Talk III: The h-principle for isometric embeddings

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1 From the Convex Integration Theory to the Nash-Kuiper Theorem

The goal of this text is to recover the Nash-Kuiper result on C^1 isometric embeddings from the machinery of the Gromov Integration Theory.

Theorem (Nash 54, Kuiper 55).– *Let M^n be a compact[1] Riemannian manifold and $f_0 : (M^n, g) \xrightarrow{C^1} \mathbb{E}^q$ be a strictly short embedding (i.e $\Delta := g - f_0^*\langle\cdot,\cdot\rangle_{E^q}$ is a Riemannian metric). Then, for every $\epsilon > 0$, there exists a C^1 isometric embedding $f : (M^n, g) \longrightarrow \mathbb{E}^q$ such that $\|f - f_0\|_{C^0} \leq \epsilon$.*

Nevertheless, there are three major obstacles to apply the Gromov Theorem for Ample relations here. First, the isometric relation is closed, second, it is not ample, third the convex integration process produces non-injective maps in general. We have seen previously that the first obstacle can be circumvented by iteratively applying the Gromov Theorem. But to deal with the two other obtacles, we will have to *adapt* the Gromov machinery. Let us see why.

Assume, for a practical presentation, that our manifold M is a Riemannian square $([0,1]^2, g)$ where g is any metric and that $q = 3$. Our goal is to produce a map $f : ([0,1]^2, g) \longrightarrow \mathbb{E}^3$ which is ϵ-isometric for a given $\epsilon > 0$. Thus, our 1-jet space is

$$J^1([0,1]^2, \mathbb{E}^3) = [0,1]^2 \times \mathbb{E}^3 \times (\mathbb{E}^3)^2$$

and our (open) differential relation $\mathcal{R}_\epsilon$ is

$$\mathcal{R}_\epsilon = \{(c, y, v_1, v_2) \mid g_{ij}(c) - \epsilon \leq \langle v_i, v_j\rangle \leq g_{ij}(c) + \epsilon, \;\; 1 \leq i, j \leq 2\}.$$

[1] This compactness hypothesis is not essential but it will help simplifying the exposition.

Let $f_0 : ([0,1]^2, g) \longrightarrow \mathbb{E}^3$ be a strictly short map. To apply the Gromov machinery for ample relations we need to extend it to a section σ of our differential relation $\mathcal{R}_\epsilon$

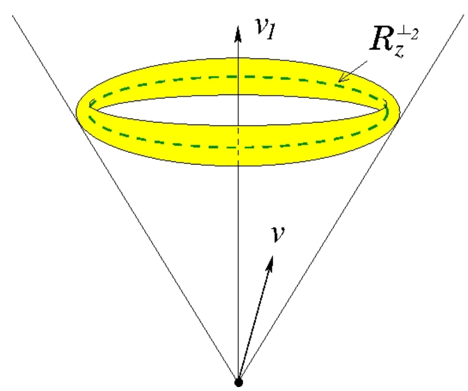
$$\begin{array}{rccc} \sigma : & [0,1]^2 & \longrightarrow & \mathcal{R}_\epsilon \subset J^1([0,1]^2, \mathbb{E}^3) \\ & c = (c_1, c_2) & \longmapsto & (c, f_0(c), v_1(c), v_2(c)). \end{array}$$

Since the topology of the base manifold is trivial, finding such a section is easy. In fact, there is a considerable latitude for the choice of (v_1, v_2) since the constraints are underdetermined : $v_1(c)$ and $v_2(c)$ must be of length approximatively $\sqrt{g_{11}(c)}$ and $\sqrt{g_{22}(c)}$ and the angle between then is approximatively $\alpha = \arccos\left(\frac{g_{12}(c)}{\sqrt{g_{11}(c)}\sqrt{g_{22}(c)}}\right)$.

The first step of the machinery is to perform a convex integration in the direction of the c_2 variable. We denote by $p^{\perp_2}$ the projection

$$(c, y, v_1, v_2) \longmapsto (c, y, v_1)$$

and we set $\mathcal{R}_z^{\perp_2} = \mathcal{R}_\epsilon \cap (p^{\perp_2})^{-1}(z)$ for every $z = (c, y, v_1)$. The space $\mathcal{R}_z^{\perp_2}$ is a thickening of a circle.



The space $\mathcal{R}_z^{\perp_2}$ in $\mathbb{E}^3$. The angle of the cone with basis $\mathcal{R}_z^{\perp_2}$ is approximatively α. We have denoted v for $\frac{\partial f_0}{\partial c_2}(c)$.

