Talk III: Regularity of Nash-Kuiper curves

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In the previous talk, we have seen that the sequence $f_1, f_2, \ldots$ of C^∞-maps generated by the Nash-Kuiper process C^1-converges toward a C^1-isometry f_∞ provided that

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty.$$

In this talk we adress the C^2-regularity of f_∞. To avoid some technicalities in the computations we still assume that the Nash-Kuiper process is applied to a smooth initial map $f_0 : \mathbb{E}/\mathbb{Z} \to \mathbb{E}^2$ such that:

- ($Cond$ 1) it is of constant speed $r_0 := \|f_0'\| < 1$
- ($Cond$ 2) it is radially symmetric, that is: $f_0'(x + \frac{1}{2}) = -f_0'(x)$

and that the target speed function is $r \equiv 1$. We also assume that the N_ks are chosen among even numbers. The general case, slightly more technical in nature, is left to the reader.

Proposition 1.– *For every $k \in \mathbb{N}^*$, f_k is of constant speed r_k and radially symmetric. In particular,*

$$f_k = IC(f_{k-1}, h_k, N_k).$$

The functions α_k are also constant and equal to $J_0^{-1}\left(\frac{r_{k-1}}{r_k}\right)$.

Proof.– By induction. Assume that f_{k-1} satisfies ($Cond$ 1) and ($Cond$ 2). In particular f_{k-1} is of constant speed r_{k-1} and thus the function $\alpha_k = J_0^{-1}\left(\frac{r_{k-1}}{r_k}\right)$ is constant. Since $N_k \in 2\mathbb{N}^*$, we have

$$h_k(x + \frac{1}{2}, \{N_k(x + \frac{1}{2})\}) = -h_k(x, \{N_k x\})$$

and consequently

$$\int_0^1 h_k(s, \{N_k s\})\mathrm{d}s = 0.$$

It ensues that

$$IC(f_{k-1}, h_k, N_k) = \widetilde{IC}(f_{k-1}, h_k, N_k)$$

and therefore f_k is of constant speed $\|f_k'(x)\| = \|h_k(x, \{Nx\})\| = r_k$. It is also radially symmetric since $f_k(x) = h(x, \{N_k x\})$. $\square$.

1 C^2-regularity

Proposition 1.– *For every $x \in \mathbb{E}/\mathbb{Z}$, we have*

$$f_k''(x) = (-2\pi\alpha_k N_k \sin 2\pi N_k x + r_{k-1} scal_{k-1}(x))\, ir_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)$$

where $scal_k$ denotes the signed curvature of f_k. Moreover

$$r_k scal_k(x) = r_0 scal_0(x) - 2\pi \sum_{l=1}^{k} \alpha_l N_l \sin(2\pi N_l x).$$

Remark 1.– Recall that two regular C^2 curves differ by a rigid motion if and only if they have the same signed curvature function.

Remark 2.– Let $\mu_k(x) := scal_k(x)\|f_k'(x)\|dx$ be the signed curvature measure of f_k. We have

$$\mu_k(x) = \mu_{k-1}(x) - 2\pi\alpha_k N_k \sin(2\pi N_k x)\mathrm{d}x.$$

The convex integration process modifies the curvature mesure in the simplest way by a cosine term with frequency N_k.

Remark 3.– More generally, if $h_k(x, s) = r_k e^{i\psi_k(s)}\mathbf{t}_{k-1}$, one can show that

$$\mu_k(x) = N_k \psi_k'(N_k x) + \mu_{k-1}(x).$$

Thus, if we require to modify the curvature mesure by a single term of frequency N_k, we have to choose

$$\psi_k'(x) := \alpha_k \sin(2\pi x + \text{phase difference}).$$

This is the reason why in Talk II we have chosen

$$\psi_k(x) = \alpha_k \cos 2\pi x.$$

Proof.– We have

$$\begin{aligned}
f_k''(x) &= \frac{\partial}{\partial x}\left(r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)\right)\\
&= \frac{\partial}{\partial x}\left(r_k(\cos(\alpha_k \cos 2\pi N_k x)\mathbf{t}_{k-1}(x) + \sin(\alpha_k \cos 2\pi N_k x)\mathbf{n}_{k-1}(x)\right)\\
&= \frac{r_k}{r_{k-1}}\frac{\partial}{\partial x}\left(\cos(\alpha_k \cos 2\pi N_k x) f_{k-1}'(x) + \sin(\alpha_k \cos 2\pi N_k x) i f_{k-1}'(x)\right)\\
&= -2i\pi\alpha_k N_k \sin(2\pi N_k x) r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)\\
&\quad + \frac{r_k}{r_{k-1}}\left(\cos(\alpha_k \cos 2\pi N_k x) f_{k-1}''(x) + \sin(\alpha_k \cos 2\pi N_k x) i f_{k-1}''(x)\right)
\end{aligned}$$

