MEMOIRE D'HABILITATION

présenté devant

l'UNIVERSITE CLAUDE BERNARD-LYON I

par

Vincent BORRELLI

Plongements lagrangiens et totalement réels

Energie et volume des champs de vecteurs

soutenu le 8 Octobre 2004.

JURY : M. Jean-Pierre BOURGUIGNON (président)
M. Jacques LAFONTAINE
M. Rémi LANGEVIN
M. François LAUDENBACH
M. Jean-Marie MORVAN
M. Pierre PANSU

RAPPORTEURS : M. Yakov ELIASHBERG,
M. Jacques LAFONTAINE,
M. François LAUDENBACH.

Remerciements

Ma reconnaissance va tout d'abord à Jean-Marie Morvan, sa façon d'aborder les problèmes, son originalité et son ouverture d'esprit m'ont profondément influencé.

Je suis très touché de l'honneur que me font Yakov Eliahberg, Jacques Lafontaine et François Laudenbach en ayant accepté de rapporter sur ce mémoire. Leurs remarques m'ont aussi permis d'en améliorer grandement la rédaction.

Je suis extrêmement honoré que Jean-Pierre Bourguignon ait accepté de présider le jury, je suis également très reconnaissant à Rémi Langevin et Pierre Pansu pour leur participation à ce jury.

Mes remerciements vont également à Fabiano Brito, Olga Gil-Medrano et Claudio Gorodski avec qui j'ai collaboré, à Leonid Polterovich qui m'a fortement encouragé dans mes travaux «symplectiques» et à David Spring pour de très enrichissantes discussions.

Je n'oublie pas, sans pouvoir les citer tant ils sont nombreux, tous ceux qui en prenant le temps de répondre à mes questions, ont apporté leur pierre à ce mémoire.

Je salue aussi mes amis du laboratoire de mathématiques pour avoir contribué à rendre agréables et néanmoins studieuses ces années : Moulay Benameur, Stéphane Lamy, Jean-Paul Moshen, Jean-François Quint, Jean-Luc Rullière, Andrea Sambusetti et les autres...

Enfin j'adresse toute ma tendresse à Sophie et à mes deux filles Claire et Julie.

Organisation du mémoire

◊ ◊
◊

Présentation générale

Ce mémoire d'habilitation est la compilation de six articles publiés ou acceptés pour publication. Il s'agit de :

[Bo.1] *Linking classes of Lagrangian or totally real embeddings,* Ann. Global. Anal. Geom. 17 (1999), 371-384.

[Bo.2] *New examples of Lagrangian rigidity,* Israel J. of Math. 125 (2001), 221-235.

[Bo.3] *On totally real isotopy classes,* Int. Math. Res. Not. (2002) 2, 89-109

[Bo.4] avec C. Gorodski, *Minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$ and absolutely area-minimizing cones*, à paraître dans Diff. Geom. and its Appl.

[Bo.5] avec F. Brito et O. Gil-Medrano, *On the infimum of the Energy of unit vector fields on odd-dimensional spheres*, Ann. Global Anal. Geom. 23 (2003), 129-140.

[Bo.6] avec O. Gil-Medrano, *A critical radius for unit Hopf vector fields on spheres*, accepté à Math. Ann. sous réserve de modifications mineures.

Ces travaux succèdent à ceux réalisés pendant la thèse de troisième cycle et dont les références sont les suivantes :

[Th.1] avec B.-Y. Chen et J.-M. Morvan, *Une caractérisation géométrique de la sphère de Whitney*, C. R. Acad. Sc. Paris, Série I, 321 (1995), 1485-1490.

[Th.2] *Classe de Maslov des sous-variétés totalement réelles*, C. R. Acad. Sc. Paris, Série I, 323 (1996), 1035-1038.

[Th.3] *Maslov form and J-volume of totally real immersions*, J. of Geo. and Phy. 25 (1998), 271-290.

Comme le laisse supposer le titre de ce mémoire, tous ces travaux concernent soit *les plon-*

gements lagrangiens et totalement réels, soit *l'énergie et le volume des champs de vecteurs.* Ils s'organisent autour de trois thématiques que nous allons développer :

1. – Plongements lagrangiens et totalement réels et phénomènes de rigidité, [Bo.1], [Bo.2] et [Bo.3].

2. – Plongements lagrangiens et totalement réels et minimalité, [Th.1],[Th.2],[Th.3] et [Bo.4].

3. – Energie et volume des champs de vecteurs, [Bo.5] et [Bo.6].

1. – Plongements lagrangiens et totalement réels et phénomènes de rigidité

Les sous-variétés lagrangiennes jouent un rôle central en géométrie symplectique ce qui justifie qu'elles soient très largement étudiées. Pourtant ces objets restent encore mal connus, par exemple, à l'exception des dimensions 1 et 2, on ne sait pas classer les sous-variétés lagrangiennes de $\mathbb{C}^n$ à isotopie lagrangienne près, ni même caractériser les variétés qui admettent un plongement lagrangien dans $\mathbb{C}^n$. Rappelons qu'une sous-variété L^n d'un espace symplectique (W^{2n}, ω) est dite *lagrangienne* si la forme induite sur L par ω est identiquement nulle. Dans le cas où la variété ambiante est kählérienne, cette définition est équivalente à la condition :

$$\forall p \in L, \quad JT_pL = T_p^{\perp}L$$

autrement dit, la structure complexe J réalise un isomorphisme entre l'espace tangent de la sous-variété et son espace normal. On peut alors distinguer une classe plus large de sous-variétés $M^n \subset W^{2n}$, celles pour lesquelles l'image par J du plan tangent est en somme directe avec le plan tangent :

$$\forall p \in M, \quad JT_pM \oplus T_pM = T_pW.$$

Ainsi, l'espace tangent ne contient aucune droite complexe, de telles sous-variétés sont dites *totalement réelles* et apparaissent en géométrie presque complexe. Tout plongement lagrangien est totalement réel, car comme on le constate, la condition «totalement réelle» est une relaxation de la condition «lagrangienne».