Even if f_0 is strictly short, there is no reason why the vector $\dfrac{\partial f_0}{\partial c_2}(c)$ should be in the convex hull of $\mathcal{R}_{z(c)}^{\perp_2}$ with $z(c) = (c, f_0(c), v_1(c))$, $c \in [0,1]^2$. Note also that the natural choice

$$v_1(c) := \sqrt{g_{11}(c)} \frac{\frac{\partial f_0}{\partial c_1}(c)}{\|\frac{\partial f_0}{\partial c_1}(c)\|}$$

is of no help since $\dfrac{\partial f_0}{\partial c_2}(c)$ will then lies in the convex hull of the cone with basis $\mathcal{R}^{\perp_2}_{z(c)}$ but not in the convex hull of $\mathcal{R}^{\perp_2}_{z(c)}$ in general. In short, a direct application of the Gromov Theorem for Ample Relations fails.

2 A strategy to solve the relation of isometric maps

Recall that we already have found a strategy to solve the closed relation of isometric maps $\mathcal{R}$ by iteratively solving a sequence $(\widetilde{\mathcal{R}}_k)_{k\in\mathbb{N}^*}$ of open differential relations converging toward $\mathcal{R}$.

Let $\Delta := g - f_0^*\langle\cdot,\cdot\rangle_{\mathbb{E}^q}$ and $(\delta_k)_{k\in\mathbb{N}^*}$ be a strictly increasing sequence of positive numbers converging toward 1. We set

$$g_k := f_0^*\langle\cdot,\cdot\rangle_{E^q} + \delta_k\Delta.$$

Obviously $(g_k)_{k\in\mathbb{N}^*} \uparrow g$. The relation $\mathcal{R}_k$ is defined to be the relation of g_k-isometric maps and $\widetilde{\mathcal{R}}_k$ is a thickning of $\mathcal{R}_k$. The strategy is to start with the strictly short map f_0, then to solve $\widetilde{\mathcal{R}}_1$ to get a new map f_1 which is strictly short for $\widetilde{\mathcal{R}}_2$, then to solve $\widetilde{\mathcal{R}}_2$, etc. Let f_{k-1} be a strictly short embedding for g_{k-1}, the fundamental step is thus to build a new map f_k such that

1) f_k is a solution of $\widetilde{\mathcal{R}}_k$,
2) f_k is C^0-close to f_{k-1},
3) $\|f_k - f_{k-1}\|_{C^1}$ is under control,
4) f_k is an embedding.

Note that since the sequence of metrics $(g_k)_{k\in\mathbb{N}^*}$ is strictly increasing, all the $\mathcal{R}_k$ are disjoint. Thus, provided the thickning $\widetilde{\mathcal{R}}_k$ is small enough, the map f_k will be short for g_{k+1}.

In what follows we describe how to buid such a map f_k from f_{k-1} since, as we have just seen, we can not apply directly the Gromov Theorem for Ample Relations.

3 How to adapt the Gromov machinery

For simplicity, we first assume that M^n is the cube $[0,1]^n$. The metric distorsion induced by f_{k-1} is measured by a field of bilinear forms obtained as the difference

$$\Delta_k := g_k - f_{k-1}^*\langle .,.\rangle_{\mathbb{E}^q}.$$

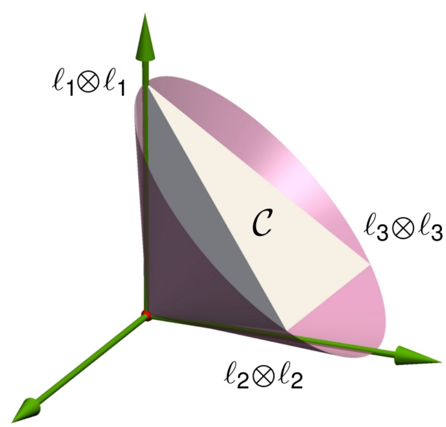
This difference is a metric since f_{k-1} is strictly short, thus the image of the map

$$\Delta_k : M^n \longrightarrow \mathcal{M} \subset (\mathbb{E}^n \otimes \mathbb{E}^n)^*$$

lies inside the positive cone $\mathcal{M}$ of inner products of $\mathbb{E}^n$. There exist $S_k \geq \frac{n(n+1)}{2}$ linear forms $\ell_{k,1}, \ldots \ell_{k,S_k}$ of $\mathbb{E}^n$ such that

$$g_k - f_{k-1}^*\langle .,.\rangle_{\mathbb{E}^q} = \sum_{j=1}^{S_k} \rho_{k,j}\ell_{k,j} \otimes \ell_{k,j}$$

where the coefficients $\rho_{k,j}$, $j \in \{1, ..., S_k\}$ are positive functions. Indeed, there exist (constant) inner products I_1, ..., I_{L_k} on $\mathbb{R}^n$ such that the image of Δ_k lies inside the positive cone generated by the constant bilinear forms I_j. Each inner product I_j is a sum of primitive forms $\ell_{i,j} \otimes \ell_{i,j}$, hence the desired decomposition.



In that illustration the space of symmetric bilinear forms of $\mathbb{R}^3$ is identified with $\mathbb{R}^3$ via the basis $(e_1^* \otimes e_2^* + e_2^* \otimes e_1^*, e_2^* \otimes e_2^*, e_1^* \otimes e_1^*)$. A cone $\mathcal{C}$ (grey-white) spanned by three bilinear forms $\ell_i \otimes \ell_i$, $i \in \{1, 2, 3\}$, is pictured inside the cone of inner products $\mathcal{M}$ (purple).