Since f_{k-1} is of constant speed r_{k-1} we have

$$f_{k-1}''(x) = r_{k-1} scal_{k-1}(x) i f_{k-1}'(x)$$

therefore

$$\begin{aligned}
f_k''(x) &= -2i\pi\alpha_k N_k \sin(2\pi N_k x) r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)\\
&\quad + r_k r_{k-1} scal_{k-1}(x) i e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x)
\end{aligned}.$$

Finally,

$$f_k''(x) = \left(-2\pi\alpha_k N_k \sin 2\pi N_k x + r_{k-1} scal_{k-1}(x)\right) i r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x).$$

Because f_k is of constant arc length we also have

$$f_k''(x) = r_k scal_k(x) i f_k'(x) = r_k scal_k(x) i r_k e^{i\alpha_k \cos 2\pi N_k x}\mathbf{t}_{k-1}(x).$$

From this we deduce

$$r_k scal_k(x) = r_{k-1} scal_{k-1}(x) - 2\pi\alpha_k N_k \sin(2\pi N_k x)$$

and by induction

$$r_k scal_k(x) = r_0 scal_0(x) - 2\pi\sum_{l=1}^{k}\alpha_l N_l \sin(2\pi N_l x).$$

□

Lemma 2 (Amplitude Lemma).– *We have*

$$\alpha_k \sim \sqrt{2(1-r_0^2)}\sqrt{\delta_k - \delta_{k-1}}$$

where $\sim$ *denotes the equivalence of sequences.*

Proof.– By definition $\alpha_k = J_0^{-1}(\frac{r_{k-1}}{r_k})$. Recall that the Taylor expansion of $J_0(\alpha)$ up to order 2 is

$$w = 1 - \frac{\alpha^2}{4} + o(\alpha^2).$$

Let $y = 1 - w$ and $X = \alpha^2$, we have $y = \frac{X}{4} + o(X)$ thus $X = 4y + o(y)$ and so $X \sim 4y$. We finally get

$$\alpha \sim 2\sqrt{1-w} \quad \text{and} \quad \alpha_k \sim 2\sqrt{1 - \frac{r_{k-1}}{r_k}}.$$

Since $r_0^2 + (1 - r_0^2) = 1$, we have

$$r_k^2 = r_0^2 + \delta_k(1 - r_0^2) = 1 + (\delta_k - 1)(1 - r_0^2)$$

so

$$r_k^2 - r_{k-1}^2 = (\delta_k - \delta_{k-1})(1 - r_0^2)$$

and

$$1 - \frac{r_{k-1}^2}{r_k^2} = \frac{(\delta_k - \delta_{k-1})(1 - r_0^2)}{1 - (1 - \delta_k)(1 - r_0^2)} \sim (\delta_k - \delta_{k-1})(1 - r_0^2).$$

In an other hand

$$1 - \frac{r_{k-1}^2}{r_k^2} = \left(1 - \frac{r_{k-1}}{r_k}\right)\left(1 + \frac{r_{k-1}}{r_k}\right) \sim 2\left(1 - \frac{r_{k-1}}{r_k}\right).$$

Thus

$$\left(1 - \frac{r_{k-1}}{r_k}\right) \sim \frac{1}{2}(\delta_k - \delta_{k-1})(1 - r_0^2).$$

and

$$\alpha_k \sim 2\sqrt{1 - \frac{r_{k-1}}{r_k}} \sim \sqrt{2(1 - r_0^2)}\sqrt{\delta_k - \delta_{k-1}}.$$

□

Proposition 2.– *If* $\sum_{k\in\mathbb{N}^*} \sqrt{\delta_k - \delta_{k-1}} N_k < +\infty$ *then* f_∞ *is* C^2.