Dans un soucis de simplicité, on va désormais supposer que l'espace ambiant W est tout simplement $\mathbb{C}^n$ avec sa structure kählérienne standard, c'est d'ailleurs dans ce cadre que se situent la plupart des travaux présentés dans cette habilitation.

Une question naturelle est la suivante : étant donnée une sous-variété totalement réelle, peut-on la déformer en une sous-variété lagrangienne ? Une conséquence importante des idées développées par M. Gromov autour de la notion de h-principe est que la réponse à cette question est positive à condition de se placer, non pas dans le cadre des sous-variétés, mais dans celui des immersions. Plus précisément, si l'on note $I_{Lag}(M)$ (resp. $I_{TR}(M)$)

l'espace des immersions lagrangiennes (resp. totalement réelles) d'une variété compacte sans bord M^n dans $\mathbb{C}^n$, alors l'inclusion naturelle :

$$I_{Lag}(M) \subset I_{TR}(M)$$

est une équivalence d'homotopie faible (ces deux espaces sont munis de la topologie de la convergence uniforme). En particulier cette inclusion induit une bijection au niveau du π_0 ce qui signifie que toute immersion totalement réelle est régulièrement homotope, parmi les immersions totalement réelles, à une immersion lagrangienne. De manière assez surprenante, il n'y a donc pas de réelle différence entre le monde des immersions lagrangiennes, régi par une condition fermée – une égalité – et celui des immersions totalement réelles, régi par une condition ouverte – une transversalité –. Pour observer une différence, il faut s'intéresser non plus aux immersions, mais aux plongements.

Théorème (Gromov, 85, [43]).– *Soit M une variété compacte sans bord et $j : M \to \mathbb{C}^n$ un plongement lagrangien, alors $H^1(M, \mathbb{R}) \neq 0$.*

Ainsi une sous-variété lagrangienne de $\mathbb{C}^n$ ne peut être simplement connexe, par exemple, les sphères $\mathbb{S}^n$ $(n > 1)$ n'admettent pas de plongement lagrangien dans $\mathbb{C}^n$. Du côté des plongements totalement réels, la situation est différente, P. Ahern et W. Rudin ont construit en 1985 un plongement totalement réel explicite de $\mathbb{S}^3$ dans $\mathbb{C}^3$, [1][1] plus généralement, les travaux de M. Audin [3] montrent qu'il existe de nombreuses variétés simplement connexes qui admettent un plongement totalement réel dans $\mathbb{C}^n$. Par conséquent les deux théories divergent, l'inclusion :

$$E_{Lag}(M) \subset E_{TR}(M)$$

de l'espace des plongements lagrangiens dans celui des plongements totalement réels n'induit pas une bijection entre les ensembles des composantes connexes par arcs de chacun des espaces.

Voici deux autres exemples frappants de théorème relatifs aux plongements lagrangiens et qui révèlent un phénomène de rigidité.

Théorème (Viterbo, 90, [89]).– *Si $j : \mathbb{T}^n \to \mathbb{C}^n$ est un plongement lagrangien, alors sa classe de Maslov $\mu(j) \in H^1(M, \mathbb{Z})$ est non nulle [2].*

Théorème (Eliashberg et Polterovich, 94, [28]). – *Si $j : \mathbb{T}^2 \to \mathbb{C}^2$ est un plongement lagrangien et si v est un champ de vecteurs partout non nul de $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, tangent aux feuilles $\mathbb{S}^1 \times \{*\}$, alors sa classe d'enlacement $\sigma(j, v) \in H^1(\mathbb{T}^2, \mathbb{Z})$ est nulle.*

La classe de Maslov et la classe d'enlacement sont des classes de cohomologie qui sont définies aussi bien dans le monde lagrangien que dans le monde totalement réel et les

[1] M. Gromov a montré que la sphère $\mathbb{S}^n$ admet un plongement totalement réel dans $\mathbb{C}^n$ si et seulement si $n = 1$ ou 3 [45], [44] p. 193

[2] Il s'agit ici d'une version affaiblie du théorème de C. Viterbo. Notons que ce résultat avait été établi indépendamment par L. Polterovich dans le cas de la dimension deux

théorèmes ci-dessus sont des théorèmes de rigidité dans le sens où ils ne sont plus vrais si l'on remplace «plongement lagrangien» par «plongement totalement réel», il est en effet très facile de construire des plongements totalement réels de tores $\mathbb{T}^n$ dans $\mathbb{C}^n$ dont la classe de Maslov soit nulle, ou des plongements totalement réels de $\mathbb{T}^2$ dans $\mathbb{C}^2$ dont la classe d'enlacement soit non nulle.

En fait, la comparaison entre plongements lagrangiens et plongements totalement réels s'inscrit dans une problématique plus large. Il s'agit de distinguer, en la géométrie symplectique, les phénomènes de rigidité des phénomènes dits «flexibles». Cette question est tout à fait centrale dans la théorie, Y. Eliashberg et N. Mishachev comparent même la géométrie symplectique à une «longue bataille» pour tracer la frontière qui sépare le territoire de la rigidité de celui de la flexibilité ([27], p. 61). Les articles [Bo.1], [Bo.2] et [Bo.3] apportent une modeste contribution dans cette «bataille».

