mu-2m\mu+2m)\|V\|^2 - \|\nabla^\mu V\|_\mu^2) dv_{g_\mu}.$$

In an other hand, from Step 1:

$$\int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(\bar{L}^\mu V, V) dv_{g_\mu} \geq (2m + \frac{1}{\mu}) \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu}.$$

Putting these together in the relation (*) we obtain

$$\int_{\mathbb{S}^{2m+1}/\Gamma} g_\mu(L^\mu V, V) dv_{g_\mu} \geq C(\mu, m) \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu}.$$

STEP 3: ENERGY.— Since $0 < \mu \leq 1$ we have:

$$\begin{aligned} (Hess E)_{H_\Gamma^\mu}(V) &= \int_{\mathbb{S}^{2m+1}/\Gamma} (-2m\mu\|V\|^2 + \|\nabla^\mu V\|_\mu^2) dv_{g_\mu} \\ &\geq (1-\mu) \frac{2m+2m\mu(m-2)+\mu+2}{m\mu-\mu+1} \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu} \\ &\geq 0. \end{aligned}$$

Hence the stability for E of the Hopf field H_Γ^μ on \mathbb{S}^{2m+1}/Γ for $0 < \mu \leq 1$ and $m \geq 1$.

STEP 4: VOLUME.— Since $0 < \mu \leq 1$ we have:

$$\begin{aligned} (Hess Vol)_{H_\Gamma^\mu}(V) &= (1+\mu)^{m-2} \int_{\mathbb{S}^{2m+1}/\Gamma} (\|\nabla^\mu V\|_\mu^2 + \mu\|\nabla_{H_\Gamma^\mu}^\mu V + \sqrt{\mu}JV\|_\mu^2 \\ &\quad + \mu(2-2m\mu-2\mu)\|V\|^2) dv_{g_\mu} \\ &\geq (1+\mu)^{m-2} \int_{\mathbb{S}^{2m+1}/\Gamma} (\|\nabla^\mu V\|_\mu^2 + \mu(2-2m\mu-2\mu)\|V\|^2) dv_{g_\mu} \\ &\geq C_1(\mu, m) \int_{\mathbb{S}^{2m+1}/\Gamma} \|V\|^2 dv_{g_\mu} \geq 0 \end{aligned}$$

where

$$C_1(\mu, m) = (1+\mu)^{m-2}(1-\mu) \frac{2\mu m^2(1+\mu) + 2(\mu+m)(1-\mu) + \mu + 2}{m\mu - \mu + 1}.$$

Hence the stability for Vol of the Hopf field H_Γ^μ on \mathbb{S}^{2m+1}/Γ for $0 < \mu \leq 1$ and $m \geq 1$. \square

4.3 Unstability for Large μ

Theorem 3.— *Let $m > 1$. Every non identically zero vector field V in the orthogonal distribution H^\perp gives an unstable direction of the Hopf field H_Γ^μ in $(\mathbb{S}^{2m+1}/\Gamma, g_\mu)$ for both energy and volume H_Γ^μ provided μ is large enough.*

In particular these two functionals are unstable at H_Γ^μ for large μ .

Proof of Theorem 3.— Let $V \in \mathcal{E}$ be a non zero vector field. Theorem 3 follows from the expansions in μ of the Hessians $(Hess E)_{H_\Gamma^\mu}(V)$ and $(Hess Vol)_{H_\Gamma^\mu}(V)$ since the coefficients of the leading terms when μ is large are negative. Here are some details.

The point is, in the expressions of the two Hessians, to get rid of the μ in the $\|\cdot\|_\mu$'s and in H_Γ^μ . We have:

$$\begin{aligned} \|\nabla^\mu V\|_\mu^2 &= \sum_{i=1}^{2m} \|\nabla_{E_i}^\mu V\|_\mu^2 + \|\nabla_{H_\Gamma^\mu}^\mu V\|_\mu^2 \\ &= \sum_{i,j} \langle \nabla_{E_i} V, E_j \rangle^2 + \mu \|V\|^2 + \|\nabla_{H_\Gamma^\mu}^\mu V\|^2 \\ &= \sum_{i,j=1}^{2m} \langle \nabla_{E_i} V, E_j \rangle^2 - 2\|V\|^2 + 2\langle \nabla_{H_\Gamma} V, JV \rangle + 2\mu \|V\|^2 \\ &\quad + \frac{1}{\mu} (\|\nabla_{H_\Gamma} V\|^2 + \|V\|^2 - 2\langle \nabla_{H_\Gamma} V, JV \rangle). \end{aligned}$$

Substituting in the expression of the Hessian of E at H_Γ^μ , we obtain:

$$\begin{aligned} (Hess E)_{H_\Gamma^\mu}(V) &= \int_{\mathbb{S}^{2m+1}} (2\mu(1-m)\|V\|^2 + \frac{1}{\mu}\|\nabla_{H_\Gamma} V\| - \|JV\|^2 \\ &\quad + \sum_{i,j=1}^{2m} \langle \nabla_{E_i} V, E_j \rangle^2 - 2\|V\|^2 + 2\langle \nabla_{H_\Gamma} V, JV \rangle) dv_{g_\mu} \end{aligned}$$

