A critical radius for unit Hopf vector fields on spheres

Vincent Borrelli and Olga Gil-Medrano

Abstract. – The volume of a unit vector field V of the sphere $\mathbb{S}^n$ (n odd) is the volume of its image $V(\mathbb{S}^n)$ in the unit tangent bundle. Unit Hopf vector fields, that is, unit vector fields that are tangent to the fibre of a Hopf fibration $\mathbb{S}^n \to \mathbb{C}P^{\frac{n-1}{2}}$, are well known to be critical for the volume functional. Moreover, Gluck and Ziller proved that these fields achieve the minimum of the volume if $n = 3$ and they opened the question of whether this result would be true for all odd dimensional spheres. It was shown to be inaccurate on spheres of radius one. Indeed, Pedersen exhibited smooth vector fields on the *unit* sphere with less volume than Hopf vector fields for a dimension greater than five. In this article, we consider the situation for any odd dimensional spheres, but not necessarily of radius one. We show that the stability of the Hopf field *actually depends on radius,* instability occurs precisely if and only if $r > \frac{1}{\sqrt{n-4}}$. In particular, the Hopf field cannot be minimum in this range. On the contrary, for r small, a computation shows that the volume of vector fields built by Pedersen is greater than the volume of the Hopf one thus, in this case, the Hopf vector field remains a candidate to be a minimizer. We then study the asymptotic behaviour of the volume; for small r it is ruled by the first term of the Taylor expansion of the volume. We call this term the *twisting* of the vector field. The lower this term is, the lower the volume of the vector field is for small r. It turns out that unit Hopf vector fields are absolute minima of the twisting. This fact, together with the stability result, gives two positive arguments in favour of the Gluck and Ziller conjecture *for small r*.

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1 Introduction and main results

The volume of a unit vector field V on a compact oriented Riemannian manifold M can be defined (see [8]) as the volume of the submanifold $V(M)$ of the unit tangent bundle equipped with the restriction of the Sasaki metric.

It is given by

$$Vol(V) = \int_M \sqrt{\det(Id + {}^T\nabla V \circ \nabla V)}\, dvol$$

where *dvol* is the volume element determined by the metric and ∇ is the Levi-Civita connection. There is a trivial absolute minimum of the volume functional when unit parallel vector fields exist, but this is not always the case, since such a vector field will determine two mutually orthogonal, complementary and totally geodesic foliations.

An odd dimensional sphere admits unit vector fields but not parallel ones. A natural unit vector field is then given by the tangents to the fibers of the Hopf fibration and it is shown in [9] that this field is critical for the volume functional. In fact, we can go further, for [8] Gluck and Ziller proved that Hopf vector fields achieved the minimum of the volume among all unit vector fields of the unit sphere of dimension three. The method they used could not be extended to higher dimensions and they opened the question of whether that result was still true for all odd-dimensional spheres. This was shown to be inaccurate on spheres of radius one. Indeed, Pedersen [10] exhibited smooth vector fields on the unit sphere with less volume than Hopf vector fields for a dimension greater than five.

One remarkable fact with the volume functional is that it is not homogeneous with a dilatation of the metric. The influence of radius on the index and on the nullity of Hopf vector fields of the sphere is studied in [6], [7]. It is shown that Hopf vector fields remain critical for any radius and that, for $n \geq 5$, instability occurs if the radius is strictly greater than $\frac{1}{\sqrt{n-4}}$ (or equivalently, if the curvature k is less than $n-4$). Whether Hopf vector fields are unstable for $k \geq n-4$ was an open question. In this article, we completely solve the stability problem, which *actually depends on curvature.*

Stability Theorem. – *Let* $n \geq 5$. *The Hopf vector field is stable if and only if* $k \geq n-4$.

Note that the situation for $n = 3$ is independent of curvature. Hopf vector fields are not only stable but absolute minimizers of the volume [8], [1].

We then study the asymptotic behaviour of the volume functional with the curvature. If V is a unit vector field on $\mathbb{S}^n(1)$, we consider the function $k \mapsto Vol(V^k)$ where $Vol(V^k)$ is the volume of the corresponding unit vector field V^k on the sphere $\mathbb{S}^n(r)$ of radius $r = k^{-\frac{1}{2}}$. If k is small, a Taylor expansion shows that this behaviour is ruled by the *bending* $B(V)$ of V, a notion previously introduced by Wiegmink [12] (see the end of section 3).

If k is large, this behaviour is ruled by another quantity which we call the *twisting* of V

$$Tw(V) = \int_{\mathbb{S}^n(1)} \sqrt{\sigma_{n-1}(\,{}^T\nabla V \circ \nabla V)}\, dvol$$

(in this expression, σ_{n-1} denotes the $(n-1)$-th elementary symmetric polynomial function).

Twisting Theorem. –

1) For every unit vector field V of $\mathbb{S}^n(1)$, $Tw(V) \geq Tw(H)$.

2) If $Tw(V) > Tw(H)$ then there is $k_0 > 0$ such that for all $k > k_0$, we have $Vol(V^k) > Vol(H^k)$.

As a consequence, vector fields constructed by Pedersen cannot achieve the minimum of the volume functional for large values of curvature since their twisting is strictly greater than that of the Hopf vector field (see Lemma 4). In fact, we do not know of any unit vector field with less volume than the Hopf one for these large values. So Gluck and Ziller's conjecture is still open for spheres of small radii.

Besides the Hopf field, there is another field which plays an important role in that problem, namely the radial field R (see the end of section 3 for a description of it). Indeed, it is shown in [3] that the volume of any smooth unit vector field is greater than or equal to the volume of R and that equality uniquely holds for this vector field. Nevertheless, since it has two singularities this vector field is not the minimum of the volume functional among globally defined smooth unit vector fields. We conclude this article with an appendix in which we consider a family of globally defined unit vector fields (R_ϵ) which is built from the radial field. We explicitly compute its volume and compare it with the volume of the Hopf field H and the volume of the field P obtained by Pedersen in [10]. Lower values of the volume are reached by R_ϵ for small k, P for medium k and H for large k.

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2 Hessian of the volume functional

In this section, we state three different expressions of the Hessian, each of them giving interesting insight into the stability problem. Nevertheless, only the two last are needed to prove the Stability Theorem.

Let $\mathbb{S}^{2m+1}(r) \subset \mathbb{C}^{m+1}$ be the sphere of curvature $k = r^{-2}$, $N(p) = \sqrt{k}p$ the outwards unit normal at p and J the usual complex structure on $\mathbb{C}^{m+1}$, so the Hopf vector field can then be written in the form $H = JN$.

We denote by $\bar{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{2m+2}$, the Levi-Civita connection ∇ on $\mathbb{S}^{2m+1}(r)$ is $\nabla_X Y = \bar{\nabla}_X Y - < \bar{\nabla}_X Y, N > N$ and $\bar{\nabla}_X H = J\bar{\nabla}_X N = \frac{1}{r} JX$. Therefore $\nabla_H H = 0$ and if $< X, H >= 0$ then $\nabla_X H = \frac{1}{r} JX$.

Let $W : \mathcal{U} \subset \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be a vector field, we put $D^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W - J\bar{\nabla}_X W$ and $\bar{D}^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W + J\bar{\nabla}_X W$. Recall that W is *holomorphic* (resp.*anti-holomorphic*) if for all X, $D^{\mathbb{C}}_X W = 0$ (resp. $\bar{D}^{\mathbb{C}}_X W = 0$).

