Talk II: The Nash-Kuiper process for curves

Vincent Borrelli

July 2, 2013

The Nash and Kuiper process [1, 2] converts a short embedding of a Riemannian manifold into an Euclidean space into a C^1-isometric embedding. A short embedding is a map $f_0 : (M^m, g) \longrightarrow \mathbb{E}^q = (\mathbb{R}^q, \langle .,. \rangle)$ that shortens the length of curves, that is:

$$Length(f \circ \gamma) = Length(\gamma)$$

for every rectifiable curve $\gamma : [a, b] \longrightarrow M^m$. Since these maps are plentiful, the Nash and Kuiper process must be thought of as a machinery to produce C^1-isometric maps in great profusion.

Here we present the Nash and Kuiper process in the simplest situation where the source manifold is a Riemannian segment or a circle and the target manifold is the Euclidean 2-plane $\mathbb{E}^2$.

1 One dimensional Isometric Problem

Let us begin with the following elementary problem.

One dimensional Isometric Problem, first version.– *Let* $f_0 : [0,1] \stackrel{C^\infty}{\longrightarrow} \mathbb{E}^2 \simeq \mathbb{C}$ *be a given regular curve and let* $r : [0,1] \longrightarrow \mathbb{R}^*_+$ *be any* C^∞ *map such that*

$$\forall x \in [0,1], \quad r(x) > \|f_0'(x)\|.$$

Build $f : [0,1] \stackrel{C^\infty}{\longrightarrow} \mathbb{E}^2$ *such that:*

i) $\forall\, x \in [0,1], \quad \|f'(x)\| = r(x)$
ii) $\|f - f_0\|_{C^0} < \delta$
where $\delta > 0$ *is given.*

We would like to solve this problem by using the One-dimensional Convex Integration Theory. To do so, we need to find a C^∞ family of loops

$$\begin{array}{rccc} h: & [0,1] & \longrightarrow & C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{C}) \\ & x & \longmapsto & h(x,.) \end{array}$$

such that

$$\forall x \in [0,1], \quad \int_0^1 h(x,s)\mathrm{d}s = f_0'(x)$$

and

$$\forall x \in [0,1], \forall x \in \mathbb{E}/\mathbb{Z}, \quad \|h(x,s)\| = r(x).$$

We will then simply have to define f by

$$\forall x \in [0,1], \quad f(x) := f_0(0) + \int_0^x h(s,\{Ns\})\mathrm{d}s$$

where $N \in \mathbb{N}^*$. By the very definition of f we will have

$$\forall x \in [0,1], \quad \|f'(x)\| = \|h(x,\{Nx\})\| = r(x),$$

and by the C^0-density property, we will also have

$$\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$

Therefore, if N is large enough, $f := IC(f_0, h, N)$ will be a solution of our problem.

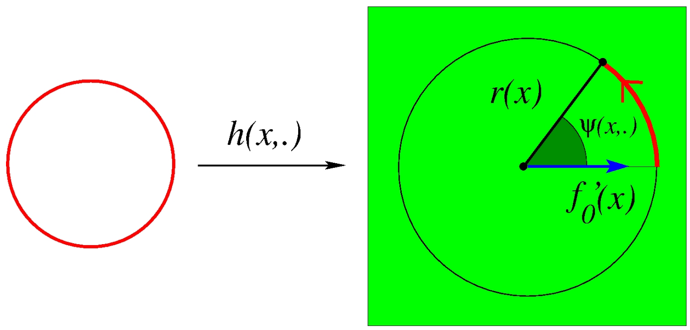
The choice of h.– In that construction, we have a huge freedom in the choice of the family of loops h and we must be careful to make a choice as natural as possible. It sounds clear to choose h in the form

$$h(x,s) := r(x)e^{i\psi(x,s)}\mathbf{t}_0(x)$$

where $t_0(x) := \frac{f_0'(x)}{\|f_0'(x)\|}$ and $\psi : [0,1] \times \mathbb{E}/\mathbb{Z} \longrightarrow \mathbb{R}$ is any map such that

$$\int_0^1 e^{i\psi(x,s)}\mathrm{d}s = \frac{\|f_0'(x)\|}{r(x)}.$$

But there is no obvious choice for ψ.



