Talk II: The h-principle for ample relations

Vincent Borrelli

September 5, 2012

In this document we move from the one to the multi-dimensional h-principle (the usual one). We state the Gromov theorem regarding ample and open differential relation and we give the main ideas of its proof. We then focus on closed differential relations and see, through the example of isometric immersions, how to deal with some of them.

1 Ample differential relations

Let $A \subset \mathbb{R}^n$, recall that we denote by $IntConv(A, a)$ the interior of the convex hull of the component of A to which a belongs. The subset $A \subset \mathbb{R}^n$ is said to be *ample* if for every $a \in A$ we have $IntConv(A, a) = \mathbb{R}^n$. In particular, $A = \emptyset$ is ample.

Let M and N be two manifolds. We denotes by $J^1(M, N)$ the 1-jet space of maps from M to N. This space is a natural fiber bundle over $M \times N$

$$\mathcal{L}(T_xM, T_yN) \longrightarrow J^1(M, N) \xrightarrow{p} M \times N.$$

We denote by J the natural inclusion

$$\begin{array}{rccc} J: & C^1(M, N) & \longrightarrow & J^1(M, N) \\ & f & \longmapsto & j^1 f. \end{array}$$

Ample relations in $J^1(M, N)$. − Locally, we identify $J^1(M, N)$ with

$$J^1(\mathcal{U}, \mathcal{V}) = \mathcal{U} \times \mathcal{V} \times \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) = \mathcal{U} \times \mathcal{V} \times \prod_{i=1}^m \mathbb{R}^n,$$

where $\mathcal{U}$ and $\mathcal{V}$ are charts of M and N. We denote by $(x, y, v_1, ..., v_m)$ an element of $J^1(\mathcal{U}, \mathcal{V})$ and we set:

$$J^1(\mathcal{U}, \mathcal{V})^\perp := \{(x, y, v_1, ..., v_{m-1})\},$$

thus $J^1(\mathcal{U},\mathcal{V}) = J^1(\mathcal{U},\mathcal{V})^\perp \times \mathbb{R}^n$. We denote by $p^\perp$ the projection over the first factor and by $\mathcal{R}_{\mathcal{U},\mathcal{V}} \subset J^1(\mathcal{U},\mathcal{V})$ the image of $\mathcal{R} \subset J^1(M,N)$ by our local identification. In a diagram, we have

$$\begin{array}{ccc} \mathcal{R}_{\mathcal{U},\mathcal{V}} & \longrightarrow & J^1(\mathcal{U},\mathcal{V}) \\ & & \downarrow p^\perp \\ & & J^1(\mathcal{U},\mathcal{V})^\perp. \end{array}$$

Finally, if $z \in J^1(\mathcal{U},\mathcal{V})^\perp$, we set $\mathcal{R}_z^\perp = (p^\perp)^{-1}(z) \cap \mathcal{R}_{\mathcal{U},\mathcal{V}}$. Note that $\mathcal{R}^\perp$ is a differential relation of the bundle $J^1(\mathcal{U},\mathcal{V}) \xrightarrow{p^\perp} J^1(\mathcal{U},\mathcal{V})^\perp$.

Definition. − A differential relation $\mathcal{R} \subset J^1(M,N)$ is *ample* if for every local identification $J^1(\mathcal{U},\mathcal{V})$ and for every $z \in J^1(\mathcal{U},\mathcal{V})^\perp$, the space $\mathcal{R}_z^\perp$ is ample in $(p^\perp)^{-1}(z) \simeq \mathbb{R}^n$.

Remark. − Obviously, this definition does not depend on the chosen chart since we take them all...

Proposition. − *The differential relation $\mathcal{I}$ of immersions from M^m to N^n is ample if $n > m$.*

Proof. − Let us reprent locally $J^1(M,N)$ by $J^1(\mathcal{U},\mathcal{V}) = \mathcal{U} \times \mathcal{V} \times \prod_{i=1}^m \mathbb{R}^n$. We have

$$(x, y, v_1, ..., v_m) \in \mathcal{R}_{\mathcal{U},\mathcal{V}} \iff (v_1, ..., v_m) \text{ est libre dans } \mathbb{R}^n.$$

Let $z = (x, y, v_1, ..., v_{m-1}) \in J^1(\mathcal{U},\mathcal{V})^\perp$.