The way to adapt the Gromov machinery to the isometric relation is to apply the successive convex integrations not along the n directions of the coordinates in $[0, 1]^n$ but rather along the S_k directions corresponding to the S_k linear forms $\ell_1, ..., \ell_{S_k}$. This will produce S_k intermediary maps

$$f_{k,1}, ..., f_{k,S_k}$$

such that

$$\begin{array}{rcr} g_k - f^*_{k,1}\langle .,.\rangle_{\mathbb{E}^q} & \approx & \rho_{k,2}\ell_{k,2}\otimes\ell_{k,2}+\rho_{k,3}\ell_{k,3}\otimes\ell_{k,3}+...+\rho_{k,S_k}\ell_{k,S_k}\otimes\ell_{k,S_k} \\ g_k - f^*_{k,2}\langle .,.\rangle_{\mathbb{E}^q} & \approx & \rho_{k,3}\ell_{k,3}\otimes\ell_{k,3}+...+\rho_{k,S_k}\ell_{k,S_k}\otimes\ell_{k,S_k} \\ \vdots & & \vdots \\ g_k - f^*_{k,S_k-1}\langle .,.\rangle_{\mathbb{E}^q} & \approx & \rho_{k,S_k}\ell_{k,S_k}\otimes\ell_{k,S_k} \\ g_k - f^*_{k,S_k}\langle .,.\rangle_{\mathbb{E}^q} & \approx & 0. \end{array}$$

In orther words, the isometric default Δ_k is going to be reduced step by step by a succession of convex integrations that detroy the coefficients in the decomposition one by one. The map $f_k := f_{k,S_k}$ is then a solution of $\widetilde{\mathcal{R}}_k$.

Note that

$$\begin{array}{rcl} f^*_{k,j}\langle .,.\rangle_{\mathbb{E}^q} - f^*_{k,j-1}\langle .,.\rangle_{\mathbb{E}^q} & = & (g_k - f^*_{k,j-1}\langle .,.\rangle_{\mathbb{E}^q}) - (g_k - f^*_{k,j}\langle .,.\rangle_{\mathbb{E}^q}) \\ & \approx & \rho_{k,j}\ell_{k,j}\otimes\ell_{k,j}. \end{array}$$

Hence, in that new approach, the fundamental problem is the following :

Fundamental problem.– *Given a positive function ρ, a linear form $\ell \neq 0$ and an embedding f_0 how to build an other embedding f such that*

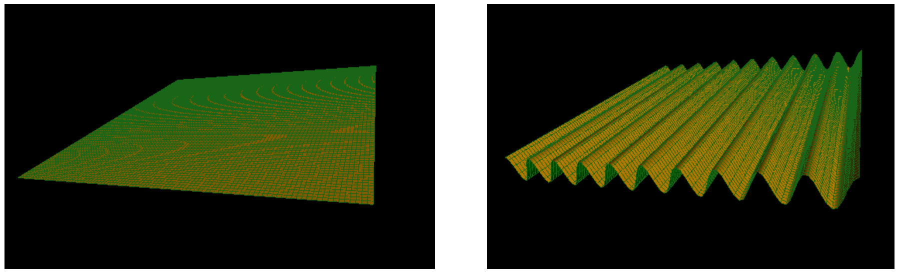
$$f^*\langle .,.\rangle_{\mathbb{E}^q} \approx \mu$$

where $\mu := f_0^*\langle .,.\rangle_{\mathbb{E}^q} + \rho\,\ell\otimes\ell$?

We are going to solve this problem thanks to a convex integration process described below.

4 The one dimensional case

In the above fundamental problem, the embedding f_0 is isometric in the directions lying inside $\ker\ell$ and short in the directions transverse to $\ker\ell$. We redress this defect by elongating f_0 in a direction transverse to $\ker\ell$. The elongation is generated by a normal deformation which gives to the image submanifold a corrugated shape.



A plane and a corrugated plane.

One difficulty in the construction of f rests in the choice of a good transversal direction to perform the convex integration process. This difficulty obviously vanishes in the case $n=1$, that is why we begin by considering this case first.

One dimensional fundamental problem.– Let $f_0 : [0,1] \longrightarrow \mathbb{E}^q$ be an embedding, ρ a positive function, $\ell \neq 0$ a linear form on $\mathbb{R}$, how to build an other embedding f such that

$$\forall c \in [0,1], \quad \|f'(c)\|^2 \approx \|f_0'(c)\|^2 + \rho(c)\ell^2(\partial_c) \quad ?$$

Of course that one dimensional problem is trivial, even if the approximation symbol ($\approx$) is replaced by a true equality ($=$). But the interest lies elsewhere: the convex integration theory offers a way to solve that problem that can be generalized to any dimension.