A choice in the form $h(x,s) := r(x)e^{i\psi(x,s)}\mathbf{t}_0(x)$

For some reasons that will become clear latter (in Talk III), we decide to take

$$\psi(x,s) := \alpha(x)\cos 2\pi s$$

where $\alpha(x)$ must be chosen such that

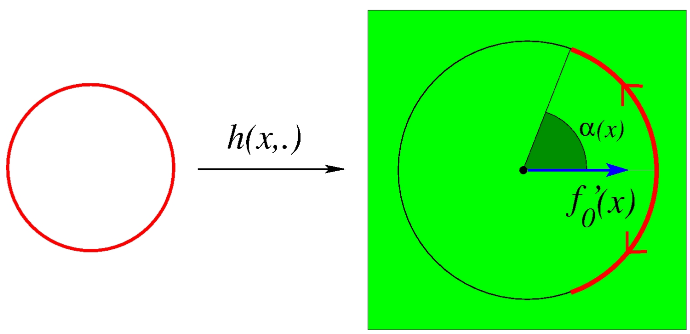
$$\int_0^1 e^{i\alpha(x)\cos 2\pi s}\mathrm{d}s = \frac{\|f_0'(x)\|}{r(x)}.$$

Note that

$$\int_0^1 \sin(\alpha(x)\cos 2\pi s)\mathrm{d}s = 0$$

thus

$$\int_0^1 e^{i\alpha(x)\cos 2\pi s}\mathrm{d}s = \int_0^1 \cos(\alpha(x)\cos 2\pi s)\mathrm{d}s$$

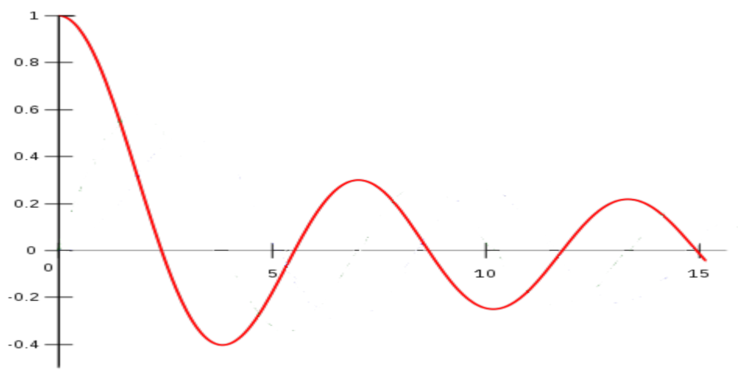


The loop $h(x,.)$ *is round-trip*

Definition.– The map

$$\begin{array}{rccl} J_0: & \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ & \alpha & \longmapsto & \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(\alpha \sin s)\mathrm{d}s \end{array}$$

is called the *Bessel function of the first kind and of order 0.*



Graph of the Bessel function J_0.

Lemma 1.– *We have:*

$$J_0(\alpha) = \int_0^1 \cos(\alpha \cos 2\pi s) \mathrm{d}s.$$

Proof.– Indeed

$$\begin{aligned}
J_0(\alpha) &= \frac{1}{\pi}\int_0^{\pi} \cos(\alpha \sin s)\mathrm{d}s \\
&= \frac{2}{\pi}\int_0^{\pi/2} \cos(\alpha \sin s)\mathrm{d}s && (\text{since } \sin(\pi - s) = \sin s) \\
&= \frac{2}{\pi}\int_0^{\pi/2} \cos(\alpha \sin u)\mathrm{d}u && (u = \tfrac{\pi}{2} - s) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha \sin u)\mathrm{d}u \\
&= \int_0^{1} \cos(\alpha \cos 2\pi s)\mathrm{d}s && (u = 2\pi s).
\end{aligned}$$

□

Let $\kappa = 2.404...$ be the first positive zero of J_0. The restricted Bessel function

$$J_0 : [0, \kappa] \longrightarrow [0, 1]$$

is 1:1. We denote by J_0^{-1} its inverse. We must choose

$$\alpha(x) := J_0^{-1}\left(\frac{\|f_0'(x)\|}{r(x)}\right)$$

to ensure

$$\int_0^1 e^{i\alpha(x)\cos 2\pi s}\mathrm{d}s = \frac{\|f_0'(x)\|}{r(x)}.$$

Solution of the One dimensional Isometric Problem.– *Obviously, the map f defined by*

$$\forall x \in [0,1], \quad f(x) := f_0(0) + \int_0^x r(x)\mathbf{e}^{i\alpha(x)\cos 2\pi N s}\mathrm{d}s$$