- If $(v_1, ..., v_{m-1})$ are linearly independent then

$$\begin{array}{rcl} v_m \in (p^\perp)^{-1}(z) \text{ lies inside } \mathcal{R}_{\mathcal{U},\mathcal{V}} & \iff & v_m \notin Vect(v_1, ..., v_{m-1}) =: \Pi \\ & \iff & v_m \in \mathbb{R}^n \setminus \Pi. \end{array}$$

Therefore $\mathcal{R}_z^\perp = \mathcal{R}_{\mathcal{U},\mathcal{V}} \cap (p^\perp)^{-1}(z) = \mathbb{R}^n \setminus \Pi$. Since the codimension of Π is $n - (m-1) \geq 2$, it ensues that $\mathcal{R}_p^\perp$ is ample.

- If $(v_1, ..., v_{m-1})$ are linearly dependent then$\mathcal{R}_p^\perp = \emptyset$ and thus $\mathcal{R}_p^\perp$ is ample. $\square$

2 H-principle for ample relations

Theorem (Gromov 69-73 [2]). – *Let $\mathcal{R} \subset J^1(M,N)$ be an open and ample differential relation. Then $\mathcal{R}$ satisfies the parametric h-principle i. e.*

$$J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$$

is a weak homotopy equivalence.

One immediate consequence.– It ensues from the above proposition and from this theorem that the parametric h-principle holds for the differential relation of immersions of M^m into N^n with $n > m$. A homotopic computation shows that if $M^m = \mathbb{S}^2$ and $N^n = \mathbb{R}^3$ then

$$\pi_0(I(\mathbb{S}^2, \mathbb{R}^3)) = \pi_2(Gl_+(3, \mathbb{R})) = 0.$$

Thus there is only one class of immersions of the sphere inside the three dimensional space and in particular, the sphere can be everted among immersions (Smale's paradox, [6]).

Guidelines of the proof.– We first work locally over a cubic chart $C = [0,1]^m$ of M and an open $\mathcal{V} \approx \mathbb{R}^n$ of N. A section $\sigma \in \mathcal{R}_{C,\mathbb{R}^n} \subset J^1(C, \mathbb{R}^n)$ has the following expression

$$\sigma : c \longmapsto (c, f_0(c), v_1(c), ..., v_m(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

Let us denote by $p^{\perp m}$ the projection

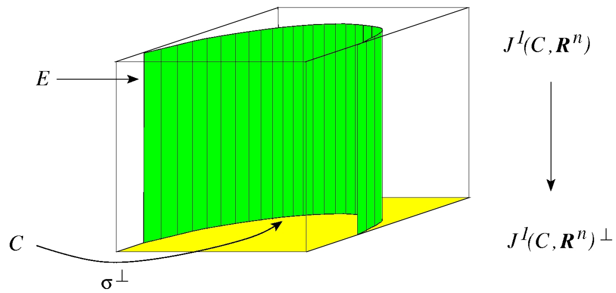
$$(c, y, v_1, ..., v_m) \longmapsto (c, y, v_1, ..., v_{m-1})$$

then $\mathcal{R}_z^{\perp m} = \mathcal{R}_{C,\mathbb{R}^n} \cap (p^{\perp m})^{-1}(z)$ for every $z = (b, y, v_1, ..., v_{m-1}) \in J^1(C, \mathbb{R}^n)^{\perp m}$.
We set

$$\begin{array}{rccc} \sigma^{\perp m} : & C & \longrightarrow & J^1(C, \mathbb{R}^n)^{\perp m} \\ & c & \longmapsto & (c, f_0(c), v_1(c), ..., v_{m-1}(c)) \end{array}$$

and we denote by E the pull-back bundle $(p^{\perp m}, J^1(C, \mathbb{R}^n), J^1(C, \mathbb{R}^n)^{\perp m})$:

$$\begin{array}{rcc} E & \longrightarrow & J^1(C, \mathbb{R}^n) \\ \pi \downarrow & & \downarrow p^{\perp m} \\ C & \xrightarrow{\sigma^{\perp m}} & J^1(C, \mathbb{R}^n)^{\perp m} \end{array}$$