Dans l'article [Bo.1], on construit une classe d'enlacement qui généralise à tout plongement totalement réel $j : M \to \mathbb{C}^n$ d'une variété compacte sans bord orientée M, la classe d'enlacement qu'utilise Y. Eliashberg et L. Polterovich dans le cas d'un plongement lagrangien du tore $\mathbb{T}^2$. L'idée est de pousser les $(n-1)$-cycles hors de la sous-variété image $j(M)$, de calculer des enlacements et de définir ainsi une classe $\sigma_{n-1}(j, v) \in H^{n-1}(M^n, \mathbb{Z})$. Plus précisément, si v est un champ de vecteurs partout non nul sur M^n – et un tel champ existe car la caractéristique d'Euler d'une variété admettant un plongement totalement réel est nulle – on pose :

$$\begin{array}{rccl} j_\epsilon : & M^n & \longrightarrow & \mathbb{C}^n \\ & x & \longmapsto & j(x) + \epsilon J dj_x(v_x). \end{array}$$

Bien sûr, puique v est un vecteur tangent à M alors $Jdj(v)$ «pointe» hors de $j(M)$ et par conséquent j_ϵ est disjoint de $j(M)$ pour ϵ suffisamment petit. Ainsi $j_\epsilon(M)$ représente un n-cycle du complémentaire de $j(M)$, il définit donc un élément $[j_\epsilon(M)]$ de $H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z})$. Or, on a les identifications canoniques suivantes :

$$H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H_{n+1}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

La première provient de la suite exacte longue associée à la paire $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ et la seconde, de la dualité d'Alexander. On définit la classe d'enlacement $\sigma_{n-1}(j, v) \in H^{n-1}(M^n, \mathbb{Z})$ comme l'image par cette suite d'identifications du n-cycle $[j_\epsilon(M)]$.

Cette classe d'enlacement est évidemment invariante par isotopie totalement réelle, on montre qu'elle est en fait fortement reliée à la classe d'isotopie *ordinaire* du plongement :

Théorème.– *Soit M^n $(n \geq 4)$ une variété compacte sans bord parallélisable. Soient j_0, j_1 deux plongements totalement réels (resp. lagrangiens) régulièrement homotopes parmi les immersions totalement réelles (resp. parmi les immersions lagrangiennes). Les propositions suivantes sont équivalentes :*

1. *j_0 et j_1 sont isotopes parmi les plongements ordinaires,*

2. Il existe un champ de vecteur partout non nul v sur M^n tel que :
$$\begin{cases} \sigma_{n-1}(j_0,v) & = & \sigma_{n-1}(j_1,v) & \textit{si } n \textit{ est impair,} \\ \mu^*(\sigma_{n-1}(j_0,v)) & = & \mu^*(\sigma_{n-1}(j_1,v)) & \textit{si } n \textit{ est pair,} \end{cases}$$
(on a noté $\mu^* : H^{n-1}(M^n,\mathbb{Z}) \longrightarrow H^{n-1}(M^n,\mathbb{Z}_2)$ l'homomorphisme des cœfficients)[3].

Grosso modo, ce théorème dit la chose suivante : si la dimension est impaire, la classe d'enlacement ne distingue que les classes d'isotopie de plongements *ordinaires.* En revanche, si la dimension est paire, on a un espoir de distinguer deux plongements lagrangiens, isotopes au sens ordinaire, regulièrement homotopes parmi les immersions lagrangiennes mais non lagrangiennement isotopes.

La classification des plongements (ordinaires) d'une variété M^n dans $\mathbb{R}^N$ à isotopie près a été obtenue dans le domaine stable ($N > \frac{3}{2}(n+1)$) par A. Haefliger en 1962 [47]. Ce travail a été ensuite revisité par A. Haefliger et M. Hirsch [50] dans le cas où $N = 2n$ (et donc $n \geq 4$) afin de dégager une obstruction «géométrique» à l'existence d'une isotopie entre deux plongements donnés. L'idée à l'origine du théorème de [Bo.1] a consisté à reformuler cette obstruction dans le cas de plongements totalement réels. On constate alors qu'elle s'exprime en fonction de la classe d'enlacement. Voyons cela un peu plus en détails : soient $f : M^n \to \mathbb{C}^n$ un plongement et ν un champ de vecteurs normal, nul en un point $f(p)$ de $f(M)$ et de norme ϵ sur $f(M \setminus D_1)$ où D_1 est un disque centré en p. On note f_ν l'application :
$$p \longmapsto f(p) + \nu(p).$$
La remarque fondamentale d'A. Haefliger et M. Hirsch est qu'il existe un unique champ ν, à homotopie convenable près, telle que le n-cycle $[f_\nu(M)]$ soit nul dans $H_n(\mathbb{C}^n \setminus f(M \setminus D_2))$ où D_2 est un disque contenant D_1. Ce champ ν permet ensuite de classer les plongements à isotopie près. Le n-cycle $[f_\nu(M)]$ fait bien entendu immédiatement penser à la classe d'enlacement...

Un prolongement naturel du travail précédent est de chercher à obtenir, au moyen des classes d'enlacement, des résultats de rigidité. C'est ce qui est réalisé dans [Bo.2] où l'on établit pour les variétés $\mathbb{S}^1 \times \mathbb{S}^3$ et $\mathbb{S}^1 \times \mathbb{S}^7$ un analogue du théorème de rigidité de Y. Eliashberg et L. Polterovich concernant le tore $\mathbb{T}^2$.