where

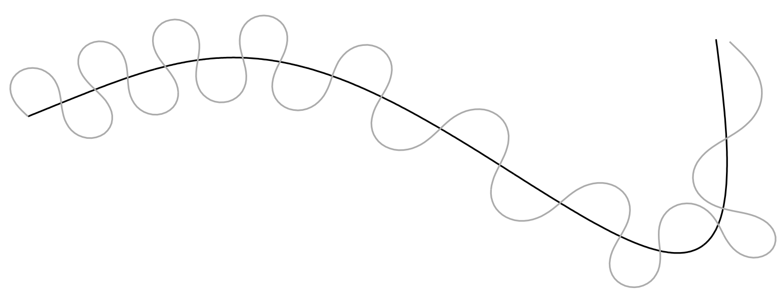
$$\mathbf{e}^{i\theta} := \cos\theta\,\mathbf{t}_0 + \sin\theta\,\mathbf{n}_0, \quad \mathbf{n}_0 = i\mathbf{t}_0$$

is a solution of problem 1.

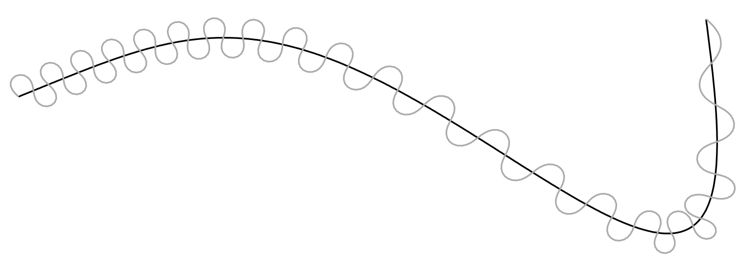
The two following pictures shows the image of $f = IC(f_0, h, N)$ with

$$x \mapsto f_0(x) := \left(x, \frac{1}{\pi}\cos\left(\frac{3}{2}\pi x^2\right)\right)$$

and $r \equiv 4$.



The number N of oscillation is 9



Beware that $f(0) = f_0(0)$ but $f(1) \neq f_0(1)$ even if it seems that the endpoints coincide on the picture. Here N is 20

2 Isometric Problem for $\mathbb{E}/\mathbb{Z}$

We now adress the question of the isometric problem for the circle $\mathbb{S}^1 = \mathbb{E}/\mathbb{Z}$.

Isometric Problem for $\mathbb{E}/\mathbb{Z}$.– *Let $f_0 : \mathbb{E}/\mathbb{Z} \xrightarrow{C^\infty} \mathbb{E}^2 \simeq \mathbb{C}$ be a given regular closed curve and let $r : \mathbb{E}/\mathbb{Z} \longrightarrow \mathbb{R}_+^*$ be any C^∞ map such that*

$$\forall x \in [0,1], \quad r(x) > \|f_0'(x)\|.$$

Build $f : \mathbb{E}/\mathbb{Z} \xrightarrow{C^\infty} \mathbb{E}^2$ such that:

i) $\forall\, x \in [0,1], \quad \|f'(x)\| = r(x)$
ii) $\|f - f_0\|_{C^0} < \delta$
where $\delta > 0$ is given.

Not that, even if f_0 is defined over $\mathbb{E}/\mathbb{Z}$, the curve

$$x \longmapsto f_0(0) + \int_0^x r(x)\mathbf{e}^{i\alpha(x)\cos 2\pi N s}\mathrm{d}s$$

is not closed in general. This defect can be easily corrected by the following modification of the convex integration formula:

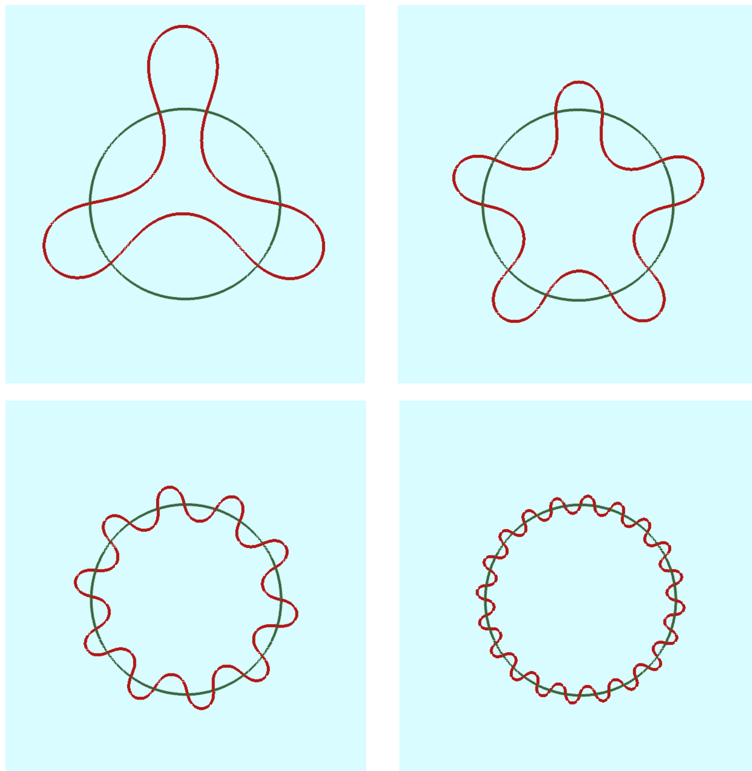
$$f(x) := f_0(0) + \int_0^x r(x)\mathbf{e}^{i\alpha(x)\cos 2\pi N s}\mathrm{d}s - x\int_0^1 r(x)\mathbf{e}^{i\alpha(x)\cos 2\pi N s}\mathrm{d}s.$$