The pull-back bundle E

Let $\mathcal{S}^m \subset E$ be the pull-back of the relation $\mathcal{R}^{\perp_m}$. The relation $\mathcal{S}^m$ is obviously open and ample. Note also that $v_m : C \longrightarrow \mathbb{R}^n$ provides a section of $\mathcal{S}^m$ over C. We now use the C^∞-parametric Fundamental Lemma with $C := [0,1]^m$ as parameter space and with $\mathcal{S}^m$ as differential relation. There exists $h : C \times [0,1] \xrightarrow{C^\infty} \mathcal{S}^m$ such that

$$h(.,0) = h(.,1) = v_m \in \Gamma^\infty(\mathcal{S}^m), \quad \forall c \in C,\ h(c,.) \in \Omega^{AR}_{v_m(c)}(\mathcal{S}^m_c)$$

and

$$\forall c \in C,\ \int_0^1 h(c,s)ds = \frac{\partial f_0}{\partial c_m}(c).$$

We set

$$F_1(c) := f_0(c_1, ..., c_{m-1}, 0) + \int_0^{c_m} h(c_1, ..., c_{m-1}, s, N_1 s)ds.$$

We then have

$$\|F_1 - f_0\| = O(\frac{1}{N_1})$$

and even more,

$$\|F_1 - f_0\|_{C^1,\widehat{m}} = O(\frac{1}{N_1})$$

where

$$\|f\|_{C^1,\widehat{m}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, ..., \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0})$$

is the C^1 norm without the $\|\frac{\partial f}{\partial c_m}\|_{C^0}$ term. This last subtlety will help us at the next step. By the very definition of $\mathcal{S}^m$, the section

$$c \mapsto (c, f_0(c), v_1(c), ..., v_{m-1}(c), \frac{\partial F_1}{\partial c_m}(c))$$

lies inside the relation $\mathcal{R}_{C,\mathbb{R}^n}$. Since $\mathcal{R}_{C,\mathbb{R}^n}$ is open and F_1 is C^0-close to f_0, even if it means to increase N_1, we can assume that

$$c \mapsto (c, F_1(c), v_1(c), ..., v_{m-1}(c), \frac{\partial F_1}{\partial c_m}(c))$$

is a section of $\mathcal{R}_{C,\mathbb{R}^n}$. We then repeat the same process with respect to the next to last variable to obtain

$$c \mapsto (c, F_1(c), v_1(c), ..., v_{m-2}(c), \frac{\partial F_2}{\partial c_{m-1}}(c), \frac{\partial F_1}{\partial c_m}(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

Noticing that $\mathcal{R}_{C,\mathbb{R}^n}$ is open and that F_2 and F_1 are $(C^1, \widehat{c_{m-1}})$-close, we can assume that:

$$c \mapsto (c, F_2(c), v_1(c), ..., v_{m-2}(c), \frac{\partial F_2}{\partial c_{m-1}}(c), \frac{\partial F_2}{\partial c_m}(c)) \in \mathcal{R}_{C,\mathbb{R}^n},$$

and so on until the section is completly holonomic, that is, until a solution $F := F_m$ over C and C^0-close to f_0 is obtained:

$$\|F - f_0\|_{C^0} = O(\frac{1}{N_1} + ... + \frac{1}{N_m}).$$

In order to build a solution globally defined over M^m, we first perform a cubic decomposition of the manifold and we then recursively apply the preceding process over every cube. Of course the real problem is the one of the sticking of the solutions together. Precisely if C is an open cube, K a compact subset of C and f_0 a solution over an open neighborhood $Op(K)$ of K, the point is to construct a solution f extending f_0 on C. To achieve this goal, we need to modify every convex integrations $F_1, ..., F_m$. Let $\lambda_1 : C \longrightarrow [0, 1]$ be a compactly supported C^∞ function such that

$$\lambda_1(c) = \begin{cases} 1 & \text{if } c \in Op_1(K) \subset Op(K) \\ 0 & \text{if } c \in C \setminus Op(K). \end{cases}$$

where $Op_1(K) \subset Op(K)$ is an open neighborhood of K. Let F_1 be the preceding solution over C obtained from

$$\sigma : c \longmapsto (c, f_0(c), v_1(c), ..., v_m(c)) \in \mathcal{R}_{C,\mathbb{R}^n}.$$