Let $H^{\perp}$ be the distribution $Span(x, Jx)^{\perp}$ on $\mathbb{C}^{m+1} \setminus \{0\}$ and $\pi : T(\mathbb{C}^{m+1} \setminus \{0\}) \to H^{\perp}$ be the natural projections $\{x\} \times \mathbb{C}^{m+1} \to H^{\perp}_x$. We denote by $\|\pi \circ D^{\mathbb{C}} W\|_{H^{\perp}}$ the norm of $\pi \circ D^{\mathbb{C}} W_{|H^{\perp}} : H^{\perp} \to H^{\perp}$ that is

$$\|\pi \circ D^{\mathbb{C}} W\|^2_{H^{\perp}} = \sum_{i=1}^{2m} \|\pi \circ D^{\mathbb{C}}_{E_i} W\|^2$$

where $E_1, ..., E_{2m}$ is a local orthonormal frame of $H^{\perp}$. Similarly

$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{H^{\perp}} = \sum_{i=1}^{2m} \|\pi \circ \bar{D}^{\mathbb{C}}_{E_i} W\|^2,$$

but in that case

$$\pi \circ \bar{D}^{\mathbb{C}} W_{|H^{\perp}} = \bar{D}^{\mathbb{C}} W_{|H^{\perp}} : H^{\perp} \to H^{\perp}$$

so that

$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{H^{\perp}} = \|\bar{D}^{\mathbb{C}} W\|^2_{H^{\perp}}.$$

Recall that, for every r, the unit Hopf vector field H is critical for the volume functional defined on the space $\Gamma^{\infty}(T^1\mathbb{S}^{2m+1}(r))$ of smooth unit vector fields of $\mathbb{S}^{2m+1}(r)$ (see [7]). An element of the tangent space to the Fréchet manifold $\Gamma^{\infty}(T^1\mathbb{S}^{2m+1}(r))$ at the point H is a smooth vector field A everywhere orthogonal to H. In [6] Lemma 10, the Hessian of the volume at H is computed and the following result gives a new expression of it in which the role of radius (equivalently the role of curvature) becomes clearer.

Main proposition. − *Let A be a smooth vector field on $\mathbb{S}^{2m+1}(r)$ such that $< A, H >= 0$. Then*

$$i)\ (Hess\ Vol)_H(A) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)} \Big(-2mk\|A\|^2 + \|\nabla A\|^2 + k\|\nabla_H A + \sqrt{k}JA\|^2\Big)\, dvol.$$

$$ii)\ (Hess\ Vol)_H(A) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)} \bigg(\frac{k(1-m)}{k+1}(3+4k+m)\|A\|^2 + (k+1)\|\nabla_H A + \sqrt{k}\frac{k+m}{k+1}JA\|^2 + \frac{1}{2}\|\pi\circ D^{\mathbb{C}}A\|^2_{H^\perp}\bigg)\, dvol.$$

$$iii)\ (Hess\ Vol)_H(A) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)} \bigg(-k\frac{(m+1)^2}{k+1}\|A\|^2 + (k+1)\|\nabla_H A + \sqrt{k}\frac{k-m}{k+1}JA\|^2 + \frac{1}{2}\|\bar{D}^{\mathbb{C}}A\|^2_{H^\perp}\bigg)\, dvol.$$

Proof. − Let $(M, <.,.>)$ be a Riemannian manifold and let W be a unit Killing vector field which is critical for the volume functional V and let A be a vector field orthogonal to W. Then the Hessian of the volume at W can be expressed using Theorem 7 and Lemma 9 of [6] as

$$(Hess\ Vol)_W(A) = \int_M \Big(\|A\|^2\omega_W(W) + f(W)\mathrm{tr}(\mathrm{L}_W^{-1}\circ^{\mathrm{T}}\nabla \mathrm{A}\circ \mathrm{L}_W^{-1}\circ\nabla\mathrm{A}) + \frac{2}{f(W)}\sigma_2(K_W\circ\nabla A)\Big)dvol.$$

where $L_W = Id +^T\nabla W\circ\nabla W$, $f(W) = \sqrt{\det \mathrm{L_W}}$, $K_W = f(W)L_W^{-1}\circ^T\nabla W$, $\omega_W = C_1^1\nabla K_W$ is the tensor contraction of ∇K_W, and $2\sigma_2(C) = \mathrm{tr(C)}^2 - \mathrm{tr(C^2)}$. This expression is useful for our purpose since Hopf vector fields can also be characterized as the unit Killing vector fields of the sphere $\mathbb{S}^{2m+1}(r)$. In that case $L_H|_{H^\perp} = (k+1)Id$ and $L_H(H) = H$ (see [6]). So

$$f(H) = (1+k)^m,\ K_H = (1+k)^{m-1}\,{}^T(\nabla H),\ \omega_H(H) = -2mk(1+k)^{m-1}.$$

Moreover, $K_H H = 0$ and if $< X, H >= 0$ then

$$< K_H(X), Y >= (1+k)^{m-1} < X, \nabla_Y H >= -\sqrt{k}(1+k)^{m-1} < JX, Y > .$$

So, $K_H(X) = -\sqrt{k}(1+k)^{m-1}J(X- < X, H > H)$ and therefore

$$K_H(\nabla_X A) = -\sqrt{k}(1+k)^{m-1}J(\nabla_X A- < \nabla_X A, H > H).$$

Since $\nabla_X JA = J\nabla_X A - \sqrt{k} < X, A > H + \sqrt{k} < JA, X > N$, then

$$(K_H\circ\nabla A)(X) = -\sqrt{k}(1+k)^{m-1}((\nabla JA)(X) + \sqrt{k} < X, A > H).$$

To compute $\sigma_2(K_H \circ \nabla A)$ in terms of $\sigma_2(\nabla JA)$, we choose a local orthonormal frame $(E_1, ..., E_{2m+1})$ such that $E_{2m+1} = H$ and $E_{m+j} = JE_j$, for $1 \leq j \leq m$. So, $(E_1, ..., E_m)$ is an orthonormal J-frame of $H^\perp$. It is easy to see that

$$2\sigma_2(K_H \circ \nabla A) = k(1+k)^{2m-2}\Big(2\sigma_2(\nabla JA) \\ -2\sqrt{k}\sum_{i=1}^{2m} < E_i, A >< \nabla_H JA, E_i > \Big),$$

thus

$$2\sigma_2(K_H \circ \nabla A) = k(1+k)^{2m-2}(2\sigma_2(\nabla JA) + 2\sqrt{k} < \nabla_H A, JA >).$$

On the other hand

$$\mathrm{tr}(\mathrm{L_H^{-1}} \circ^{\mathrm{T}} \nabla \mathrm{A} \circ \mathrm{L_H^{-1}} \circ \nabla \mathrm{A}) = (1+\mathrm{k})^{-2}(\|\nabla \mathrm{A}\|^2 + \mathrm{k}^2\|\mathrm{JA}\|^2 + \mathrm{k}\|\nabla_{\mathrm{H}}\mathrm{A}\|^2).$$

Consequently, we have

$$(Hess\ Vol\,)_H(A) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)}\Big(-2mk(1+k)\|A\|^2 + 2k\sigma_2(\nabla JA) \\ +2k\sqrt{k} < \nabla_H A, JA > + \|\nabla A\|^2 + k^2\|JA\|^2 + k\|\nabla_H A\|^2\Big)dvol.$$

Now for any vector field X on a Riemannian manifold M, we have (see, for example, [11] p. 170)

$$\int_M Ricci(X,X)dvol = 2\int_M \sigma_2(\nabla X)dvol,$$

from where part i follows immediately. To show ii and iii we compute $\|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp}$ and $\|\bar{D}^{\mathbb{C}} A\|^2_{H^\perp}$ in terms of the matrix B of ∇A in a local frame, i. e. $B_i^j =< \nabla_{E_i} A, E_j >$, obtaining

$$\begin{aligned}\frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp} &= \sum_{i,j=1}^{m}(B_{i+m}^{j+m} - B_i^j)^2 + (B_{i+m}^j + B_i^{j+m})^2 \\ &= \sum_{i,j=1}^{2m}(B_i^j)^2 + 2\sum_{i,j=1}^{m}(B_{i+m}^j B_i^{j+m} - B_{i+m}^{j+m} B_i^j),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\|\bar{D}^{\mathbb{C}} A\|^2_{H^\perp} &= \sum_{i,j=1}^{m}(B_{i+m}^{j+m} + B_i^j)^2 + (B_{i+m}^j - B_i^{j+m})^2 \\ &= \sum_{i,j=1}^{2m}(B_i^j)^2 - 2\sum_{i,j=1}^{m}(B_{i+m}^j B_i^{j+m} - B_{i+m}^{j+m} B_i^j).\end{aligned}$$

To go further, we need the following lemma which relates H with vectors on the orthogonal distribution.