Note that the differential relation $\mathcal{S}$ of that problem depends on the point $c \in [0,1]$, precisely

$$\mathcal{S} = \{(c, y, v_1) \mid \|v_1\| = r(c)\} \subset J^1([0,1], \mathbb{R}^q)$$

where $r(c) := \sqrt{\|f_0'(c)\|^2 + \rho(c)\ell^2(\partial_c)}$. We thus have to find a family of loops $(h_c)_{c \in [0,1]}$:

$$h_c : [0,1] \longrightarrow \mathbb{S}^{q-1}(r(c)) \subset \mathbb{E}^q$$

such that

$$f_0'(c) = \int_0^1 h_c(s)\mathrm{d}s.$$

Let $\mathbf{n} : [0,1] \longrightarrow \mathbb{E}^q$ a unit normal to the curve f_0. We set

$$h_c(s) := r(c)\mathbf{e}^{i\alpha(c)\cos 2\pi s}$$

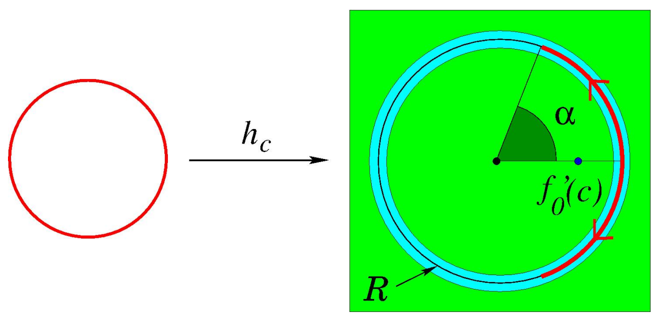
where $\mathbf{e}^{i\theta} := \cos\theta\,\mathbf{t} + \sin\theta\,\mathbf{n}$ and $\mathbf{t} := \frac{f_0'}{\|f_0'\|}$. It is easily checked that

$$\int_0^1 r(c)\mathbf{e}^{i\alpha(c)\cos 2\pi s}\mathrm{d}s = r(c)J_0(\alpha(c))\,\mathbf{t}(c)$$

where J_0 is the Bessel function of order 0. We thus have to choose

$$\alpha(c) := J_0^{-1}\left(\frac{\|f_0'(c)\|}{r(c)}\right)$$

(recall that J_0 is invertible on $[0,\kappa]$ where $\kappa \approx 2.4$ is the smallest positive root of J_0).



The loop h_c.

We now define f by the following **one dimensional convex integration formula**:

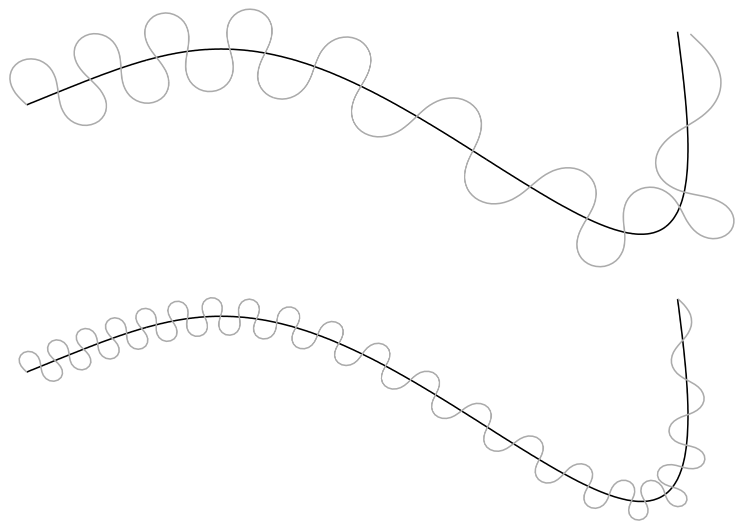
$$f(c) := f_0(0) + \int_0^c r(u)\mathbf{e}^{i\alpha(u)\cos 2\pi N u}\,\mathrm{d}u.$$

with $N \in \mathbb{N}^*$.

Observation.– We call N the *number of corrugations* of the convex integration formula.

Lemma.– *The map f solves the one dimensional fundamental problem. Its speed $\|f'\|$ is equal to the given function $r = (\|f_0'\|^2 + \rho\ell^2(\partial_c))^{\frac{1}{2}}$. Moreover $\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$ and if N is large enough f is an embedding.*

Proof.– The relation $\|f'\|^2 = \|f_0'\|^2 + \rho\ell^2(\partial_c)$ ensues from the very definition of f. If N is large enough, the image of f lies inside a small tubular neighborhood of f_0. Since f is a normal deformation of f_0, it is embedded. $\square$.



A short curve f_0 (black) and the curve f obtained with the one dimensional convex integration formula (grey, $N = 9$ and $N = 20$).