Théorème. – *Soient $n = 4$ ou 8 et $j: \mathbb{S}^1 \times \mathbb{S}^{n-1} \longrightarrow \mathbb{C}^n$ un plongement lagrangien. Soit v un champ de vecteurs partout non nul sur $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ et tangent aux feuilles $\mathbb{S}^1 \times \{*\}$, alors : $\sigma_{n-1}(j,v) = 0$.*

Dans ce théorème, on ne peut bien sûr pas remplacer «plongement lagrangien» par «plongement totalement réel». Il est en effet aisé de construire des plongements totalement réels de $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ dans $\mathbb{C}^n$, $n = 4$ ou 8, dont la classe d'enlacement soit non triviale.

[3] En fait, si $\sigma_{n-1}(j_0,v) = \sigma_{n-1}(j_1,v)$ pour un champ de vecteur partout non nul v, alors $\sigma_{n-1}(j_0,w) = \sigma_{n-1}(j_1,w)$ pour n'importe quel champ de vecteurs partout non nul w.

Un théorème profond dû à Y. Eliashberg, A. Floer, M. Gromov et D. McDuff [68] [4] énonce que si V est une variété symplectique satisfaisant $H_2(V,\mathbb{R}) = 0$ et dont la structure symplectique coïncide, hors d'une boule, avec celle de $\mathbb{C}^n$ alors V est difféomorphe à $\mathbb{C}^n$. Ainsi la connaissance de la structure symplectique «à l'infini» force la topologie. Ce fait va impliquer le phénomène de rigidité lagrangienne observé pour $\mathbb{S}^1\times\mathbb{S}^3$ et $\mathbb{S}^1\times\mathbb{S}^7$, voici comment.

Etant donné un plongement lagrangien $j : M \to \mathbb{C}^n$ d'une variété compacte sans bord, on réalise une chirurgie de $\mathbb{C}^n$ en ôtant un voisinage tubulaire N de $j(M)$ et en le recollant au moyen d'un difféomorphisme f du bord ∂N. On choisit ce difféomorphisme de telle manière que l'espace résultant V_f satisfasse $H_2(V_f,\mathbb{R}) = 0$, et que de plus, les structures symplectiques soient respectées de façon à ce que V_f soit une variété symplectique. Le but est de montrer que si j a un enlacement non nul, l'espace V_f n'a pas la topologie de $\mathbb{C}^n$, ce qui est contradictoire avec le théorème d'Y. Eliashberg, A. Floer, M. Gromov et D. McDuff. Dans ce programme, toute la difficulté réside dans le choix du difféomorphisme, il doit être suffisamment compliqué pour créer de la topologie, mais pas trop, pour respecter les conditions imposées sur V_f. Il est mené à bien pour les variétés $\mathbb{S}^1\times\mathbb{S}^3$ et $\mathbb{S}^1\times\mathbb{S}^7$, mais je reste persuadé qu'il doit être possible d'aller beaucoup plus loin.

Le résultat principal de [Bo.1] met en évidence le lien entre la classe d'enlacement et la classe d'isotopie du plongement. Ce lien permet ici de mieux comprendre l'inclusion :

$$i : E_{Lag}(\mathbb{S}^1 \times \mathbb{S}^{n-1}) \subset E(\mathbb{S}^1 \times \mathbb{S}^{n-1})$$

entre l'espace des plongements lagrangiens et l'espace des plongements ordinaires de $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ dans $\mathbb{C}^n$, $n = 4$ ou 8.

Théorème. – *L'application*

$$i_\sharp\colon \pi_0(E_{Lag}(\mathbb{S}^1 \times \mathbb{S}^{n-1})) \longrightarrow \pi_0(E(\mathbb{S}^1 \times \mathbb{S}^{n-1})) \simeq \mathbb{Z}_2$$

$$\textit{est} \begin{cases} \textit{surjective} & \textit{si } n = 4 \\ \textit{constante} & \textit{si } n = 8. \end{cases}$$

Ainsi tout plongement de $\mathbb{S}^1 \times \mathbb{S}^3$ dans $\mathbb{C}^4$ est isotope à un plongement lagrangien alors que ce n'est pas le cas pour les plongements de $\mathbb{S}^1 \times \mathbb{S}^7$ dans $\mathbb{C}^8$. Dans le cas où $n = 4$, étant donnés deux plongements lagrangiens $j_0, j_1 : \mathbb{S}^1 \times \mathbb{S}^3 \to \mathbb{C}^4$, la question se pose donc de savoir si l'on peut réaliser où non une isotopie (ordinaire) qui joigne ces deux plongements. Cette question est résolue dans [Bo.2], on définit une classe de cohomologie $\epsilon(j_0, j_1) \in H^3(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{Z}_2) = \mathbb{Z}_2$ dont la nullité est une condition nécessaire et suffisante à l'existence de l'isotopie. L'intérêt de cette classe est qu'elle résulte d'un calcul homotopique direct ne faisant intervenir que les différentielles de j_0 et j_1.

La démonstration du théorème ci-dessus n'est *pas* une application du théorème de [Bo.1] car l'hypothèse principal n'est pas satisfaite. En effet, étant donné deux plongements

[4] Voir [26] pour connaître la contribution de chacun.

lagrangiens de $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ dans $\mathbb{C}^n$, $n = 4$ ou 8, rien ne garantit qu'ils soient régulièrement homotopes parmi les immersions lagrangiennes, c'est même faux en général. Néanmoins, la nullité de la classe d'enlacement donne un renseignement très fort sur l'espace normal et cela permet, en utilisant les résultats d'A. Haefliger et M. Hirsch [50] de connaître la classe d'isotopie ordinaire. En fait, une fois connu la classe d'enlacement, le résultat découle d'un calcul purement homotopique. On montre que l'image de l'application $i_\sharp$ s'identifie à l'image de la composition :

$$\mathbb{Z} = \pi_{n-1}(U(n)) \xrightarrow{l_\sharp} \pi_{n-1}(SO(2n)) = \mathbb{Z} \xrightarrow{mod.\ 2} \mathbb{Z}_2.$$

Or si $n = 4$, le morphisme $l_\sharp$ est un isomorphisme tandis que si $n = 8$, c'est le monomorphisme $n \longmapsto 2n$. Notons que c'est une raison de même nature, impliquant le même morphisme $l_\sharp$, qui empêche $\mathbb{S}^7$ d'admettre un plongement totalement réel dans $\mathbb{C}^7$ alors que $\mathbb{S}^3$ en admet un dans $\mathbb{C}^3$ ([45], p. 193).