For short, we write $f = \widetilde{IC}(f_0, h, N)$ with

$$h(x,s) = r(x)\mathbf{e}^{i\alpha(x)\cos 2\pi s}.$$

Let $F = IC(f_0, h, N)$, from the C^0-density property we have $\|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$ and since $f(x) = F(x) - x(F(1) - F(0))$ we also have

$$\|f - f_0\|_{C^0} \leq 2\|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$



Modified integration convex formula with $N = 3, 5, 10$ *and* 20

The speed $\|f'\|$ is only approximately equal to $r(x)$, precisely

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \big|\|f'(x)\| - r(x)\big| \leq \|F - f_0\|_{C^0} = O\left(\frac{1}{N}\right).$$

We shall obtain a map $f_\infty : \mathbb{E}/\mathbb{Z} \longrightarrow \mathbb{E}^2$ such that

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \|f'_\infty(x)\| = r(x)$$

by iteratively applying the modifying convex integration formula. The idea behind the construction of f_∞ is due to Nash.

The Nash and Kuiper Process.– Let $(\delta_k)_{k\in\mathbb{N}^*}$ such that $\delta_k \uparrow 1$ and $\delta_k > 0$. For all $k \in \mathbb{N}^*$ and for all $x \in \mathbb{E}/\mathbb{Z}$ we set

$$r_k^2(x) := r_0^2(x) + \delta_k \left(r^2(x) - r_0^2(x)\right)$$

with $r_0(x) := \|f_0'(x)\|$ and then define

$$f_k := \widetilde{IC}(f_{k-1}, h_k, N_k)$$

with

$$h_k(x,s) := r_k(x)\mathbf{e}^{i\alpha_k(x)\cos 2\pi s}$$

where

$$\alpha_k(x) = J_0^{-1}\left(\frac{\|f'_{k-1}(x)\|}{r_k(x)}\right) \text{ and } \mathbf{e}^{i\theta} := \cos\theta\, \mathbf{t}_{k-1} + \sin\theta\, \mathbf{n}_{k-1}.$$

Each f_k has a speed which is approximately r_k:

$$\big|\|f'_k(x)\| - r_k(x)\big| = O\left(\frac{1}{N_k}\right).$$

Since the sequence $r_k(x)$ is strictly increasing for every $x \in \mathbb{E}/\mathbb{Z}$, the number N_k can be chosen large enough such that