Proof.– Since we already know that the sequence $(f_k)_{k\in\mathbb{N}}$ C^1 converges, it is enough to prove that $(f_k'')_{k\in\mathbb{N}}$ is a Cauchy sequence. From

$$f_k''(x) = r_k scal_k(x) i f_k'(x)$$

we deduce

$$
\begin{aligned}
\|f_k''(x)-f_{k-1}''(x)\|_{C^0} &\leq \|r_k scal_k(x) f_k'(x) - r_{k-1} scal_{k-1}(x) f_{k-1}'(x)\|_{C^0} \\
&\leq \|r_{k-1} scal_{k-1}(x) f_k'(x) - r_{k-1} scal_{k-1}(x) f_{k-1}'(x)\|_{C^0} \\
&\quad + |r_k scal_k(x) - r_{k-1} scal_{k-1}(x)| \|f_k'(x)\|_{C^0} \\
&\leq r_{k-1}|scal_{k-1}(x)| \|f_k'(x) - f_{k-1}'(x)\|_{C^0} \\
&\quad + r_k |r_k scal_k(x) - r_{k-1} scal_{k-1}(x)|.
\end{aligned}
$$

Since

$$r_k scal_k(x) = r_0 scal_0(x) - 2\pi \sum_{l=1}^{k} \alpha_l N_l \sin(2\pi N_l x)$$

we have

$$|r_k scal_k(x) - r_{k-1} scal_{k-1}(x)| \leq 2\pi \alpha_k N_k$$

and

$$r_k |scal_k(x)| \leq r_0 |scal_0(x)| + 2\pi \sum_{l\in\mathbb{N}^*} \alpha_l N_l.$$

In particular the $r_k|scal_k(x)|$ are uniformly bounded by

$$M := \|r_0 scal_0(x)\|_{C^0} + 2\pi \sum_{k\in\mathbb{N}^*} \alpha_k N_k.$$

Note that $M < +\infty$. Indeed $\alpha_k \sim \sqrt{2(1-r_0^2)}\sqrt{\delta_k - \delta_{k-1}}$ therefore

$$\sum_{k\in\mathbb{N}^*} \sqrt{\delta_k - \delta_{k-1}} N_k < +\infty \quad \Longrightarrow \quad \sum_{k\in\mathbb{N}^*} \alpha_k N_k < +\infty.$$

We deduce

$$\|f_k''(x) - f_{k-1}''(x)\|_{C^0} \leq M \|f_k'(x) - f_{k-1}'(x)\|_{C^0} + 2\pi \alpha_k N_k.$$

Let $p < q$, we thus have

$$
\begin{aligned}
\|f_q''(x) - f_p''(x)\|_{C^0} &\leq M \sum_{k=p}^{q} \sqrt{\delta_k - \delta_{k-1}} + 2\pi \sum_{k=p}^{q} \alpha_k N_k \\
&\leq M \sum_{k=p}^{\infty} \sqrt{\delta_k - \delta_{k-1}} + 2\pi \sum_{k=p}^{\infty} \alpha_k N_k.
\end{aligned}
$$

Hence $\left(f_k''\right)_{k\in\mathbb{N}}$ is a Cauchy sequence. □

2 The normal map

2.1 Analogy with a Riesz product

Theorem 1.– *Let* $\mathbf{n}_k$ *be the normal map of* f_k*. We have*

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_k(x) = e^{i\alpha_k \cos(2\pi N_k x)}\mathbf{n}_{k-1}(x)$$

where α_k *is the amplitude of the loop used in the convex integration to build* f_{k-1} *from* f_k *and* $N_k \in 2\mathbb{N}^*$ *is the number of corrugations of* f_k *(precise definitions below). In particular, the normal map* $\mathbf{n}_\infty$ *of* f_∞ *has the following expression*

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_\infty(x) = \left(\prod_{k=1}^{+\infty} e^{i\alpha_k \cos(2\pi N_k x)}\right)\mathbf{n}_0(x).$$

Proof.– This is straightforward from Lemma 3, Talk II and the fact that $\mathbf{n}_k = i\mathbf{t}_k$. □