Dans [Bo.3], on s'intéresse au problème de la classification des plongements totalement réels à isotopie totalement réelle près, le problème de l'existence ayant été en grande partie résolue par M. Audin [3]. On obtient en particulier le résultat suivant :

Théorème. – *Soit M^n, $n \equiv 3$ [4], une variété connexe, compacte, sans bord, orientable et admettant un plongement totalement réel j dans $\mathbb{C}^n$ alors :*

$$\pi_0(E_{TR}) \simeq \pi_0(E) \times \pi_0(I^0_{TR})$$

et la bijection est induite par l'application diagonale $E_{TR} \longrightarrow E \times I^0_{TR}$.

Dans ce théorème, on a noté E_{TR} l'espace des plongements totalement réels de M dans $\mathbb{C}^n$, E celui des plongements ordinaires et I^0_{TR} celui des immersions totalement réels qui sont régulièrement homotopes à un plongement ordinaire. La topologie est celle de la convergence uniforme. Les ensembles $\pi_0(E)$ et $\pi_0(I^0_{TR})$ peuvent effectivement être déterminés à partir des travaux d'A. Haefliger sur la classification des plongements [47] et ceux de M. Gromov sur le h-principe [45]. Dans [Bo.3], des calculs explicites du $\pi_0(E_{TR})$ sont d'ailleurs menés pour certaines variétés. Voici maintenant quelques commentaires sur ce résultat et les techniques qui permettent de l'obtenir.

Toute immersion $j : M \to N$ induit, au travers de sa différentielle, un monomorphisme entre les fibrés tangents TM et TN. Supposons que N soit muni d'une structure presque complexe J et que j soit une immersion totalement réelle. Le monomorphisme induit entre les espaces tangents est alors totalement réel, on a une flèche :

$$\begin{array}{ccc} I_{TR}(M,N) & \longrightarrow & Mono_{TR}(TM,TN) \\ j & \longmapsto & dj \end{array}$$

entre l'espace des immersions totalement réelles et l'espace des monomorphismes totalement réels de TM dans TN. Dans [45] (voir aussi [44], [27] ou [84]) M. Gromov montre que cette flèche est une équivalence d'homotopie faible, on dit alors que les immersions totalement réelles satisfont à un h-principe [5], une notion mis en évidence dans les années 70 par M. Gromov et qui signifie, lorsqu'elle est vérifiée, que l'on peut déduire les propriétés homotopiques d'un espace d'applications satisfaisant à une relation différentielle de celles d'un sous-espace de l'espace des jets déterminé par cette même relation différentielle. Ici, l'espace d'applications est $I_{TR}(M,N)$, la relation différentielle est donnée par la relation définissant les immersions totalement réelles, et l'on déduit les propriétés homotopiques de $I_{TR}(M,N)$ de celles de l'espace des sections de la relation différentielle, c'est-à-dire $Mono_{TR}(TM,TN)$. De cette façon, on réduit une question de topologie différentielle à son simple aspect topologique. Par exemple, la détermination des classes d'homotopie régulière parmi les immersions totalement réelles se ramène à compter les composantes connexes par arcs de l'espace des sections d'un fibré. Si $N = \mathbb{C}^n$, on en déduit qu'il existe une bijection [6] entre l'ensemble des composantes connexes par arcs de l'espace des immersions totalement réelles $\pi_0(I_{TR}(M,\mathbb{C}^n))$ et $[M,U(n)]$, les classes d'homotopie d'applications de M dans $U(n)$.

Que se passe-t-il maintenant si l'on s'intéresse non plus aux immersions mais aux plongements totalement réels ? Dans [44], M. Gromov montre qu'il existe une sorte de h-principe pour les plongements totalement réels.

Théorème (Gromov, [44], p. 192). – *Soit $f : M \to N$ un plongement. S'il existe un chemin continu dans $Mono(TM,TN)$ joignant df à un élément de $Mono_{TR}(TM,TN)$ alors f est isotope à un plongement totalement réel.*

La réciproque est bien sûr évidente. Ce théorème donne donc une condition nécessaire et suffisante à l'existence d'un plongement totalement réel et que celle-ci dépend de l'inclusion :

$$Mono_{TR}(TM,TN) \subset Mono(TM,TN).$$

Le problème de l'existence d'un plongement totalement réel est ainsi ramené à un problème de topologie.

Il est facile d'obtenir une version à paramètre de ce théorème, on en trouvera une dans le livre récent de Y. Eliashberg et N. Mishachev [27] ou encore au paragraphe 3.2.1 de [Bo.3]. Elle permet d'avoir accès aux propriétés homotopiques de l'espace $E_{TR}(M,N)$ des plongements totalement réels.