5 A Kuiper-like convex integration process

5.1 A first attempt

We now come back to the n-dimensional case and we assume for simplicity that $\ker\ \ell = Span(e_2, ..., e_n)$ and $\ell(e_1) = 1$ where $(e_1, ..., e_n)$ is the standard basis of $[0,1]^n$. The previous convex integration formula can be easily generalized to the n-dimensional case by setting:

$$f(s,c) := f_0(0,c) + \int_0^s r(u,c)\mathbf{e}^{i\alpha(u,c)\cos 2\pi N u}\, \mathrm{d}u$$

with $s \in [0,1]$, $c = (c_2, ..., c_n) \in [0,1]^{n-1}$, $N \in \mathbb{N}^*$, $r = \sqrt{\mu(e_1, e_1)} = \sqrt{\|f_0'\|^2 + \rho}$, $\alpha = J_0^{-1}\left(\frac{\|df_0(e_1)\|}{r}\right)$, $\mathbf{e}^{i\theta} = \cos\theta\, \mathbf{t} + \sin\theta\, \mathbf{n}$, $\mathbf{t} = \frac{df_0(e_1)}{\|df_0(e_1)\|}$ and $\mathbf{n}$ is any unit normal to f_0.

But surprisingly the resulting map f **does not solve the fundamental problem**. Let us see why. The isometric relation

$$f^*\langle .,. \rangle_{\mathbb{E}^q} = f_0^*\langle .,. \rangle_{\mathbb{E}^q} + \rho\, \ell \otimes \ell$$

is equivalent to the following system of equations:

$$\begin{array}{lcll}\langle df(e_1), df(e_1)\rangle_{\mathbb{E}^q} & = & \langle df_0(e_1), df_0(e_1)\rangle_{\mathbb{E}^q} + \rho & \\ \langle df(e_1), df(e_j)\rangle_{\mathbb{E}^q} & = & \langle df_0(e_1), df_0(e_j)\rangle_{\mathbb{E}^q} & \text{if } j \neq 1 \\ \langle df(e_i), df(e_j)\rangle_{\mathbb{E}^q} & = & \langle df_0(e_i), df_0(e_j)\rangle_{\mathbb{E}^q} & \text{with } i > 1 \text{ and } j > 1.\end{array}$$

In the one hand we have

$$\frac{\partial f}{\partial s}(s,c) = r(s,c)\mathbf{e}^{i\alpha(s,c)\cos 2\pi Ns}$$

thus $\|df(e_1)\|^2 = r^2(s,c)$ and the first equation is fulfilled. In the other hand, the $C^{1,\hat{1}}$ closeness of f to f_0:

$$\|f - f_0\|_{C^{1,\hat{1}}} = O\left(\frac{1}{N}\right)$$

implies that $\|df(e_j) - df_0(e_j)\| = O\left(\frac{1}{N}\right)$ for every $j \neq 1$. In particular

$$\langle df(e_i), df(e_j)\rangle_{\mathbb{E}^q} = \langle df_0(e_i), df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right)$$

for every $i > 1$, $j > 1$. The problem arises with the mixted term $\langle df(e_1), df(e_j)\rangle_{\mathbb{E}^q}$, $j > 1$. Indeed

$$\begin{array}{lcl}\langle df(e_1), df(e_j)\rangle_{\mathbb{E}^q} & = & \langle df(e_1), df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right) \\ & = & \langle r(s,c)\mathbf{e}^{i\alpha(s,c)\cos 2\pi Ns}, df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right) \\ & = & \langle r(s,c)\cos(\alpha(s,c)\cos 2\pi Ns)\mathbf{t}, df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right) \\ & = & \frac{r(s,c)}{\|df_0(e_1)\|}\cos(\alpha(s,c)\cos 2\pi Ns)\langle df_0(e_1), df_0(e_j)\rangle_{\mathbb{E}^q} + O\left(\frac{1}{N}\right)\end{array}$$

Thus, unless $\mu(e_1, e_j) = \langle df_0(e_1), df_0(e_j)\rangle_{\mathbb{E}^q}$ is null, there is no reason why $\langle df(e_1), df(e_j)\rangle_{\mathbb{E}^q}$ should be equal to $\langle df_0(e_1), df_0(e_j)\rangle_{\mathbb{E}^q}$.

5.2 Adjusting the convex integration formula

To correct this default we need to adjust our convex integration formula: rather than performing the normal deformation along straight lines we are going to follow the integral lines of some well chosen vector field. Let

$$W(s,c) := e_1 + \sum_{j=2}^{n} \zeta_j(s,c)e_j$$

be a vector field μ-orthogonal to ker ℓ, that is $\mu(W, e_j) = 0$ for $j \in \{2, ..., n\}$. Let $s \mapsto \varphi(s, c)$ be the integral curve of W issuing from $(0, c)$ that is

$$\frac{\partial \varphi}{\partial s}(s, c) = W(\varphi(s, c)) \quad \text{and} \quad \varphi(0, c) = (0, c).$$