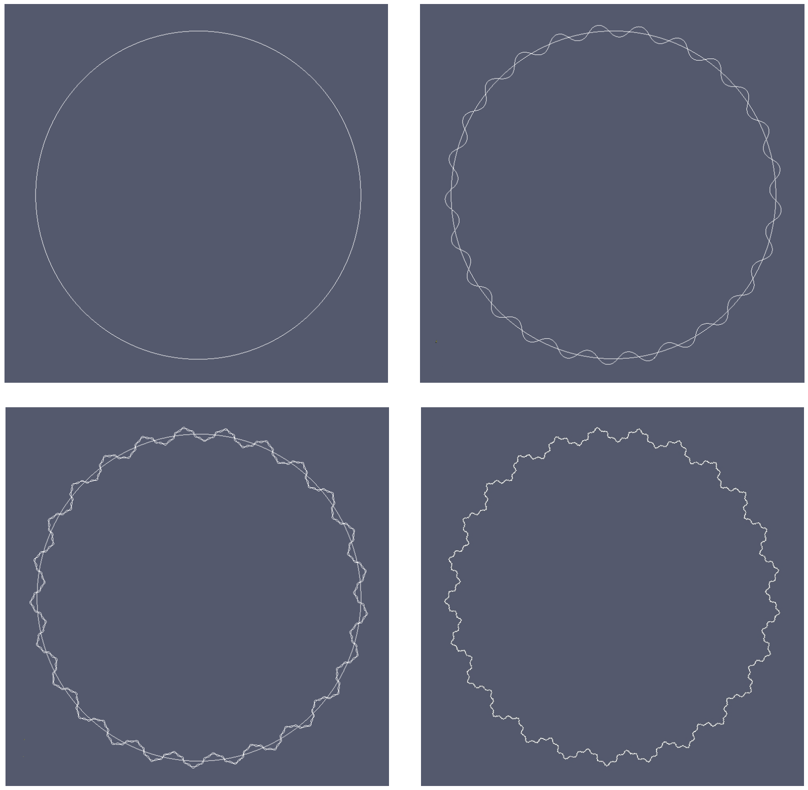
$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad r_{k+1}(x) > \|f_k(x)\|.$$

From now on, we implicitely assume that every N_k is iteratively chosen such that to fulfill the above inequality. This is crucial to define f_{k+1} as $\widetilde{IC}(f_k, h_{k+1}, N_{k+1})$.

Theorem (solution of the Isometric Problem for $\mathbb{E}/\mathbb{Z}$).– *If the sequence $(\delta_k)_{k\in\mathbb{N}^*}$ is chosen so that*

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$$

then there exists a sequence of integers $(N_k)_{k\in\mathbb{N}^*}$ such that $f_k := \widetilde{IC}(f_{k-1}, h_k, N_k)$ is C^1 converging toward a C^1 limit f_∞ with speed $\|f'_\infty\| = r$.



The Nash-Kuiper process, f_0, f_1, f_2 and f_∞.

3 Proof of the Theorem

Lemma 2.– *The inequality*

$$1 + J_0^2(\alpha) - 2J_0(\alpha)\cos(\alpha) \leq 7(1 - J_0^2(\alpha))$$

holds for every $\alpha \in [0, \kappa]$ (recall that κ is the first positive root of J_0).

Proof of lemma 2.– Subtracting the right hand side from the left hand side, we rewrite this inequality as

$$4J_0^2(\alpha) - J_0(\alpha)\cos(\alpha) - 3 \leq 0.$$

By considering the alternating Taylor series of J_0 and cos, we get

$$J_0(\alpha) \leq 1 - \frac{\alpha^2}{4} + \frac{\alpha^4}{64} \quad \text{and} \quad \cos(\alpha) \geq 1 - \frac{\alpha^2}{2}.$$

Whence

$$0 \leq 4J_0(\alpha) - \cos(\alpha) \leq 3 - \frac{\alpha^2}{2} + \frac{\alpha^4}{16} \leq 3 + \frac{\alpha^2}{2},$$

where the last inequality follows from $-\frac{\alpha^2}{2} + \frac{\alpha^4}{16} \leq \frac{\alpha^2}{2}$ for all $\alpha \in [0, \kappa]$. We can now write

$$\begin{array}{rcl} 4J_0^2(\alpha) - J_0(\alpha)\cos(\alpha) - 3 & = & J_0(\alpha)(4J_0(\alpha) - \cos(\alpha)) - 3 \\ & \leq & (1 - \frac{\alpha^2}{4} + \frac{\alpha^4}{64})(3 + \frac{\alpha^2}{2}) - 3. \end{array}$$

Putting $x = \alpha^2/4$, this last polynomial can be rewritten

$$(1 - x + \frac{x^2}{4})(3 + 2x) - 3 = \frac{x}{2}(x - x_1)(x - x_2),$$

where $x_1 < 0 < \kappa^2/4 < x_2$. It ensues that this polynomial is negative for $\alpha \in [0, \kappa]$. □

Lemma 3.– *For all $x \in \mathbb{E}/\mathbb{Z}$, we have*

$$\begin{array}{rcl} \|f_k'(x) - f_{k-1}'(x)\| & \leq & \sqrt{7} r(x) \sqrt{\delta_k - \delta_{k-1}} \\ & & + \sqrt{14 r(x)} \|F_{k-1} - f_{k-2}\|_{C^0}^{1/2} + \|F_k - f_{k-1}\|_{C^0} \end{array}$$