Il est aussi important de remarquer que l'on ne peut pas énoncer ce théorème dans le cadre lagrangien. En effet, il existe un plongement totalement réel $f : \mathbb{S}^3 \to \mathbb{C}^3$ (c'est justement une conséquence de ce théorème) et il est facile de voir que $Mono_{TR}(TM,TN)$ se retracte par déformation forte sur $Mono_{Lag}(TM,TN)$. Donc il existe un chemin dans $Mono_{TR}(T\mathbb{S}^3,T\mathbb{C}^3)$ qui joint df à un élément de $Mono_{Lag}(T\mathbb{S}^3,T\mathbb{C}^3)$. Pourtant, f ne peut être isotope à un plongement lagrangien puisque, d'après un corollaire du théorème

[5] Exactement, à un h-principe paramétrique.

[6] Cette bijection n'est cependant pas canonique.

de Gromov sur la classe de Liouville [43] il n'existe pas de plongement lagrangien de $\mathbb{S}^n$ dans $\mathbb{C}^n$ sauf si $n = 1$.

Notons enfin un dernier point important : le théorème de M. Gromov ci-dessus suppose l'existence d'un chemin continu dans $Mono(TM, TN)$, ceci a pour conséquence que la seule obstruction non triviale à réaliser une isotopie totalement réelle entre deux plongements totalement réels j_0 et j_1 réside dans le groupe fondamental de l'espace des immersions I. Voyons pourquoi.

Une condition nécessaire évidente à l'existence d'une isotopie totalement réelle joignant j_0 et j_1 est qu'il doit exister une isotopie ordinaire f_t ainsi qu'une homotopie régulière totalement réelle g_t qui joignent j_0 à j_1. Autrement dit j_0 et j_1 doivent habiter la même composante (connexe par arcs) de E ainsi que la même composante de I_{TR}. Notons $E(j_0)$ la composante de E contenant j_0, $I_{TR}(j_0)$ la composante de I_{TR} qui contient j_0, etc. On vient simplement d'observer que :

$$E_{TR}(j_0) = E_{TR}(j_1) \Rightarrow \begin{cases} I_{TR}(j_0) & = & I_{TR}(j_1), \\ E(j_0) & = & E(j_1). \end{cases}$$

A-t-on la réciproque ? Une reformulation du h-principe pour les plongements totalement réels permet d'affirmer que s'il existe une homotopie de chemins de l'espace des immersions ordinaires I à bord fixés joignant le chemin $t \mapsto f_t$ au chemin $t \mapsto g_t$, alors j_0 et j_1 sont isotopes parmi les plongements totalement réels. Autrement dit, si la concaténation $f_t \bullet g_t^{-1}$ est homotope dans I au lacet constant, alors $E_{TR}(j_0) = E_{TR}(j_1)$. On s'aperçoit par conséquent qu'il y a une obstruction à obtenir une réciproque et que celle-ci se trouve dans le groupe fondamental de I. Notons $\iota = I_{TR}(j_0)$, $\epsilon = E(j_0)$ et $E_{TR}(\epsilon, \iota) = E_{TR} \cap \iota \cap \epsilon$. Ce dernier espace est donc l'ensemble des plongements totalement réels qui sont dans la même composante de I_{TR} et de E que j_0. Des arguments combinatoires permettent alors de montrer que :

$$N(\epsilon, \iota) := |\pi_0(E_{TR}(\epsilon, \iota))| \leq |\pi_1(I)|$$

et bien sûr $N(\epsilon, \iota) = 1$ signifie que la réciproque dans l'implication ci-dessus est vraie.

En général, le groupe fondamental de I n'est pas trivial (i. e. contient plus d'un élément) et la majoration ci-dessus est trop grossière pour obtenir un résultat de classification. Pour l'améliorer, on remarque que l'on possède une certaine liberté dans le choix des chemins $t \mapsto f_t$ et $t \mapsto g_t$. On peut éventuellement changer leurs classes d'homotopie afin d'obtenir, par concaténation, un lacet trivial dans I. On est donc amené à étudier les inclusions $l : I_{TR} \to I$ et $i : E \to I$ et, plus précisément, les morphismes qu'elles induisent entre les groupes fondamentaux. Les flèches $\pi_k(l)$ induites par l'inclusion $l : I_{TR} \to I$ sont accessibles grâce au h-principe, en revanche, pour les flèches induites par l'inclusion $i : E \to I$, les choses sont plus délicates. D'abord, on ne connaît que les tous premiers groupes d'homotopie de E (*cf.* [47], [25] et [33]), ensuite les descriptions dues respectivement à A. Haefliger et M. Hirsch des espaces E et I en termes de sections d'un fibré ne sont pas compatibles. Plus exactement, il n'y a pas d'inclusion naturelle entre les espaces de sections qui approximent E et I, par conséquent, on ne peut déduire de ces «approximations» la description des flèches $\pi_k(i)$. Ce défaut a été corrigé par A. Haefliger et M.

Hirsch dans un travail commun [49] proposant une approximation de I par un espace de morphismes $\mathbb{Z}_2$-équivariants entre espaces tangents. Dans cette nouvelle description l'inclusion $i : E \to I$ s'interprète simplement, ce qui permet de faire des calculs effectifs au moins dans les situations qui nous intéressent.

Pour les dimensions paires, le cardinal $N(\epsilon, \iota)$ peut être infini. Il est construit dans [Bo.3] une famille dénombrable de plongements totalement réels de $\mathbb{T}^n$ dans $\mathbb{C}^n$ qui appartiennent tous à la même composante de E et à la même composante de I_{TR}. On montre qu'ils ne sont pas isotopes parmi les plongements totalement réels en vérifiant que leurs classes d'enlacement diffèrent deux à deux.