We now define f by the following **convex integration formula**:

$$f(\varphi(s, c)) := f_0(c) + \int_0^s r(\varphi(u, c))\mathbf{e}^{i\theta(\varphi(u,c)),u)}\, \mathrm{d}u$$

with $N \in \mathbb{N}^*$, $\theta(q, u) := \alpha(q)\cos(2\pi N u)$, $\mathbf{t} = \frac{df_0(W)}{\|df_0(W)\|}$, $\mathbf{n}$ is any unit normal to f_0 and $c = (c_2, ..., c_n)$. By differentiating this formula with respect to s we get

$$\|df(W)\|^2_{\mathbb{E}^q} = r^2$$

hence we must choose $r = \sqrt{\mu(W, W)}$ and $\alpha = J_0^{-1}\left(\frac{\|df_0(W)\|}{r}\right)$. Of course, these expressions should be considered at the point $\varphi(u, c)$ (or $\varphi(s, c)$). The above formula defines f over $[0, 1]^n$ as long as

$$\varphi : \quad [0, 1] \times [0, 1]^{n-1} \quad \longrightarrow \quad [0, 1]^n$$

is a diffeomorphism. We will ignore this technicality here and will assume that φ is indeed a diffeomorphism.

Proposition.– *The map f solves the fundamental problem. Precisely*

$$\|f^*\langle ., .\rangle_{\mathbb{E}^q} - \mu\| = O\left(\frac{1}{N}\right)$$

where $\mu = f_0^*\langle ., .\rangle_{\mathbb{E}^q} + \rho\, \ell \otimes \ell$. Moreover

1) $\|f - f_0\|_{C^0} = O\left(\frac{1}{N}\right)$,

2) $\|df - df_0\|_{C^0} \leq \frac{Cte}{N} + \sqrt{7}\rho^{\frac{1}{2}}|\ell(W)|$,

and if N is large enough, f is an embedding.

Proof.– Let us check that the mixed term vanishes when performing a convex integration along the integral lines of W. We have

$$\begin{aligned}
\langle d(f\circ\varphi)(e_1), d(f\circ\varphi)(e_j)\rangle_{\mathbb{E}^q} &= \langle d(f\circ\varphi)(e_1), d(f_0\circ\varphi)(e_j)\rangle_{\mathbb{E}^q} + O\left(\tfrac{1}{N}\right) \\
&= \langle r(\varphi(s,c))\mathbf{e}^{i\alpha(\varphi(s,c))\cos 2\pi N s}, d(f_0\circ\varphi)(e_j)\rangle_{\mathbb{E}^q} + O\left(\tfrac{1}{N}\right) \\
&= \lambda\langle d(f_0\circ\varphi)(e_1), d(f_0\circ\varphi)(e_j)\rangle_{\mathbb{E}^q} + O\left(\tfrac{1}{N}\right) \\
&= \lambda\, f_0^*\langle ., .\rangle_{\mathbb{E}^q}(d\varphi(e_1), d\varphi(e_j)) + O\left(\tfrac{1}{N}\right)
\end{aligned}$$

where $\lambda = \frac{r(\varphi(s,c))}{\|d(f_0\circ\varphi)(e_1)\|}\cos(\alpha(\varphi(s,c))\cos 2\pi Ns)$. From the definition of W we deduce that $\varphi(s,c) = se_1 + \psi(s,c)$ with $\psi(s,c) \in \ker\ \ell$. Therefore

$$\forall\ j \in \{2,...,n\},\quad d\varphi(e_j) \in \ker\ \ell.$$

In particular

$$f_0^*\langle .,.\rangle_{\mathbb{E}^q}(d\varphi(e_1), d\varphi(e_j)) = \mu(d\varphi(e_j), d\varphi(e_1))$$

and since $d\varphi(e_1) = W$ which is μ-orthogonal to ker ℓ we have

$$\mu(d\varphi(e_j), d\varphi(e_1)) = 0.$$

As a consequence $\langle d(f\circ\varphi)(e_1), d(f\circ\varphi)(e_j)\rangle_{\mathbb{E}^q} = O\left(\frac{1}{N}\right)$ and

$$\|(f\circ\varphi)^*\langle .,.\rangle_{\mathbb{E}^q}(e_1,e_j) - (\varphi^*\mu)(e_1,e_j)\| = O\left(\frac{1}{N}\right)$$

for $j \in \{2,...,n\}$. It is straightforward to check that

$$\|(f\circ\varphi)^*\langle .,.\rangle_{\mathbb{E}^q}(e_i,e_j) - (\varphi^*\mu)(e_i,e_j)\| = O\left(\frac{1}{N}\right)$$

for $i > 1, j > 1$ and the equality $\|df(W)\|^2_{\mathbb{E}^q} = r^2$ means that

$$\|(f\circ\varphi)^*\langle .,.\rangle_{\mathbb{E}^q}(e_1,e_1) - (\varphi^*\mu)(e_1,e_1)\| = 0.$$

Therefore

$$\|(f\circ\varphi)^*\langle .,.\rangle_{\mathbb{E}^q} - \varphi^*\mu\| = O\left(\frac{1}{N}\right)$$