Lemma 1. – *We have*

$$a)\quad 2m\sqrt{k}H = -\sum_{i=1}^{m}[E_i, JE_i] + \sum_{i=1}^{m}\mathrm{div(JE_i)E_i} - \sum_{\mathrm{i=1}}^{\mathrm{m}}\mathrm{div(E_i)JE_i}.$$

$$b)\quad 2m\sqrt{k}\int_{\mathbb{S}^{2m+1}(r)} < \nabla_H A, JA > \;=\; -2k\int_{\mathbb{S}^{2m+1}(r)} \|A\|^2 + 2\int_{\mathbb{S}^{2m+1}(r)}\sum_{i=1}^{m} < J\nabla_{E_i}A, \nabla_{JE_i}A > .$$

Proof. – A simple computation shows that

$$\sum_{i=1}^{m}[E_i, JE_i] = \sum_{i,j}^{2m+1}(< E_i, \nabla_{E_i}JE_j > E_j + < JE_i, \nabla_{JE_i}JE_j > E_j)$$

$$-\sum_{i,j}^{2m+1}(< E_i, \nabla_{E_i}E_j > JE_j + < JE_i, \nabla_{JE_i}E_j > JE_j) - 2m\sqrt{k}H$$

hence a) and then

$$2m\sqrt{k} <\nabla_H A, JA>= \sum_{i=1}^{m}\Big(- <\nabla_{[E_i,JE_i]}A, JA> +\mathrm{div(JE_i)} <\nabla_{\mathrm{E_i}}\mathrm{A, JA}>\Big)$$

$$-\sum_{i=1}^{m}\mathrm{div(E_i)} < \nabla_{\mathrm{JE_i}}\mathrm{A, JA} > .$$

On the other hand

$$k\|A\|^2 = \sum_{i=1}^{m} R(E_i, JE_i, A, JA) = \sum_{i=1}^{m}(JE_i)(< \nabla_{E_i}A, JA >)$$

$$-\sum_{i=1}^{m} E_i(< \nabla_{JE_i}A, JA >) + \sum_{i=1}^{m} < \nabla_{[E_i,JE_i]}A, JA >$$

$$-\sum_{i=1}^{m} < \nabla_{E_i}A, \nabla_{JE_i}JA > + \sum_{i=1}^{m} < \nabla_{JE_i}A, \nabla_{E_i}JA > .$$

By integrating and using Stokes Theorem and the identity $\mathrm{div(fX) = fdiv(X) + X(f)}$, we get

$$2m\sqrt{k}\int_{\mathbb{S}^{2m+1}(r)} < \nabla_H A, JA >=$$

$$\int_{\mathbb{S}^{2m+1}(r)} \Big(-k\|A\|^2 - \sum_{i=1}^{m} < \nabla_{E_i} A, \nabla_{JE_i} JA > + \sum_{i=1}^{m} < \nabla_{JE_i} A, \nabla_{E_i} JA > \Big)$$

hence the result. □

Using the lemma

$$\int_{\mathbb{S}^{2m+1}(r)} \frac{1}{2} \|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp} dvol =$$

$$\int_{\mathbb{S}^{2m+1}(r)} \Big(\|\nabla A\|^2 - 3k\|A\|^2 - \|\nabla_H A\|^2 - 2m\sqrt{k} < \nabla_H A, JA > \Big) dvol,$$

and

$$\int_{\mathbb{S}^{2m+1}(r)} \frac{1}{2} \|\bar{D}^{\mathbb{C}} A\|^2_{H^\perp} dvol =$$

$$\int_{\mathbb{S}^{2m+1}(r)} \Big(\|\nabla A\|^2 + k\|A\|^2 - \|\nabla_H A\|^2 + 2m\sqrt{k} < \nabla_H A, JA > \Big) dvol.$$

From here

$$(Hess\ Vol)_H(A) = \int_{\mathbb{S}^{2m+1}(r)} \Big((1+k)^{m-2} \frac{1}{2} \|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp}$$

$$+(-2m+3+k)k\|A\|^2 + (k+1)\|\nabla_H A\|^2 + 2(k+m)\sqrt{k} < \nabla_H A, JA > \Big) dvol,$$

and

$$(Hess\ Vol)_H(A) = \int_{\mathbb{S}^{2m+1}(r)} \Big((1+k)^{m-2} \frac{1}{2} \|D^{\mathbb{C}} A\|^2_{H^\perp}$$

$$+(-2m-1+k)k\|A\|^2 + (k+1)\|\nabla_H A\|^2 + 2(k-m)\sqrt{k} < \nabla_H A, JA > \Big) dvol.$$

That the Hessian admits expressions *ii* and *iii* is now an easy computation.
□

Remarks. – We will use *ii* and *iii* to show the Stability Theorem. Expression *i)* relates the Hessian of the volume with the Hessian of the energy functional that is the functional

$$V \overset{E}{\longmapsto} \frac{1}{2} \int_{\mathbb{S}^{2m+1}(r)} \mathrm{tr}(\mathrm{Id} + {}^{\mathrm{T}}\nabla \mathrm{V} \circ \nabla \mathrm{V}) \mathrm{dvol}.$$

Just as for the volume, Hopf vector fields are critical for the energy and from Lemma 10 of [6] we have

$$(Hess\ E)_H(A) = \int_{\mathbb{S}^{2m+1}(r)} -2mk\|A\|^2 + \|\nabla A\|^2.$$

This gives us some idea why the curvature could have an influence on the signature of $(Hess\ Vol)_H$; the term $-2mk\|A\|^2 + \|\nabla A\|^2$ is of order k and

$k\|\nabla_H A + \sqrt{k} JA\|^2$ is of order k^2.

Putting $m = 1$ in the expression ii gives

$$(Hess\ Vol)_H(A) = \frac{1}{1+k}\int_{\mathbb{S}^3(r)}\Big((k+1)\|\nabla_H A+\sqrt{k}JA\|^2+\frac{1}{2}\|\pi\circ D^{\mathbb{C}}A\|^2_{H^\perp}\Big)dvol$$

so we immediately get that the Hopf vector field on $\mathbb{S}^3(r)$ is stable for any radius.

3 Proof of the Stability Theorem

The Stability Theorem will follow from the main proposition of the preceding section and the two lemmas below.

Let $A : \mathbb{S}^{2m+1}(r) \to H^\perp \subset \mathbb{C}^{m+1}$ be a smooth vector field orthogonal to H, that is $A(p) \in Span(p, Jp)^\perp$ for every $p \in \mathbb{S}^{2m+1}(r)$. We set

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta}d\theta \in H_p^\perp$$

so that the Fourier serie of this smooth map A is

$$A(p) = \sum_{l\in\mathbb{Z}} A_l(p).$$

Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$, we have

$$\nabla_H A = \bar{\nabla}_H A = \sum_{l\in\mathbb{Z}} i\sqrt{k} l A_l$$

and, if $\mathcal{C}(p)$ denotes the fiber of the Hopf fibration $\mathbb{S}^{2m+1} \to \mathbb{C}P^m$ passing through p,

$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0,$$

if $l \neq q$. We denote the symmetric bilinear form associated to $(Hess\ Vol\,)_H$ by B_{Vol}, that is $B_{Vol}(A, A) = (Hess\ Vol\,)_H(A)$.