We set

$$f_1 := F_1 + \lambda_1(f_0 - F_1).$$

Let $j \in \{1, ..., m\}$, we have

$$\frac{\partial f_1}{\partial c_j} = \frac{\partial F_1}{\partial c_j} + \lambda_1 . \left(\frac{\partial f_0}{\partial c_j} - \frac{\partial F_1}{\partial c_j} \right) + \frac{\partial \lambda_1}{\partial c_j} . (f_0 - F_1).$$

Since λ_1 is compactly supported, the $\dfrac{\partial \lambda_1}{\partial c_j}$ is bounded for every $j \in \{1, ..., m\}$. In the one hand, since F_1 and f_0 are $(C^1, \widehat{m})$-close, it ensues that for every $j \in \{1, ..., m-1\}$, we have

$$\|f_1 - F_1\|_{C^1, \widehat{m}} = O(\frac{1}{N_1}).$$

In the other hand, regarding the

$$\frac{\partial f_1}{\partial c_m} - \frac{\partial F_1}{\partial c_m}$$

term, there is no reason why it could be small in general. But it is the relevant term if we want

$$c \longmapsto \left(c, \frac{\partial f_1}{\partial c_m}(c) \right)$$

to be a solution of $\mathcal{S}^m$. Indeed, since $\mathcal{S}^m$ is open and $c \longmapsto \dfrac{\partial F_1}{\partial c_m}(c)$ is a solution of $\mathcal{S}^m$, it would be enough to have $\dfrac{\partial f_1}{\partial c_m}$ and $\dfrac{\partial F_1}{\partial c_m}$ C^0-close together to conclude. The smallness of

$$\left\| \frac{\partial f_1}{\partial c_m} - \frac{\partial F_1}{\partial c_m} \right\|_{C^0}$$

relies on the one of

$$\left\| \frac{\partial f_0}{\partial c_m} - \frac{\partial F_1}{\partial c_m} \right\|$$

over $Op(K)$. It turns out that we can always choose the family of loops h globally with the extra constraint that, over $Op(K)$, it is equal to the family of constant loops $\dfrac{\partial f_0}{\partial c_m}$ *i. e.*

$$\forall c \in Op(K),\ h(c, s) = \frac{\partial f_0}{\partial c_m}(c).$$

Since

$$\begin{aligned}\frac{\partial F_1}{\partial c_m}(c) &= h(c_1, ..., c_{m-1}, c_m, N_1 c_m) \\ &= \frac{\partial f_0}{\partial c_m}(c)\end{aligned}$$

the difference $\dfrac{\partial f_0}{\partial c_m} - \dfrac{\partial F_1}{\partial c_m}$ vanishes over $Op(K)$ and thus $\left\| \dfrac{\partial f_1}{\partial c_m} - \dfrac{\partial F_1}{\partial c_m} \right\|_{C^0}$ is small. For more details see [7] p. 51-60. Note that there is no difficulty to move from a h-principle to a parametric h-principle. □

Theorem (Gromov [2], [3]). – *Let $\mathcal{R} \subset J^1(M, N)$ be open and ample, then $\mathcal{R}$ satisfies to the C^0-dense h-principle .*

This means that if P is a compact manifold seen as a parameter space and if $\sigma : P \longrightarrow \Gamma(\mathcal{R})$, then for every $\epsilon > 0$ there exists a homotopy $\sigma_u : [0, 1] \times P \longrightarrow \Gamma(\mathcal{R})$ such that $\sigma_0 = \sigma$ and

$$\begin{array}{rccl}\sigma_1 : & P & \longrightarrow & J(\mathcal{S}ol(\mathcal{R})) \subset \Gamma(\mathcal{R}) \\ & p & \longmapsto & j^1 f_p.\end{array}$$

Moreover $\max_{p\in P} \|g_p - f_p\|_{C^0} < \epsilon$, where $g_p = bs(\sigma) : P \longrightarrow C^\infty(M, N)$.