Theorem 1 puts into light some resemblance of $\mathbf{n}_\infty$ with a *Riesz product*, that is, an infinite product

$$p(x) := \prod_{j=1}^{\infty}(1 + \alpha_j \cos(2\pi N_j x)),$$

where $(\alpha_j)_{j\in\mathbb{N}}$ is a sequence of real numbers such that for every $j \in \mathbb{N}^*$, $|\alpha_j| \leq 1$, and

$$\forall j \in \mathbb{N}^*, \quad \frac{N_{j+1}}{N_j} \geq 3 + q$$

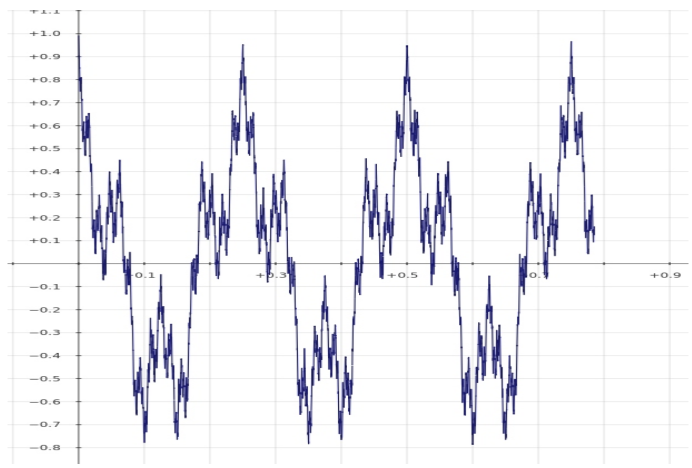
for some fixed $q > 0$. In particular, if

$$p(x) = 1 + \sum_{\nu=1}^{\infty} \gamma_\nu \cos(2\pi\nu x)$$

is the Fourier expansion of p, then $\gamma_{N_j} = \alpha_j$ and $\gamma_\nu = 0$ if ν is not of the form $N_{j_1} \pm N_{j_2} \pm ... \pm N_{j_k}$, $j_1 > j_2 > ... > j_k$ [3]. Riesz products are well known to have a fractal structure. Precisely, their Riesz measures $p(x)dx$ have a fractionnary Hausdorff dimension [4].

An interesting case of a Riesz structure occurs for $\alpha_j = a^j$ and $N_j = b^j$ for some positive numbers a, b with $a < 1$ and $ab > 1$. Indeed, in that case, $A_\infty := \sum_j \alpha_j \cos(2\pi N_j x)$ is the well-known Weierstrass function:

$$A_\infty(x) = \sum_j a^j \cos(2\pi b^j x).$$



Graph of a Weierstrass function with $a = 0.5$ and $b = 4$.

Although its exact value is conjectural, the Hausdorff dimension of its graph is larger than one [2]. It follows that the Hausdorff dimension of the graph of

$$\mathbf{n}_\infty = \left(\prod_{j=1}^{+\infty} e^{ia^j \cos(2\pi b^j x)} \right) \mathbf{n}_0(x).$$

is also larger than one.

2.2 Spectrum

The normal map $\mathbf{n}_\infty$ can be thought of as a 1-periodic map from $\mathbb{R}$ to $\mathbb{C}$. Let

$$\forall x \in \mathbb{E}/\mathbb{Z}, \quad \mathbf{n}_k(x) = \sum_{p \in \mathbb{Z}} a_p(k) e^{2i\pi p x}$$

denotes the Fourier series expansion of the normal map $\mathbf{n}_k$. We derive from Theorem 1 the following inductive formula.

Fourier series expansion of $\mathbf{n}_k$.– *We have*

$$\forall p \in \mathbb{Z}, \quad a_p(k) = \sum_{n \in \mathbb{Z}} u_n(k) a_{p-nN_k}(k-1)$$

where $u_n(k) = i^n J_n(\alpha_k)$.