Lemma 2. – *If $l \neq q$ then $B_{Vol}(A_l, A_q) = 0$, thus*

$$(Hess\ Vol\,)_H(A) = \sum_{l\in\mathbb{Z}} (Hess\ Vol\,)_H(A_l).$$

Proof. $-$ We write

$$\begin{aligned} B_{Vol}(A_l, A_q) = \quad & (1+k)^{m-2} \int_{\mathbb{S}^{2m+1}(r)} \Big(-k\frac{(m+1)^2}{k+1} < A_l, A_q > \\ & +(k+1) < \nabla_H A_l + \sqrt{k}\frac{k-m}{k+1} JA_l, \nabla_H A_q + \sqrt{k}\frac{k-m}{k+1} JA_q > \\ & +\frac{1}{2}\sum_{j=1}^{2m} < \bar{D}^{\mathbb{C}}_{E_j} A_l, \bar{D}^{\mathbb{C}}_{E_j} A_q > \Big) dvol. \end{aligned}$$

Let $l \neq q$, we have

$$\int_{\mathcal{C}(p)} -k\frac{(m+1)^2}{k+1} < A_l, A_q >= 0$$

and

$$\int_{\mathcal{C}(p)} (k+1) < \nabla_H A_l + \sqrt{k}\frac{k-m}{k+1} JA_l, \nabla_H A_q + \sqrt{k}\frac{k-m}{k+1} JA_q >= 0.$$

Let $(E_1, ..., E_m)$ be an orthonormal J-frame of $H^\perp$ along $\mathcal{C}(p)$ satisfying

$$\forall\, 1 \leq j \leq m, \qquad E_j(e^{i\theta}p) = e^{i\theta}E_j(p).$$

Then

$$(\bar{D}^{\mathbb{C}}_{E_j} A)_l = \bar{D}^{\mathbb{C}}_{E_j} A_l.$$

Indeed, let $X \in H^\perp$ such that $X(e^{i\theta}p) = e^{i\theta}X(p)$, and γ_{p,X_p} be a path on $\mathbb{S}^{2m+1}(r)$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$, we have

$$\begin{aligned} (\bar{\nabla}_{X_p} A_l)(p) &= \frac{d}{dt}\left(A_l \circ \gamma_{p,X_p}(t)\right)_{|t=0} \\ &= \frac{d}{dt}\left(\frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}\gamma_{p,X_p}(t))e^{-il\theta}d\theta\right)_{|t=0} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dt}\left(A \circ \gamma_{e^{i\theta}p, e^{i\theta}X_p}(t)\right)_{|t=0} \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\bar{\nabla}_{e^{i\theta}X_p} A)(e^{i\theta}p)e^{-il\theta}d\theta. \end{aligned}$$

On the other hand

$$(\bar{\nabla}_X A)_l(p) = \frac{1}{2\pi}\int_0^{2\pi} (\bar{\nabla}_{X_{e^{i\theta}p}} A)(e^{i\theta}p)e^{-il\theta}d\theta.$$

Thus, if $l \neq q$

$$\int_{\mathcal{C}(p)} < D^{\mathbb{C}}_{E_j} A_l, D^{\mathbb{C}}_{E_j} A_q >= \int_{\mathcal{C}(p)} < (\bar{D}^{\mathbb{C}}_{E_j} A)_l, (\bar{D}^{\mathbb{C}}_{E_j} A)_q >= 0$$

$\square$

Lemma 3. – *If $k \geq 2m-3$ then for all $l \in \mathbb{Z}$, $(Hess\ Vol\,)_H(A_l) \geq 0$.*

Proof. – Note that

$$\int_{\mathcal{C}(p)} \|\nabla_H A_l + \alpha\sqrt{k} J A_l\|^2 = (l+\alpha)^2 k \int_{\mathcal{C}(p)} \|A_l\|^2.$$

Thus, from expression ii of the main proposition, we have

$$(Hess\ Vol\,)_H(A_l) = (1+k)^{m-2} \int_{\mathbb{S}^{2m+1}(r)} \Big(\frac{1}{2}\|\pi \circ D^{\mathbb{C}} A_l\|^2_{H^\perp}$$

$$(3-2m+k+l((k+1)l+2(k+m)))k\|A_l\|^2\Big) dvol$$

Since $k \geq 2m-3$, $(Hess\ Vol\,)_H(A_l) < 0$ implies $-3 \leq \frac{-2(k+m)}{k+1} < l < 0$, hence $l = -1$ or -2. But, from expression iii of the main proposition

$$(Hess\ Vol\,)_H(A_{-1}) = (1+k)^{m-2} \int_{\mathbb{S}^{2m+1}(r)} \frac{1}{2}\|\bar{D}^{\mathbb{C}} A_{-1}\|^2_{H^\perp} dvol \geq 0$$

and

$$(Hess\ Vol\,)_H(A_{-2}) =$$

$$(1+k)^{m-2} \int_{\mathbb{S}^{2m+1}(r)} \Big((k+2m+3)k\|A_{-2}\|^2 + \frac{1}{2}\|\bar{D}^{\mathbb{C}} A_{-2}\|^2_{H^\perp}\Big) dvol \geq 0$$

$\square$

Lemmas 2 and 3 prove the Stability Theorem.

4 Asymptotic behaviour of the volume

In this section we prove the Twisting Theorem and then compare the asymptotic behaviour of different vector fields, namely the Hopf vector field, the perturbed parallel field (Pedersen construction) and the perturbed radial field.

Let V be a unit vector field on $\mathbb{S}^{2m+1}(1)$, the volume of the corresponding unit vector field V^k on the sphere $\mathbb{S}^{2m+1}(r)$ with radius $r = k^{-\frac{1}{2}}$ is given by

$$\begin{aligned} Vol(V^k) =& \int_{\mathbb{S}^{2m+1}(1)} \sqrt{\det(\frac{1}{\mathrm{k}}\mathrm{Id} + \mathrm{M})}\, dvol \\ =& \int_{\mathbb{S}^{2m+1}(1)} \sqrt{\frac{1}{k^n} + \frac{1}{k^{n-1}}\sigma_1(M) + \ldots + \frac{1}{k}\sigma_{2m}(M) + \sigma_{2m+1}(M)}\, dvol \end{aligned}$$

where M denotes ${}^T\nabla V \circ \nabla V$ and the σ_j's are the elementary symmetric polynomial functions. It is well-known that, since $\|V\| = 1$, $\sigma_{2m+1}(M)$ is zero. Hence when k is large, the behaviour of the volume is ruled by

$$\int_{\mathbb{S}^{2m+1}(1)} \sqrt{\sigma_{2m}({}^T\nabla V \circ \nabla V)} dvol.$$

that is the *twisting of* V. In particular, if W is another unit vector field such that $Tw(W) > Tw(V)$ then we have $k_0 > 0$ such that for every $k > k_0$ the volume of W^k is strictly greater than that of V^k. This states part 2) of the Twisting Theorem.