Proof of lemma 3.– Since $J_0(\alpha_k(x)) = \|f_{k-1}'(x)\|/r_k(x)$, we have

$$\begin{array}{rl} \|F_k'(x) - f_{k-1}'(x)\|^2 = & r_k^2(x) + \|f_{k-1}'(x)\|^2 - 2r_k(x)\|f_{k-1}'(x)\| \cos(\alpha_k(x)\cos(2\pi N x)) \\ = & r_k^2(x)\left(1 + J_0(\alpha_k(x))^2 - 2J_0(\alpha_k(x))\cos(\alpha_k(x)\cos(2\pi N x))\right). \end{array}$$

We also have $\cos(\alpha\cos(2\pi N x)) \geq \cos(\alpha)$ for every $\alpha \in [0, \kappa] \subset [0, \pi]$. By using the previous lemma, we get

$$\begin{array}{rcl} \|F_k'(x) - f_{k-1}'(x)\|^2 & \leq & r_k^2(x)\left(1 + J_0(\alpha_k(x))^2 - 2J_0(\alpha_k(x))\cos(\alpha_k(x))\right) \\ & \leq & 7r_k^2(x)(1 - J_0(\alpha_k(x))^2) \\ & \leq & 7(r_k^2(x) - \|f_{k-1}'(x)\|^2). \end{array}$$

Since

$$\left|\|f_{k-1}'(x)\| - r_{k-1}(x)\right| \leq \|F_{k-1} - f_{k-2}\|_{C^0}$$

and

$$\|f_{k-1}'(x)\| + r_{k-1}(x) \leq 2r(x)$$

we deduce

$$\left| \|f'_{k-1}(x)\|^2 - r_{k-1}^2(x) \right| \leq 2r(x)\|F_{k-1} - f_{k-2}\|_{C^0}.$$

Therefore

$$\begin{aligned} \|F'_k(x) - f'_{k-1}(x)\|^2 &\leq 7(r_k^2(x) - r_{k-1}^2(x) + 2r(x)\|F_{k-1} - f_{k-2}\|_{C^0}) \\ &\leq 7(\delta_k - \delta_{k-1})(r^2(x) - r_0^2(x)) + 14r(x)\|F_{k-1} - f_{k-2}\|_{C^0} \end{aligned}$$

and

$$\|F'_k(x) - f'_{k-1}(x)\| \leq \sqrt{7}r(x)\sqrt{\delta_k - \delta_{k-1}} + \sqrt{14r(x)}\|F_{k-1} - f_{k-2}\|_{C^0}^{1/2}.$$

Finally,

$$\begin{aligned} \|f'_k(x) - f'_{k-1}(x)\| &\leq \sqrt{7}r(x)\sqrt{\delta_k - \delta_{k-1}} \\ &\quad + \sqrt{14r(x)}\|F_{k-1} - f_{k-2}\|_{C^0}^{1/2} + \|F_k - f_{k-1}\|_{C^0} \end{aligned}$$

□

Proof of the theorem (solution of the Isometric Problem for $\mathbb{E}/Z$).– Since $\|F_k - f_{k-1}\|_{C^0} = O\left(\frac{1}{N_k}\right)$, we can choose the sequence $(N_k)_{k\in\mathbb{N}^*}$ such that

$$\sum \|F_k - f_{k-1}\|_{C^0} < +\infty$$

and thus the maps $f_k s$ are C^0-converging toward a C^0-map f_∞. If

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty$$

and if the sequence $(N_k)_{k\in\mathbb{N}^*}$ is also chosen such that

$$\sum \|F_k - f_{k-1}\|_{C^0}^{1/2} < +\infty$$

then, by Lemma 3,

$$\sum \|f'_k - f'_{k-1}\|_{C^0} < +\infty$$

and the maps $f_k s$ are C^1-converging. In particular

$$\forall\, x \in \mathbb{E}/\mathbb{Z}, \quad \lim_{k\to+\infty} \|f'_k(x)\| = \|f'_\infty(x)\|$$

and since

$$\forall\, x \in \mathbb{E}/\mathbb{Z}, \quad r_{k-1}(x) < \|f'_k(x)\| < r_{k+1}(x)$$

we obtain

$$\forall\, x \in \mathbb{E}/\mathbb{Z}, \quad \|f'_\infty(x)\| = r(x).$$

□

References

[1] N. Kuiper, *On C^1-isometric imbeddings*, Indag. Math., 17 (1955), 545-556.

[2] J. Nash, *C^1-isometric imbeddings*, Ann. of Math. (2), 60 (1954), 383-396.