La technique utilisée dans [Bo.3], à savoir une majoration de $N(\epsilon, \iota)$, ne permet d'obtenir un résultat de classification que dans le cas où le majorant vaut 1, ce qui n'est établi que dans certaines situations particulière, par exemple si la dimension est congrue à trois modulo quatre. Pour achever la classification, il faut avoir, non pas une majoration, mais une valeur exacte de $N(\epsilon, \iota)$. Je me suis rendu compte, après avoir pris connaissance [7] de l'article [25] de J.-P. Dax, que l'on peut aller plus loin dans l'étude de l'inclusion $E \subset I$ et probablement améliorer les majorations. D'autre part, $N(\epsilon, \iota)$ peut être strictement plus grand que 1, un prolongement naturel est donc celui de la recherche d'un (ou plusieurs) invariant(s) géométrique(s) des plongements totalement réels permettant de décrire tous les éléments du $\pi_0(E_{TR}(\epsilon, \iota))$. Mon intuition est que cet invariant devrait avoir un lien avec la classe d'enlacement.

Récemment, T. Goodwillie, J. Klein et M. Weiss ont donné une description de l'espace des plongements ordinaires d'une variété dans une autre en termes de tiré en arrière homotopique de certains diagrammes [33]. Peut-on obtenir des résultats analogues pour l'espace des plongements totalement réels ? Une première approche serait de comparer le tiré en arrière homotopique du diagramme :

$$\begin{tikzcd} & I_{TR} \arrow[d, "\iota"'] \\ E \arrow[r, "i"] & I \end{tikzcd}$$

avec l'espace des plongements totalement réels E_{TR}.

2.– Plongements lagrangiens et totalement réels et minimalité

Il existe un lien remarquable, découvert en 1981 par J.-M. Morvan, entre la classe de Maslov d'une immersion lagrangienne $j : M^n \to \mathbb{C}^n$ et le vecteur courbure moyenne H de j.

Théorème (Morvan, 81, [70]). – *La classe de Maslov d'une immersion lagrangienne $j : M^n \longrightarrow \mathbb{C}^n$ est représentée par la 1-forme $\mu(j) = \frac{n}{\pi} i_H \omega$ où $\omega =< J.,. >$ est la forme*

[7] Malheureusement trop tard, c'est-à-dire après la publication de [Bo.3].

symplectique standard de $\mathbb{C}^n$.

La classe de Maslov d'une immersion lagrangienne ou totalement réelle est la classe de cohomologie obtenue an tirant en arrière, au moyen de l'application de Gauss, un générateur du groupe de cohomologie $H^1(\mathcal{T}(n), \mathbb{Z}) = \mathbb{Z}$ de la grassmannienne $\mathcal{T}(n)$ des n-plans totalement réels de $\mathbb{C}^n$. Le résultat de J.-M. Morvan a pour corollaire immédiat que toute sous-variété lagrangienne minimale a une classe de Maslov nulle.

Une question naturelle est celle de la généralisation du résultat de J.-M. Morvan au cas des immersions totalement réelles. Dans [Th.3], on établit le résultat suivant.

Théorème. – *La classe de Maslov d'une immersion totalement réelle* $j : M^n \longrightarrow \mathbb{C}^n$ *est représentée par la 1-forme* $\mu(j)$ *définie par :*

$$\forall X \in TM, \quad < \mu(j), X >= \frac{1}{\pi} tr(F^{-1} h_X)$$

où $h_X = h(X, .) : TM \longrightarrow T^{\perp}M$ *est la contraction de la seconde forme fondamentale* h *avec* X *et* $F : TM \longrightarrow T^{\perp}M$ *est la projection orthogonale de* J *sur le fibré normal, autrement dit :* $FX = (JX)^{\perp}$.

Comme on le constate, la forme $\mu(j)$ ne fait plus intervenir le seul vecteur courbure moyenne H, mais une expression plus complexe fonction de la seconde forme fondamentale h. En fait, pour une immersion totalement réelle, la forme $\frac{n}{\pi} i_H \omega$ n'est pas fermée en général, on ne pouvait donc espérer une expression aussi simple que dans le cas lagrangien. Il semble qu'avec ce résultat, le lien avec la minimalité soit rompu. On peut le voir réapparaître à condition de considérer non plus la fonctionnelle volume, mais une notion de volume qui prenne en compte la structure complexe J de $\mathbb{C}^n$: le J-volume. Le J-volume Vol_J d'une immersion totalement réelle j est défini par :

$$Vol_J(j) = \int_{M^n} \rho_J \, dvol_M \ \leq \infty$$

où $dvol_M$ est l'élément de volume donné par la métrique induite et :

$$\rho_J(p) = |e_1 \wedge ... \wedge e_n \wedge Je_1 ... \wedge Je_n|^{\frac{1}{2}}$$

avec $p \in M^n$ et $(e_1, ..., e_n)$ une base orthonormée de $dj_p(T_pM)$. On montre alors le résultat suivant.

Théorème. – *Une immersion totalement réelle* j *est critique pour le* J*-volume parmi les immersions totalement réelles laissant le bord fixe si et seulement si la forme de Maslov* $\mu(j)$ *est identiquement nulle.*

Ainsi, la nullité de la forme de Maslov est une condition de minimalité non pas du volume mais du J-volume. Ce résultat se généralise aux immersions totalement réelles d'une variété orientée M^n dans une variété kählérienne quelconque $\tilde{M}^{2n}$, la forme de Maslov n'est certes plus fermée (en général) mais sa nullité équivaut toujours au fait que l'immersion j soit critique pour la généralisation évidente du J-volume dans le cadre kählérien. Si, de plus, la variété kählérienne $\tilde{M}^{2n}$ a une courbure de Ricci négative ou nulle les immersions critiques sont stables.