Proof of the Twisting Theorem, part 1.– Since

$${}^T\nabla H \circ \nabla H = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix},$$

then $Tw(H) = \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1))$. It remains to show that if V is any unit vector field then $Tw(V) \geq \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1))$. Note that

$$\begin{aligned} \sigma_{2m}({}^T\nabla V \circ \nabla V) &= \sum_{i_1 < \ldots < i_{n-1}} \|\nabla_{e_{i_1}} V \wedge \ldots \wedge \nabla_{e_{i_{n-1}}} V\|^2 \\ &\geq \|\nabla_{e_1} V \wedge \ldots \wedge \nabla_{e_{n-1}} V\|^2. \end{aligned}$$

So, if $S = \nabla V_{|V^\perp}$, we obtain

$$\sigma_{2m}({}^T\nabla V \circ \nabla V) \geq \sigma_{2m}({}^TS. \circ S) = \det{}^2(\mathrm{S}).$$

Thus

$$\sqrt{\sigma_{2m}({}^T\nabla V \circ \nabla V)} \geq |\det(\mathrm{S})| \geq \det(\mathrm{S}) = \sigma_{2\mathrm{m}}(\mathrm{S}).$$

According to [4],

$$\int_{\mathbb{S}^{2m+1}(1)} \sigma_{2m}(S)\ dvol = \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1)),$$

henceforth

$$Tw(V) \geq \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1)).$$

Since $Tw(H) = \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1))$, the Hopf field actually achieves the minimum. $\square$

THE PEDERSEN CONSTRUCTION. – In [10], we can see the construction of smooth vector fields on $\mathbb{S}^{2m+1}(1)$ with less volume than the Hopf one. They are obtained from parallel vector fields, that is a parallel translation along the geodesics of a given vector at a point p. These parallel vector fields have a singularity of index 0 at $-p$ but a C^∞-pertubation on a small neighbourhood of $-p$ gives smooth vector fields on the whole sphere. The volume of these vector fields approaches the volume of the parallel vector field and similarly so for the twisting.

Lemma 4. – *Let P be a unit vector field obtained by the parallel transport of any given unit vector. Then*

$$Tw(P) = \frac{1}{2^{2m}} \frac{C_{4m}^{2m}}{C_{2m-1}^{m}} \operatorname{vol}(\mathbb{S}^{2m+1}(1)).$$

In particular $Tw(P) > Tw(H)$, thus the Pedersen construction cannot yield to the minimum of the volume for large k.

Proof. – Let $\mathbb{S}^n(1) \subset \mathbb{R}^{n+1}$ with $n = 2m+1$, and $S = (0, ..., 0, -1)$ be the south pole. The parallel vector field P on $\mathbb{S}^n(1) \setminus \{S\}$ obtained by the parallel transport of $\partial_n = (0, ..., 0, 1, 0)$ along geodesics has the following expression

$$P(x) = x_n(h(x)\overline{x} - \partial_{n+1}) + \partial_n$$

where $\overline{x} = \sum_{j=1}^{n} x_j \partial_j$ and $h(x) = -(1 + x_{n+1})^{-1}$. Let $E_i = \sum_{l=1}^{n}(\delta_{li} + h x_l x_i)\partial_l - x_i \partial_{n+1}$. According to [7], the $n \times n$ matrix of ∇P in the basis (E_i) is given by

$$\nabla P = h \begin{pmatrix} x_n Id & {}^T a \\ 0 & 0 \end{pmatrix}$$

with $a = (-x_1, ..., -x_{n-1})$. Thus

$$\frac{1}{k} Id + {}^T\nabla P \circ \nabla P = \begin{pmatrix} (h^2 x_n^2 + \frac{1}{k}) Id & x_n h^2 \, {}^T a \\ x_n h^2 a & h^2 a . \, {}^T a + \frac{1}{k} \end{pmatrix},$$

and a simple computation shows that

$$\begin{aligned} \det(\tfrac{1}{\mathrm{k}}\mathrm{Id} + {}^{\mathrm{T}}\nabla \mathrm{P} \circ \nabla \mathrm{P}) &= \tfrac{1}{k}(h^2 x_n^2 + \tfrac{1}{k})^{n-2}(h^2 x_n^2 + h^2\|a\|^2 + \tfrac{1}{k}) \\ &= \tfrac{1}{k}(h^2 x_n^2 + \tfrac{1}{k})^{n-2}(-2h - 1 + \tfrac{1}{k}). \end{aligned}$$

Since $\sigma_{2m}({}^T\nabla P \circ \nabla P)$ is the $\frac{1}{k}$ term of the expansion of the above determinant, we obtain

$$\sigma_{2m}({}^T\nabla P \circ \nabla P) = h^{2n-4} x_n^{2n-4}(-2h - 1).$$

Note that $h^{2n-4}(-2h-1) = (1 - x_{n+1})(1 + x_{n+1})^{3-2n}$, so

$$\sqrt{\sigma_{2k}({}^T\nabla P \circ \nabla P)} = (1 - x_{n+1})^{1/2}(1 + x_{n+1})^{3/2-n}|x_n|^{n-2}.$$

Let $x_{n+1} = \cos t$, $x_n = \sin t \cos\alpha$, $0 < t < \pi$, $0 < \alpha < \pi$, we then have

$$Tw(P) = I \cdot \operatorname{vol}(\mathbb{S}^{n-2}(1))$$

with

$$I = \int_0^\pi \int_0^\pi (1-\cos t)^{\frac{1}{2}} (1+\cos t)^{\frac{3}{2}-n} \sin^{n-2} t |\cos\alpha|^{n-2} \sin^{n-1} t \sin^{n-2} \alpha dt d\alpha.$$

Let

$$\begin{aligned} I_1 &= \int_0^\pi (1-\cos t)^{\frac{1}{2}} (1+\cos t)^{\frac{3}{2}-n} (1-\cos^2 t)^{n-3/2} dt \\ &= \int_0^\pi (1-\cos t)^{n-1} dt = \int_0^\pi 2^{n-1} \sin^{2n-2} \frac{t}{2}\, dt \\ &= 2^n \int_0^{\frac{\pi}{2}} \sin^{2n-2} t\, dt = C_{4m}^{2m} \frac{\pi}{2^{2m}}, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^\pi |\cos\alpha|^{n-2} \sin^{n-2} \alpha\, d\alpha = 2 \int_0^{\frac{\pi}{2}} \cos^{n-2} \alpha \sin^{n-2} \alpha\, d\alpha \\ &= \frac{1}{2^{n-3}} \int_0^{\frac{\pi}{2}} \sin^{n-2} 2\alpha\, d\alpha = \frac{1}{2^{n-2}} \int_0^\pi \sin^{n-2} \beta\, d\beta \\ &= \frac{1}{2^{n-3}} \int_0^{\frac{\pi}{2}} \sin^{n-2} \beta\, d\beta = \frac{1}{(2m-1) C_{2m-2}^{m-1}}. \end{aligned}$$

Since

$$\begin{aligned} Tw(P) &= I_1 I_2 \mathrm{vol}(\mathbb{S}^{\mathrm{n}-2}(1)) \\ &= C_{4m}^{2m} \frac{\pi}{2^{2m}} \frac{1}{(2m-1) C_{2m-2}^{m-1}} \frac{m}{\pi} \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1)), \end{aligned}$$

we get the result.