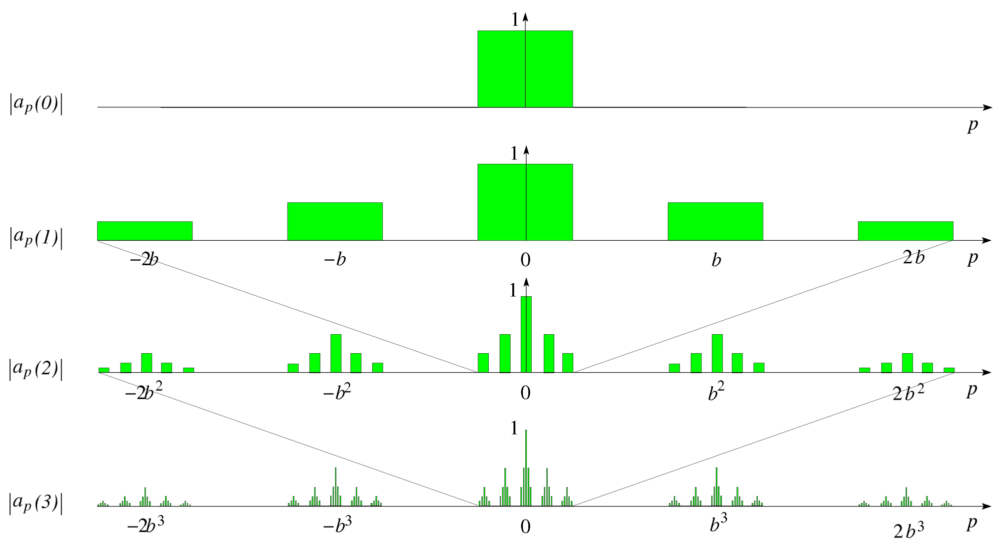
In the above formula, J_n denotes the Bessel function of order n (see [1] or [5]):

$$\alpha \longmapsto J_n(\alpha) = \frac{1}{\pi} \int_0^\pi \cos(nu - \alpha \sin u) \mathrm{d}u.$$

The Fourier expansion of n_k gives the key to understand the construction of the spectrum $(a_p(k))_{p \in \mathbb{Z}}$ from the spectrum $(a_p(k-1))_{p \in \mathbb{Z}}$. The k-th spectrum is obtained by collecting an infinite number of shifts of the previous spectrum. The n-th shift is of amplitude nN_k and weighted by $u_n(k) = i^n J_n(\alpha_k)$. Since

$$|J_n(\alpha_k)| \downarrow 0$$

the weight is decreasing with n (see the figure below).



A schematic picture of the various spectra $(a_p(k))_{p \in \mathbb{Z}}$ *with* $N_k = b^k$.

Lemma (Jacobi-Anger identity).– *For every* $x \in \mathbb{R}_+$, *we have*

$$e^{ix \cos \theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(x) e^{in\theta}.$$

Proof of the Jacobi-Anger identity. – Since $\theta \longmapsto e^{ix\cos\theta}$ is a C^∞ periodic function, it admits an expansion in Fourier series:

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{n=\infty} c_n(x) e^{in\theta}$$

with

$$c_n(x) := \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta} e^{-in\theta} \mathrm{d}\theta.$$

The change of variable $\theta \longrightarrow \pi - \theta$ shows that

$$\begin{aligned} c_n(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta - in\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos\left(x\cos\theta - n\theta\right)\mathrm{d}\theta \qquad \text{if } n \text{ is even} \\ &= \frac{1}{2\pi}\int_0^{2\pi} i\sin\left(x\cos\theta - n\theta\right)\mathrm{d}\theta \qquad \text{if } n \text{ is odd} \end{aligned}$$

Now, by arguments similar to the ones of Lemma 1 Talk II, we obtain $c_n(x) = i^n J_n(x)$. □

Proof of the Fourier series expansion of $\mathbf{n}_k$.– From the Jacobi-Anger identity

$$e^{ix\cos\theta} = \sum_{n=-\infty}^{+\infty} i^n J_n(x) e^{in\theta}$$

we deduce

$$e^{i\alpha_k\cos(2\pi N_k x)} = \sum_{n=-\infty}^{+\infty} i^n J_n(\alpha_k) e^{2i\pi n N_k x} = \sum_{n=-\infty}^{+\infty} u_n(k) e^{2i\pi n N_k x}.$$

Since the Fourier coefficients of a product of two fonctions are given by the discrete convolution product of their coefficients, the product

$$\mathbf{n}_k(x) = e^{i\alpha_k\cos(2\pi N_k x)}\mathbf{n}_{k-1}(x)$$

can be written

$$\begin{aligned} \mathbf{n}_k(x) &= \left(\sum_{n=-\infty}^{+\infty} u_n(k) e^{2i\pi n N_k x}\right)\left(\sum_{p=-\infty}^{+\infty} a_p(k-1) e^{2i\pi p x}\right) \\ &= \sum_{p=-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty} u_n(k) a_{p-nN_k}(k-1)\right) e^{2i\pi p x}. \end{aligned}$$

Therefore

$$a_p(k) = \sum_{n=-\infty}^{+\infty} u_n(k) a_{p-nN_k}(k-1). \qquad \square$$

References

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