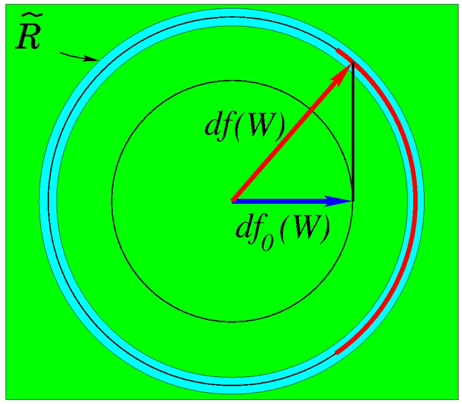
and since $[0,1]^n$ is compact

$$\|f^*\langle .,.\rangle_{\mathbb{E}^q} - \mu\| = O\left(\frac{1}{N}\right).$$

The general theory of convex integration ensures that $\|f - f_0\|_{C^{1,\hat{1}}} = O\left(\frac{1}{N}\right)$. This proves point 1 and reduces point 2 to the proof of the following inequality:

$$\|df(W) - df_0(W)\|_{C^0} \leq \sqrt{7}\rho^{\frac{1}{2}}|\ell(W)|.$$

The maximum of distance between $df(W)$ and $df_0(W)$ could be roughly estimated by using the Pythagorean Theorem (see the figure below).



The image of the loop $s \longmapsto h_{.}(s)$ is the red arc and the differential $df(W)$ lies somewhere in this arc. The maximum of the distance between $df(W)$ and $df_0(W)$ is roughly given by the lenght of the black vertical segment.

The maximum of the square of the difference $\|df(W) - df_0(W)\|^2$ is of the order of magnitude of

$$\|df(W)\|^2 - \|df_0(W)\|^2 = \rho\ell^2(W).$$

A precise computation shows that, in fact, $\|df(W) - df_0(W)\|^2 \leq 7\rho\ell^2(W)$. $\square$

6 Proof of the Nash-Kuiper Theorem

We give here the main arguments of the proof setting aside countless details.

6.1 Isometric embeddings of a Riemannian cube $M^n = [0,1]^n$

With the approach described above, we are now able to produce from a strictly short embedding f_{k-1} a sequence

$$f_{k,1}, ..., f_{k,S_k}$$

of embeddings such that

$$\|(g_k - f_{k,j}^*\langle .,.\rangle_{\mathbb{E}^q}) - (\rho_{k,j+1}\ell_{k,j+1}\otimes\ell_{k,j+1} + ... + \rho_{k,S_k}\ell_{k,S_k}\otimes\ell_{k,S_k})\|_{C^0} = \sum_{i=1}^{j} O\left(\frac{1}{N_{k,j}}\right)$$

where $N_{k,j}$ is the number of corrugations of the map $f_{k,j}$. In particular

$$\|g_k - f_k^*\langle .,.\rangle_{\mathbb{E}^q}\|_{C^0} = O\left(\frac{1}{N_{k,1}}\right) + ... + O\left(\frac{1}{N_{k,S_k}}\right)$$

where we have set $f_k := f_{k,S_k}$. Since $g_k < g_{k+1}$, the embedding $f_{k,j}$ is strictly short for g_{k+1} if $\sum_{i=1}^{j} O\left(\frac{1}{N_{k,j}}\right)$ is small enough.

By iterating the process, we generate an infinite sequence of embeddings

$$f_0, \quad f_{1,1}, ..., f_{1,S_1} =: f_1, \quad f_{2,1}, ..., f_{2,S_2} =: f_2, \quad etc.$$

Since

$$\|f_k - f_{k-1}\|_{C^0} = O\left(\frac{1}{N_{k,1}}\right) + ... + O\left(\frac{1}{N_{k,S_k}}\right)$$

this sequence will converge C^0 if we choose the $N_{k,j}$s large enough. Let f_∞ the limit map. The crucial point is that the sequence $(f_k)_{k\in\mathbb{N}}$ is also C^1-converging. Indeed we have

$$\|df_k - df_{k-1}\|_{C^0} \leq \sum_{j=1}^{S_k} O\left(\frac{1}{N_{k,j}}\right) + \sqrt{7}\sum_{j=1}^{S_k} \rho_{k,j}^{\frac{1}{2}} |\ell_{k,j}(W_{k,j})|.$$

Let us assume that the $\ell_{k,j}$s and the $W_{k,j}$s are normalized by requiring, for instance, that $\ell_{k,j}(.) = \langle U_{k,j}, .\rangle_{\mathbb{E}^n}$ with $\|U_{k,j}\|_{\mathbb{E}^n} = 1$ and $W_{k,j} = U_{k,j} + V_{k,j}$ with $V_{k,j} \in \ker \ell_{k,j}$. From the decomposition

$$g_k - f_{k-1}^* \langle ., .\rangle_{\mathbb{E}^q} = \sum_{j=1}^{S_k} \rho_{k,j} \ell_{k,j} \otimes \ell_{k,j}$$

we deduce that there exists a constant $K(\ell_1, ..., \ell_{S_k}) > 0$ depending on the chosen $\ell_1, ..., \ell_{S_k}$ such that

$$\sqrt{7}\sum_{j=1}^{S_k} \rho_{k,j}^{\frac{1}{2}} |\ell_j(W_{k,j})| \quad \leq \quad K(\ell_1, ..., \ell_{S_k}) \|g_k - f_{k-1}^* \langle ., .\rangle_{\mathbb{E}^q}\|.$$