3 H-principle for closed relations

I consider here the only example with which I am familiar: the closed differential relation of isometric immersion. Nevertheless, the study of this example gives some idea of what is needed for solving more general closed relations with a convex integration process. The two key points are the following: we have to own a subsolution of the differential relation (here a strictly short immersion) together with a control of the C^1 norm of the maps resulting from the convex integration process. Let us see this more closely with the celebrated C^1 embedding theorem of Nash and Kuiper.

Definition.– Let $f : (M^m, g) \longrightarrow \mathbb{E}^q$ be an embedding. If $g > f^*\langle\cdot,\cdot\rangle_{E^q}$, *i. e.* $\Delta := g - f^*\langle\cdot,\cdot\rangle_{E^q}$ is a metric, then f is said to be *strictly short*. If $g = f^*\langle\cdot,\cdot\rangle_{E^q}$, the embedding f is said to be *isometric*.

Theorem (Nash 54 [5], Kuiper 55 [4]).– *Let $f_0 : (M^m, g) \longrightarrow \mathbb{E}^q$ $(q > n)$ be a strictly short embedding then there exists a C^1 isometric em-*

bedding $f:(M^m,g)\longrightarrow \mathbb{E}^q$ which is C^0 close to f_0.

A rough sketch of the proof.– To simplify the presentation, we assume that M^m is compact. Let us denotes by $\mathcal{R}$ the isometric differential relation and by $(\delta_k)_k$ an increasing sequence of positive numbers converging toward 1. We set:

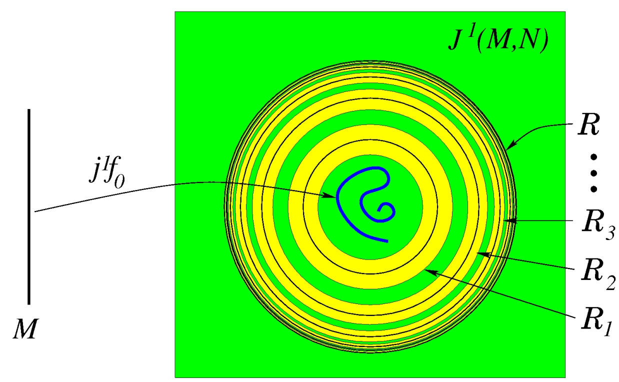
$$g_k := f_0^*\langle\cdot,\cdot\rangle_{E^q} + \delta_k\Delta.$$

We have $(g_k)_k \uparrow g$. We then define a sequence of differential relation $(\mathcal{R}_k)_k$ by the following inequations

$$g_k - \epsilon_k\Delta < f_0^*\langle\cdot,\cdot\rangle_{E^q} < g_k + \epsilon_k\Delta$$

with $\epsilon_k = \frac{\delta_{k+1}-\delta_k}{3}$. The sequence $(\mathcal{R}_k)_k$ is converging toward $\mathcal{R}$ (for the Hausdorff distance) and the $\mathcal{R}_k$s are pairewise disjoint.

The embedding f_0 is strictly short for g_1. The differential relation $\mathcal{R}_1$ is not ample (as all of the $\mathcal{R}_k$s) but, thanks to the fact that f_0 is strictly short for g_1 this obstacle can be circumvented. This will be explained in the next document. In some sense, the shortness hypothesis ensures that the 1-jet de f_0 lies inside some iterated convex hull extension of $\mathcal{R}_1$.



The 1-jet space $J^1(M,N)$ (green), the differential relations $\mathcal{R}_k$ (yellow) and the image of j^1f_0 (blue).