We know from the proof of the Twisting Theorem part 1 that $Tw(H) = \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1))$. From this, it is easily checked that $Tw(P) > Tw(H)$. □

The perturbed radial field. – Analogously, the behaviour of the volume for k small is ruled by the bending, that is

$$B(V) = \int_{\mathbb{S}^{2m+1}(1)} \sigma_1(M) dvol = \int_{\mathbb{S}^{2m+1}(1)} \|\nabla V\|^2 dvol.$$

In [3] it is shown that the volume of any smooth unit vector field is greater than or equal to the volume of the radial field R and equality uniquely holds for this vector field. This field is the one given by the unit tangent vectors to the radial geodesics issuing from a given point p. It has two singularities at points p and $-p$ of opposite index ± 1. Due to the non-vanishing of these indices, it is impossible to obtain a smooth unit vector field from the field by a C^∞-perturbation on small neighbourhoods of p and $-p$. This perturbation has to be realised on a small tubular neighbourhood of an arc joining p and $-p$. In [2] a construction following these lines is performed to obtain a family

R_ϵ of smooth unit vector fields. Unfortunately, as ϵ tends to 0, the volume of R_ϵ does not converge to the volume of R (see appendix). Although the twisting of H is equal to the twisting of R, the limit $\lim_{\epsilon\to 0} Tw(R_\epsilon)$ is strictly bigger than the twisting of H and then by the Twisting Theorem, for large k

$$\lim_{\epsilon\to 0} Vol(R_\epsilon^k) > Vol(H^k).$$

On the contrary, in [2] and [5] it is proven that for every smooth vector field V

$$\lim_{\epsilon\to 0} B(R_\epsilon) = B(R) < B(V).$$

So for any given V, we have

$$\lim_{\epsilon\to 0} Vol(R_\epsilon^k) < Vol(V^k)$$

for sufficiently small k. To sum up, we have

Proposition 5 ("Bending Theorem"). − *Let V be a unit vector field of $\mathbb{S}^{2m+1}(1)$. Then* *1) $B(V) > \lim_{\epsilon\to 0} B(R_\epsilon) = B(R)$ ([2], [5]),*

2) There are $\epsilon > 0$ and $k_0 > 0$ such that, for all $k < k_0$, we have $Vol(V^k) > Vol(R_\epsilon^k)$.

5 Appendix

In this appendix we explicitly compute and then compare the volume of the parallel field and the volume of the perturbed radial field.

Proposition 6. − *Let $m \geq 1$ and $r = k^{-\frac{1}{2}}$.*

1) $Vol(H^k) = (1+k)^m \mathrm{vol}(\mathbb{S}^{2m+1}(r)$,

2) $Vol(P^k) =$

$$\frac{k(2m+1)4^{2m}}{2\,C_{4m}^{2m}}\,{}_3F_2(\frac{1}{2}, \frac{2m+1}{2}, 2m+1, 1, 2m+\frac{1}{2}; 1-k)\mathrm{vol}(\mathbb{S}^{2m+1}(r)),$$

3) $$\lim_{\epsilon\to 0} Vol(R_\epsilon^k) = \left(\sum_{j=0}^{m} \frac{(C_m^j)^2}{C_{2m}^{2j}} k^j + \frac{4^m}{C_{2m}^m} k^m\right) \mathrm{vol}(\mathbb{S}^{2m+1}(r))$$

Remark. − In 2) ${}_3F_2$ denotes the generalized hypergeometric series. This series is convergent if $k < 2$. Since $k \mapsto Vol(P^k)$ is analytic, the formula for $k \geq 2$ is the analytic continuation of the above expression.

Let

$$\mathcal{V}(k) = \min(Vol(H^k), Vol(P^k), \lim_{\epsilon\to 0} Vol(R_\epsilon^k)).$$

The study of these three functions shows that

Corollary 7. – *For every $m \geq 2$ there is $0 < k_1(m) < k_2(m)$ such that:*

$$\begin{array}{ll} \mathcal{V}(k) = \lim_{\epsilon \to 0} Vol(R_\epsilon) & \textit{if } k \leq k_1(m), \\ \mathcal{V}(k) = Vol(P^k) & \textit{if } k_1(m) \leq k \leq k_2(m), \\ \mathcal{V}(k) = Vol(H^k) & \textit{if } k_2(m) \leq k. \end{array}$$

A numerical computation gives $k_1(2) = 0.269...$, $k_1(3) = 0.494...$, $k_1(4) = 0.618...$, $k_1(10) = 0.848...$, $k_1(100) = 0.985...$ and $k_2(2) = 1.815...$, $k_2(3) = 5.563...$, $k_2(4) = 8.729...$, $k_2(10) = 26.29...$, $k_2(100) = 286.1...$

Recall that

$$\begin{aligned} Tw(H) &= \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1)), \\ Tw(P) &= \frac{1}{2^{2m}} \frac{C_{4m}^{2m}}{C_{2m-1}^{m}} \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1)), \end{aligned}$$

(see the proof of the Twisting Theorem and Lemma 4). The mere power expansion of the expression 3) in Proposition 6 shows that

Corollary 8. – $\lim_{\epsilon \to 0} Tw(R_\epsilon) = (1 + \frac{4^m}{C_{2m}^m}) \; \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(1))$.

The next two subsections are devoted to the proof of points 2) and 3) of Proposition 6. Point 1) is well-known and easy.

5.1 Volume of the field obtained by parallel transport

The volume of the parallel field P^k on the n-dimensional sphere $\mathbb{S}^n(r)$ of radius $r = k^{-\frac{1}{2}}$ is given by

$$Vol(P^k) = \int_{\mathbb{S}^n(1)} \sqrt{\det\, (\frac{1}{\mathrm{k}}\mathrm{Id} + \ {}^{\mathrm{T}}\nabla\mathrm{P} \circ \nabla\mathrm{P})} \; dvol.$$

Let $f = \sqrt{\det\, (\frac{1}{\mathrm{k}}\mathrm{Id} + \ {}^{\mathrm{T}}\nabla\mathrm{P} \circ \nabla\mathrm{P})}$, from the proof of Lemma 4 we know that f only depends on x_n and x_{n+1}. Putting $x_{n+1} = \cos t$, $x_n = \sin t \cos \alpha$, $0 < t < \pi$, $0 < \alpha < \pi$, we get

$$g(t, \alpha) = f(\sin t \cos \alpha, \cos t) =$$

$$r(1 + \cos t)^{\frac{1-n}{2}} ((1 - \cos t)(\cos^2 \alpha - r^2) + 2r^2)^{\frac{n-2}{2}} ((r^2 - 1)(1 + \cos t) + 2)^{\frac{1}{2}}.$$

Thus

$$Vol(P^k) = \int_{\mathbb{S}^n(1)} f \; dvol = \mathrm{vol}(\mathbb{S}^{\mathrm{n}-2}(1)) \; \mathrm{I}.$$

with

$$I = \int_0^\pi \int_0^\pi g(t, \alpha) \sin^{n-1} t \sin^{n-2} \alpha \; dt d\alpha.$$

Let $u = \dfrac{1}{2}(1-\cos t)$, we obtain

$$\begin{aligned} I &= \int_0^\pi \int_0^1 r^n 2^{n-1} u^{\frac{n-2}{2}} (1-u)^{-\frac{1}{2}} (1-(1-\frac{\cos^2\alpha}{r^2})u)^{\frac{n-2}{2}} \\ &\qquad\qquad (1-(1-\frac{1}{r^2})u)^{\frac{1}{2}} \sin^{n-2}\alpha du d\alpha \\ &= \int_0^\pi F_1(\frac{n}{2}, 1-\frac{n}{2}, -\frac{1}{2}, \frac{n+1}{2}; 1-\frac{\cos^2\alpha}{r^2}, 1-\frac{1}{r^2}) \\ &\qquad\qquad r^n 2^{n-1} B(\tfrac{n}{2}, \tfrac{1}{2}) \sin^{n-2}\alpha d\alpha, \end{aligned}$$

where B is the Beta function and F_1 is the first hypergeometric function of two variables. Since

$$F_1(\alpha,\beta,\beta',\gamma;x,y) = (1-y)^{-\alpha} F_1(\alpha,\beta,\gamma-\beta-\beta',\gamma;\frac{y-x}{y-1},\frac{y}{y-1})$$

we have

$$I = 2^n r^{2n} B(\frac{n}{2},\frac{1}{2}) \int_0^{\pi/2} F_1(\frac{n}{2}, 1-\frac{n}{2}, n, \frac{n+1}{2}, \sin^2\alpha, 1-r^2) \sin^{n-2}\alpha d\alpha.$$