But we also have

$$\|g_{k-1} - f_{k-1}^* \langle ., .\rangle_{\mathbb{E}^q}\| = \sum_{j=1}^{S_{k-1}} O\left(\frac{1}{N_{k-1,j}}\right),$$

hence

$$\begin{aligned}
\sqrt{7}\sum_{j=1}^{S_k} \rho_{k,j}^{\frac{1}{2}} |\ell_j(W_{k,j})| \quad &\leq \quad K(\ell_1, ..., \ell_{S_k}) \|g_k - g_{k-1}\| + \sum_{j=1}^{S_{k-1}} O\left(\frac{1}{N_{k-1,j}}\right) \\
&\leq \quad K(\ell_1, ..., \ell_{S_k}) \sqrt{\delta_k - \delta_{k-1}} \|\Delta\| + \sum_{j=1}^{S_{k-1}} O\left(\frac{1}{N_{k-1,j}}\right).
\end{aligned}$$

In fact, since $M^n = [0,1]^n$ is compact, one can choose a stationnary sequence of set of linear forms $\{\ell_{k,1}, ..., \ell_{k,S_k}\}$ so that it is stationnary. In particular, there exists a uniform upper bound K for all the constants $K(\ell_1, ..., \ell_{S_k})$. Thus

$$\|df_k - df_{k-1}\|_{C^0} \leq K\sqrt{\delta_k - \delta_{k-1}}\|\Delta\| + \sum_{j=1}^{S_{k-1}} O\left(\frac{1}{N_{k-1,j}}\right).$$

The sequence $(f_k)_{k\in\mathbb{N}}$ can be made C^1-converging by choosing the sequence $(\delta_k)_{k\in\mathbb{N}^*}$ such that

$$\sum \sqrt{\delta_k - \delta_{k-1}} < +\infty.$$

For instance, $\delta_k = 1 - e^{-k\gamma}$ with $\gamma > 0$ is appropriate.

Once the sequence is C^1-converging, it is then straightforward to see that the limite is an isometry. Indeed from

$$g_{k-1} < f_k^*\langle ., .\rangle_{\mathbb{E}^q} < g_{k+1}$$

we deduce by taking the limite that

$$\lim_{k\longrightarrow\infty}(f_k^*\langle ., .\rangle_{\mathbb{E}^q}) = g.$$

From the C^1-convergence we have

$$\lim_{k\longrightarrow\infty}(f_k^*\langle ., .\rangle_{\mathbb{E}^q}) = (\lim_{k\longrightarrow\infty} f_k)^*\langle ., .\rangle_{\mathbb{E}^q}$$

therefore $f_\infty^*\langle ., .\rangle_{\mathbb{E}^q} = g$. Note that f_∞ is necessarily an C^1 immersion.

The fact that f_∞ is an embedding is easily obtained in the codimension one case. Indeed, f_∞ is C^0 close to a C^1 embedding f_k and is such that the tangent planes to f_∞ are C^0-close to the corresponding tangent planes of f_k. In codimension one, this implies that f_∞ is an embedding. In greater codimension, the argument is more involved, see [3], p. 393-394.

6.2 Isometric embeddings of a compact manifold

It is enough to perform a cubical decomposition of the manifold and then to glue the local solutions together with the help of the following relative version of the proof of the Nash-Kuiper theorem for the cube:

Isometric embedding of the cube: relative version .– *Let* $([0,1]^n, g)$ *be a Riemannian cube,* $A \subset [0,1]^n$ *be a polyhedron*[2]*,* $f_0 : ([0,1]^n, g) \xrightarrow{C^1} \mathbb{E}^q$ *be an embedding which is isometric over an open neighbourhood* $\mathcal{O}p\ A$ *of* A *and is strictly short elsewhere. Then, for every* $\epsilon > 0$*, there exists a* C^1 *isometric embedding* $f : ([0,1]^n, g) \longrightarrow \mathbb{E}^q$ *such that* $\|f - f_0\|_{C^0} \leq \epsilon$ *and* $f = f_0$ *over a smaller neighbourhood* $\mathcal{O}p_1\ A$*.*

References

[1] Y. Eliahsberg et N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, vol. 48, A. M. S., Providence, 2002.

[2] N. Kuiper, *On* C^1*-isometric imbeddings I, II*, Indag. Math. 17 (1955), 545-556, 683-689.

[3] F. Nash, C^1*-isometric imbeddings*, Ann. Math. 63 (1954), 384-396.

[2] That is a subcomplex of a certain smooth triangulation of $[0,1]^n$.