By applying the Gromov machinery to f_0, we then obtain a new embedding f_1 such that:

1) f_1 is a solution of $\mathcal{R}_1$ i. e. $f_1^*\langle\cdot,\cdot\rangle_{E^q}\approx g_1$

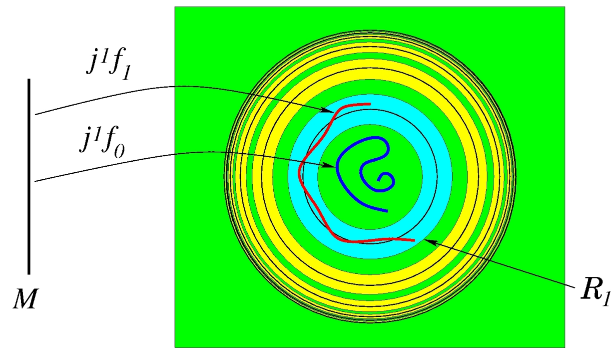
2) $\|f_1-f_0\|_{C^0} = O(\frac{1}{N_1})$

where $N_1\in\mathbb{N}^*$ is a free parameter in the convex integration formula defining

f_1. By an appropriate choice of the family of loops h appearing in that formula, it is possible to achieve the following control of the C^1 norm of f_1

3) $\|f_1 - f_0\|_{C^1} \leq C\sqrt{\delta_1}$.

where C is a universal constant (independent of N_1, f_0 and $(\delta_k)_{k\in\mathbb{N}^*}$).



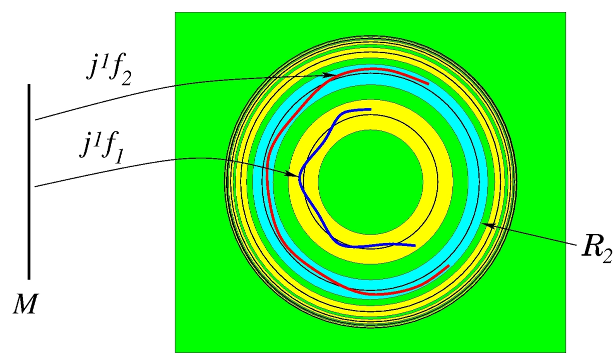
The map f_1 is a solution of $\mathcal{R}_1$. The image of its 1-jet j^1f_1 (red) lies inside $\mathcal{R}_1$ (blue).

From 1) we deduce that f_1 is strictly short for g_2. We thus apply once again the integration convex machinery to build a new map f_2 such that :

1) f_2 is a solution of $\mathcal{R}_2$ *i. e.* $f_2^*\langle\cdot,\cdot\rangle_{E^q} \approx g_2$

2) $\|f_2 - f_1\|_{C^0} = O(\frac{1}{N_2})$

3) $\|f_2 - f_1\|_{C^1} \leq C\sqrt{\delta_2 - \delta_1}$.



The Gromov machinery is applied iteratively producing a sequence of maps f_1, f_2, etc.

This builds a sequence of maps $(f_k)_k$ which is C^0 converging if the growth of the $(N_k)_k$ is fast enough, and which is C^1 converging if

$$\sum_k \sqrt{\delta_{k+1} - \delta_k} < +\infty.$$

Raising the $(N_k)_k$s costs nothing, and the choice of the sequence $(\delta_k)_k$ is a free ingredient of the proof, we thus can assume that the sequence $(f_k)_k$ is C^1 converging toward a map f which is bound to be a solution of $\mathcal{R}$, that is, a C^1 isometric immersion.

Much more details will be given in the next document. The following references should be mentionned: [5], [4], [1] p. 189-197, [3] p. 201-207, [7] p. 194-199. $\square$

References

[1] Y. Eliahsberg et N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, A. M. S., Providence, 2002.

[2] M. Gromov, *Convex integration of differential relations I*, Izv. Akad. Nauk SSSR 37 (1973), 329-343.

[3] M. Gromov, *Partial Differential Relations*, Springer-Verlag, 1986.

[4] N. Kuiper, *On C^1-isometric imbeddings I, II*, Indag. Math. 17 (1955), 545-556, 683-689.

[5] F. Nash, *C^1-isometric imbeddings*, Ann. Math. 63 (1954), 384-396.

[6] S. Smale, *A classification of immersions of the two-sphere*, Trans. of the A. M. S. 90 (1958), 281?290.

[7] D. Spring, *Convex Integration Theory*, Monographs in Mathematics, Vol. 92, Birkhäuser Verlag, 1998.