Let $x = \sin^2\alpha$, we finally obtain

$$I = 2^{n-1} r^{2n} B(\frac{n}{2},\frac{1}{2})\ J$$

with

$$J = \int_0^1 x^{\frac{n-3}{2}} (1-x)^{-\frac{1}{2}} F_1(\frac{n}{2}, 1-\frac{n}{2}, n, \frac{n+1}{2}, x, 1-r^2) dx.$$

We put $\rho - 1 = \frac{n-3}{2}$, $\sigma - 1 = -\frac{1}{2}$, $\alpha = \frac{n}{2}$, $\beta = 1-\frac{n}{2}$, $\beta' = n$, $\gamma = \frac{n+1}{2}$ and $y = 1-r^2$. As usual $(\alpha)_k$ denotes $\Gamma(\alpha+k)/\Gamma(\alpha)$. Then

$$\begin{aligned} J &= \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} F_1(\alpha,\beta,\beta',\gamma,x,y) dx \\ &= \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} \sum_{k,l} \frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_{k+l} k!\ l!} x^k y^l dx \\ &= \sum_{k,l} B(\rho+k,\sigma) \frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_{k+l} k!\ l!} y^l \\ &= \sum_{k,l} B(\rho,\sigma) \frac{(\rho)_k}{(\rho+\sigma)_k} \frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_{k+l} k!\ l!} y^l \\ &= B(\rho,\sigma) \sum_l \frac{(\alpha)_l(\beta')_l}{(\gamma)_l l!} \left(\sum_k \frac{(\alpha+l)_k(\beta)_k(\rho)_k}{(\gamma+l)_k(\rho+\sigma)_k k!} \right) y^l \\ &= B(\rho,\sigma) \sum_l \frac{(\alpha)_l(\beta')_l}{(\gamma)_l l!}\ {}_3F_2(\alpha+l,\beta,\rho,\gamma+l,\rho+\sigma;1) y^l. \end{aligned}$$

Since

$$ {}_3F_2(a,b,c,e,f;1) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(b+s)\Gamma(c+s)}\ {}_3F_2(e-a,f-a,s,s+b,s+c;1) $$

with $s = e+f-a-b-c$ we deduce

$$ {}_3F_2(\alpha+l,\beta,\rho,\gamma+l,\rho+\sigma;1) = \frac{(\gamma)_l}{(\alpha)_l}\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})}{\Gamma(n-\frac{1}{2})}\ {}_3F_2(\frac{1}{2},-l,\frac{n}{2},1,n-\frac{1}{2};1) $$

and therefore

$$ J = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n-\frac{1}{2})}K $$

where

$$ K = \sum_l \frac{(n)_l}{l!}\ {}_3F_2(\frac{1}{2},-l,\frac{n}{2},1,n-\frac{1}{2};1)y^l. $$

We have

$$ K = \sum_{k,l} \frac{(n)_l(\frac{1}{2})_k(-l)_k(\frac{n}{2})_k}{(1)_k(n-\frac{1}{2})_k l!\ k!}y^l = \sum_k \frac{(\frac{1}{2})_k(\frac{n}{2})_k}{(1)_k(n-\frac{1}{2})_k k!}\mathcal{L}_k(y) $$

with $\mathcal{L}_k(y) = \sum_l \frac{(n)_l(-l)_k}{l!}y^l$. Since $(-l)_k = (-1)^k l(l-1)...(l-k+1)$ we get

$$ \mathcal{L}_k(y) = \sum_{l\geq k} \frac{(n)_l(-l)_k}{l!}y^l = (-1)^k \sum_{l\geq k}\frac{(n)_l}{(l-k)!}y^l. $$

It is straightforward that

$$ \mathcal{L}_0(y) = (1-y)^{-n} $$

and that

$$ \mathcal{L}_k(y) = (-1)^k y^k \mathcal{L}_0^{(k)}(y) = (-1)^k (n)_k y^k (1-y)^{-(n+k)}. $$

Thus

$$ \begin{aligned} K &= \sum_k \frac{(\frac{1}{2})_k(\frac{n}{2})_k(n)_k}{(1)_k(n-\frac{1}{2})_k k!}\left(\frac{y}{y-1}\right)^k \frac{1}{(1-y)^n} \\ &= \frac{1}{(1-y)^n}\ {}_3F_2(\frac{1}{2},\frac{n}{2},n,1,n-\frac{1}{2};\frac{y}{y-1}) \\ &= \frac{1}{r^{2n}}\ {}_3F_2(\frac{1}{2},\frac{n}{2},n,1,n-\frac{1}{2};\frac{r^2-1}{r^2}). \end{aligned} $$

Putting all these successive expressions together, we obtain

$$ Vol(P^k) = C_n \cdot\ {}_3F_2(\frac{1}{2},\frac{n}{2},n,1,n-\frac{1}{2};\frac{r^2-1}{r^2}), $$

with

$$C_n = 2^{n-1}B(\frac{n}{2},\frac{1}{2})\frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n-\frac{1}{2})}\frac{n-1}{2\pi}\mathrm{vol}(\mathbb{S}^{\mathrm{n}}(1)).$$

It is easy to check that

$$C_n = \frac{4^{n-1}}{C_{2n-2}^{n-1}}\mathrm{vol}(\mathbb{S}^{\mathrm{n}}(1)),$$

thus

$$Vol(P^k) = k^{\frac{n}{2}}\frac{4^{n-1}}{C_{2n-2}^{n-1}}\,{}_3F_2(\frac{1}{2},\frac{n}{2},n,1,n-\frac{1}{2};1-k)\mathrm{vol}(\mathbb{S}^{\mathrm{n}}(\mathrm{r})).$$

The preceding computations are valid only if $\frac{1}{\sqrt{2}} < r < \sqrt{2}$. Nevertheless, since $k \mapsto Vol(P^k)$ is analytic and ${}_3F_2(\frac{1}{2},\frac{n}{2},n,1,n-\frac{1}{2};1-k)$ is converging if $k < 2$, the above expression is in fact valid as soon as $k < 2$. For $n = 2m+1$, we obtain the result stated in Proposition 6. $\square$

5.2 Volume of the perturbed radial fields

The perturbed radial field R_ϵ built in [2] is such that $R_\epsilon = R$ out of an ϵ-neighbourhood G_ϵ of a geodesic segment joining the two singular points p and $-p$ of the radial field. When computing the volume of R_ϵ^k and then passing to the limit, we obtain a result which splits in two parts, one term coming from the underlying radial field R^k and another taking into account (the limit of) the pertubation on G_ϵ.

Lemma 9. – $\lim_{\epsilon\to 0} Vol(R_\epsilon^k) = Vol(R^k) + \pi\frac{1}{\sqrt{k}}\mathrm{vol}(\mathbb{S}^{2\mathrm{m}}(1))$.