If $m \neq 1$, the leading term for μ large is

$$\int_{\mathbb{S}^{2m+1}} 2\mu(1-m)\|V\|^2 dv_{g_\mu}$$

which is negative if $m > 1$, hence the unstability of E at H_Γ^μ is μ is large enough. For the volume we have:

$$\begin{aligned} (Hess Vol)_{H_\Gamma^\mu}(V) &= (1+\mu)^{m-2} \int_{\mathbb{S}^{2m+1}} (\|\nabla^\mu V\|_\mu^2 + \mu \|\nabla_{H_\Gamma^\mu}^\mu V + \sqrt{\mu} JV\|_\mu^2 \\ &\quad + \mu(2-2m\mu-2\mu)\|V\|_\mu^2) dv_{g_\mu}. \end{aligned}$$

Using the relation $\nabla_{H_\Gamma^\mu}^\mu V = \nabla_{H_\Gamma} V + (\mu-1)\nabla_V H_\Gamma$, we obtain

$$\begin{aligned} \|\nabla_{H_\Gamma^\mu}^\mu V + \sqrt{\mu} JV\|_\mu^2 &= \frac{1}{\mu} \|\nabla_{H_\Gamma}^\mu V\|^2 + \mu \|V\|^2 + 2\langle \nabla_{H_\Gamma}^\mu V, JV \rangle \\ &= \frac{1}{\mu} (\|\nabla_{H_\Gamma} V\|^2 + \|V\|^2 - 2\langle \nabla_{H_\Gamma} V, JV \rangle) \\ &\quad + 4\mu \|V\|^2 - 4\|V\|^2 + 4\langle \nabla_{H_\Gamma} V, JV \rangle. \end{aligned}$$

It ensues that

$$\begin{aligned} (Hess Vol)_{H_\Gamma^\mu}(V) &= (1 + \mu)^{m-2} \int_{\mathbb{S}^{2m+1}} (2\mu^2(1 - m)\|V\|^2 + \frac{1}{\mu}\|\nabla_{H_\Gamma} V\|^2 \\ &\quad - \|JV\|^2 + 4\mu\langle \nabla_{H_\Gamma} V, JV \rangle + \|\nabla_{H_\Gamma} V\|^2) dv_{g_\mu}. \end{aligned}$$

If $m > 1$, the leading term for μ large is

$$(1 + \mu)^{m-2} \int_{\mathbb{S}^{2m+1}} 2\mu^2(1 - m)\|V\|^2 dv_{g_\mu}$$

hence the unstability of Vol at H_Γ^μ for large μ . \square

4.4 Critical Radii for Berger Projective Spaces

In this subsection we completely solve the stability question for $\Gamma = \mathbb{Z}_2$ in dimension $4m - 1$, that is for Berger Projective Spaces $(\mathbb{R}P^{4m-1}, g_\mu)$, $\mu > 0$ et $m \geq 1$. From theorem 3 we already know that, if $0 < \mu \leq 1$, the Hopf field H_Γ^μ is stable for both energy and volume on $(\mathbb{R}P^{4m-1}, g_\mu)$, $m \geq 1$.

Proposition 4.– *Let $\Gamma = \mathbb{Z}_2$. If $\mu > 1$, the Hopf vector field H_Γ^μ is unstable for both energy and volume on $(\mathbb{R}P^{4m-1}, g_\mu)$, $m \geq 1$.*

Proof of Proposition 4.– Consider $\mathbb{S}^{4m-1}(1) \subset \mathbb{H}^m$ as the hypersphere of the m -th Cartesian power of the quaternionic field $\mathbb{H} = Span(1, i, j, k)$. Let J_1 be the complex structure of $\mathbb{R}^{4m} = \mathbb{H}^m$ induced by the multiplication by j . We define a unit vector field V of $(\mathbb{S}^{4m-1}(1), g_\mu)$ orthogonal to H by putting

$$\forall p \in \mathbb{S}^{4m-1}(1), \quad V(p) = J_1 p.$$

This field obviously descends to the quotient $(\mathbb{R}P^{4m-1}, g_\mu)$. If $m > 1$ a direct computation then shows:

$$\begin{aligned} (Hess E)_{H_\Gamma^\mu}(V) &= \frac{4}{\sqrt{\mu}}(1 - m)(\mu - 1)\left(\mu + \frac{1}{m - 1}\right)vol_m \\ (Hess Vol)_{H_\Gamma^\mu}(V) &= \frac{4}{\sqrt{\mu}}(1 + \mu)^{2m-3}(1 - m)(\mu - 1)\left(\mu + \frac{1}{m - 1}\right)vol_m. \end{aligned}$$

where vol_m stands for the volume of $\mathbb{S}^{4m-1}(1)$ with the round metric. If $\mu > 1$ and $m > 1$ these expressions are negative.

If $m = 1$ the instability of H^μ over (\mathbb{S}^3, g_μ) , $\mu > 1$, is also obtained in the direction given by $V(p) = J_1 p$. The computations essentially reduce to the ones already done in [8]. \square

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