Since

$$Vol(R^k) = (\sum_{j=0}^{m}\frac{(C_m^j)^2}{C_{2m}^{2j}}k^j)\ \mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(\mathrm{r}))$$

and

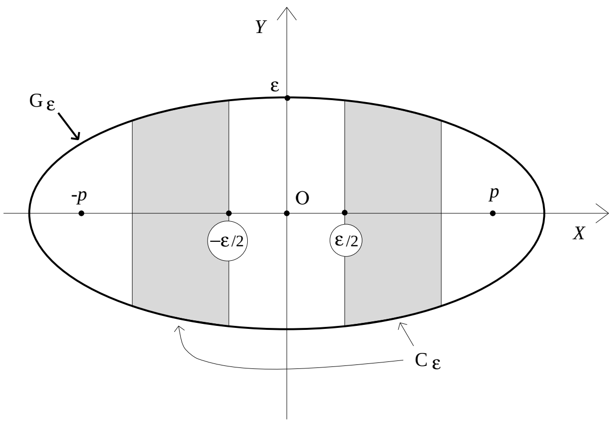
$$\frac{\mathrm{vol}(\mathbb{S}^{2\mathrm{m}}(1))}{\mathrm{vol}(\mathbb{S}^{2\mathrm{m}+1}(\mathrm{r}))} = \frac{1}{\pi}\frac{4^m}{C_{2m}^m}k^{m+\frac{1}{2}},$$

we obtain the expression 3) of Proposition 6.

Proof. – See [2] for the construction of R_ϵ (in this article, R_ϵ is denoted T_ϵ). Let O be the middle point of the geodesic segment $[-p,p]$. In the image of the stereographic projection from $-O$, let G_ϵ be the full $(2m+1)$-dimensional ellipsoid with focal points $\{-p,p\}$ defined by

$$G_\epsilon = \{q \in \mathbb{R}^{2m+1} \mid d(q,p) + d(q,-p) \leq 2\sqrt{\epsilon^2 + d^2(O,p)}\}.$$

We denote by x the coordinate on the line $D =]-p, p[$ and by $y = (y_1, ..., y_{2m})$ those of $D^{\perp}$.



Let $A_\epsilon = G_\epsilon \cap \{|x| \leq \frac{\epsilon}{2}\}$, $B_\epsilon^+ = G_\epsilon \cap \{x \geq p - \epsilon/2\}$, $B_\epsilon^- = G_\epsilon \cap \{x \geq -p + \epsilon/2\}$ and $C_\epsilon = G_\epsilon \setminus (A_\epsilon \cup B_\epsilon^+ \cup B_\epsilon^-)$. By dimensional arguments

$$\lim_{\epsilon \longrightarrow 0} Vol(R_\epsilon^k|_{A_\epsilon}) = 0 \quad \text{and} \quad \lim_{\epsilon \longrightarrow 0} Vol(R_\epsilon^k|_{B_\epsilon^\pm}) = 0.$$

Let $S_\epsilon = R_\epsilon^k(C_\epsilon)$, S_ϵ is a submanifold of $T^1\mathbb{S}^{2m+1}(r)|_{C_\epsilon}$ and obviously

$$\lim_{\epsilon \longrightarrow 0} Vol(R_\epsilon^k) = Vol(R^k) + \lim_{\epsilon \longrightarrow 0} \mathrm{vol}(\mathrm{S}_\epsilon).$$

Now let $\Sigma = \pi^{-1}(] - p, p[) \subset T^1\mathbb{S}^{2m+1}(r)$ where $\pi : T^1\mathbb{S}^{2m+1}(r) \to \mathbb{S}^{2m+1}(r)$ is the projection. Since π is a Riemannian submersion, when we consider in the first manifold the Sasaki metric, the volume of Σ is πr times the common volume of each fiber, which is equal to $\mathrm{vol}(\mathbb{S}^{2\mathrm{m}}(1))$.

To conclude, we only need to show that $\lim_{\epsilon \to 0} \mathrm{vol}(\mathrm{S}_\epsilon) = \mathrm{vol}(\Sigma)$. Let

$$\begin{array}{llll} f: & C_\epsilon & \longrightarrow & S_\epsilon \\ & (x, y) & \longmapsto & (x, y, R_\epsilon^r(x, y)), \end{array} \qquad \begin{array}{llll} h: & C_\epsilon & \longrightarrow & \Sigma \\ & (x, y) & \longmapsto & (x, R_\epsilon^r(x, y)) \end{array}$$

and g_S be the Sasaki metric on $T^1\mathbb{S}^{2m+1}(r)$. Of course $f^* g_S = h^* g_S + d^2 y$. Thus if F, H and Y are the corresponding matrices of $f^* g_S$, $h^* g_S$ and $d^2 y$, we have

$$dS_\epsilon = \mathrm{detF})^{\frac{1}{2}} \mathrm{dxdy} = \det(\mathrm{H} + \mathrm{Y})^{\frac{1}{2}} \mathrm{dxdy}.$$

Since $\|\frac{\partial h}{\partial y_i}\| = O(\frac{1}{\epsilon})$, $\|\frac{\partial h}{\partial x}\| = O(1)$ (see [2]) we deduce $\det(\mathrm{H})^{\frac{1}{2}} = \mathrm{O}(\frac{1}{\epsilon^{2\mathrm{m}}})$ and $\det(\mathrm{H} + \mathrm{Y})^{\frac{1}{2}} = \det(\mathrm{H})^{\frac{1}{2}} + \mathrm{o}(\frac{1}{\epsilon^{2\mathrm{m}}})$, thus

$$\mathrm{vol}(\mathrm{S}_\epsilon) = \mathrm{vol}(\mathrm{h}(\mathrm{C}_\epsilon)) + \mathrm{o}(1).$$

When ϵ goes to zero, $h(C_\epsilon)$ covers Σ except on a set of zero measure. Therefore, $\lim_{\epsilon\to 0} \mathrm{vol(h(C_\epsilon))} = \mathrm{vol}(\Sigma)$. $\square$

References

[1] F. Brito, *Total Bending of flows with mean curvature correction*, Diff. Geom. and its Appl. 12 (2000), 157-163.

[2] V. Borrelli, F. Brito and O. Gil-Medrano, *The infimum of the Energy of unit vector fields on odd-dimensional spheres*, Ann. Global Anal. Geom. 23 (2003) 129-140.

[3] F. Brito, P. Chacón and A. M. Naveira, *On the volume of unit vector fields on spaces of constant sectional curvature*, Comment. Math. Helv. 79 (2004) 300-316.

[4] F. Brito, R. Langevin and H. Rosenberg, *Intégrale de courbure sur des variétés feuilletées*, J. Diff. Geom. 16 (1981), 19-50.

[5] F. Brito and P. Walczak, *On the Energy of Unit vector Fields with Isolated Singularities*, Ann. Math. Polon. 73 (2000), 269-274.

[6] O. Gil-Medrano and E. Llinares-Fuster, *Second variation of Volume and Energy of vector fields. Stability of Hopf vector fields*, Math. Ann. 320 (2001), 531-545.

[7] O. Gil-Medrano and E. Llinares-Fuster, *Minimal unit vector fields*, Tohoku Math. J. 54 (2002), 71-84.

[8] H. Gluck and W. Ziller, *On the volume of a unit vector field on the three-sphere*, Comment. Math. Helv. 61 (1986), 177-192.

[9] D. L. Johnson, *Volume of flows*, Proc. of the Amer. Math. Soc. 104 (1988), 923-932.

[10] S. L. Pedersen, *Volumes of vector fields on spheres*, Trans. Amer. Math. Soc. 336 (1993), 69-78.

[11] W. A. Poor, *Differential geometric structures*, Mc Graw-Hill, New-York, 1981.

[12] G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann. 303 (1995), 325-344.

V. Borrelli
Institut Camille Jordan, Université Claude Bernard, Lyon 1

43, boulevard du 11 Novembre 1918
69622 Villeurbanne Cedex, France
email : borrelli@igd.univ-lyon1.fr

O. Gil-Medrano
Departamento de Geometría y Topología, Facultad de Matemáticas
Universidad de Valencia
46100 Burjassot, Valencia, España
email : Olga.Gil@uv.es