Théorème. – *Soient $j : M^n \longrightarrow \tilde{M}^{2n}$ une immersion totalement réelle critique pour le J-volume, $\Phi_t : M^n \longrightarrow \tilde{M}^{2n}$ une déformation qui laisse fixe le bord de M^n et telle que $j = \Phi_0$ et $\frac{\partial \Phi}{\partial t} \in JTM$, on a :*

$$\frac{d^2}{dt^2} Vol_J(\Phi_t)|_{t=0} = \int_{M^n} \Big(\frac{1}{\rho_J^2} (div\, \rho_J X)^2 + 3(d\, ln\, \rho_J(X))^2 - Ricci(X,X) \Big) \rho_J dvol_M$$

où $X = -J\frac{\partial \Phi}{\partial t}$.

Ce résultat peut être considéré comme une généralisation du travail de Y.-G Oh sur la stabilité des sous-variétés lagrangiennes dans les variétés de kähler [74].

Une conséquence de la formule de J.-M. Morvan est de montrer que les sous-variétés lagrangiennes minimales de $\mathbb{C}^n$ sont précisément les sous-variétés lagrangiennes spéciales. Pour expliquer ce que sont ces sous-variétés il est nécessaire d'introduire la notion de calibration. Une *calibration* d'une variété riemannienne est une forme fermée φ satisfaisant à l'inégalité :

$$|\varphi(e_1, ..., e_n)| \leq 1$$

pour tout n-repère orthonormé $(e_1, ..., e_n)$, n étant le degré de φ. Une sous-variété M^n orientée est dite *φ-calibrée* si $\varphi(e_1, ..., e_n) = 1$ pour tout repère orthonormé direct de TM. L'intérêt fondamental des calibrations est que toute sous-variété φ-calibrée minimise le volume dans sa classe d'homologie, autrement dit, si M' est une autre sous-variété telle que $\partial M = \partial M'$ alors :

$$Vol(M) \leq Vol(M').$$

On dit encore qu'une sous-variété φ-calibrée minimise le volume *absolument*. Dans leur article fondateur sur la théorie des calibrations [52], R. Harvey et H. B. Lawson définissent les sous-variétés lagrangiennes spéciales (de phase $\theta \in [0, 2\pi[$) comme les sous-variétés de $\mathbb{C}^n$ calibrées par la calibration :

$$\varphi_\theta = Re(e^{i\theta} dz_1 \wedge \cdots \wedge dz_n).$$

Les sous-variétés lagrangiennes minimales étant justement les sous-variétés lagrangiennes spéciales, la minimalité d'une sous-variété lagrangienne équivaut à la minimalité absolue.

L'article [Bo.4], qui est un travail commun avec C. Gorodski (Université de Saõ Paulo), donne de nouveaux exemples de sous-variétés legendriennes minimales de la sphère. Une sous-variété legendrienne minimale $L^{n-1} \subset \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ est une sous-variété minimale de la sphère satisfaisant à la relation :

$$\forall p \in L, \;\; T_pL \oplus^\perp JT_pL = (Jp)^\perp.$$

Une des propriétés remarquables d'une telle sous-variété est que le cône qu'elle engendre :

$$C(L) = \{tx \in \mathbb{C}^n \mid t \in \mathbb{R}_+^*,\, x \in L\}$$

est une sous-variété spéciale de $\mathbb{C}^n$, c'est donc un cône minimiseur. En effet, l'intersection d'un cône lagrangien avec une sphère centrée à l'origine est une sous-variété legendrienne, cette dernière est minimale si et seulement si il en est de même du cône lagrangien, et d'après la discussion précédente, la minimalité équivaut à la minimalité absolue.

Au delà de cette propriété de minimisation, l'intérêt pour les cônes lagrangiens spéciaux provient de l'étude des singularités des sous-variétés lagrangiennes spéciales. Les plus simples de ces singularités sont dites «coniques». Cela signifie que la sous-variété en ces points singuliers admet un cône tangent qui est, bien sûr, un cône lagrangien spécial. Ce type de singularité fait aujourd'hui l'objet d'une littérature relativement abondante ([53],[59],[69]) car leur compréhension semble être un préalable essentiel à la clarification d'une conjecture d' A. Strominger, S.-T. Yau et E. Zaslov relative à la symétrie miroir en théorie des cordes [86].

Il est construit dans [Bo.4] deux familles d'exemples de sous-variétés legendriennes minimales de $\mathbb{S}^{2n-1}$ et donc de cônes minimisants. La première généralise deux exemples de R. Harvey et H. B. Lawson figurant dans [52], la seconde généralise des exemples de B. Cheng [20], [21]. On fournit également un exemple isolé.

Première série d'exemples. –

$$\mathbb{S}^4 \longrightarrow \mathbb{S}^9, \quad \mathbb{S}^7 \longrightarrow \mathbb{S}^{15}, \quad \mathbb{S}^{13} \longrightarrow \mathbb{S}^{27}, \quad \mathbb{S}^{25} \longrightarrow \mathbb{S}^{51},$$

$$(\mathbb{S}^{\frac{n-1}{2}} \times \mathbb{S}^{\frac{n-1}{2}}) \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^{2n+1}, \quad n \textit{ impair},$$
$$(U(2)/\mathbb{Z}_2) \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^{11}, \quad (Sp(2)/\mathbb{T}^2) \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^{19},$$
$$(SO(4)/\mathbb{Z}_2) \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^{15}, \quad (G_2/\mathbb{T}^2) \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^{27}.$$

L'écriture $X \longrightarrow \mathbb{S}^N$ doit se lire : «il existe un plongement legendrien minimal de X dans $\mathbb{S}^N$». Bien sûr, pour la première ligne, on doit aussi comprendre que ces plongements sont non triviaux, c'est-à-dire non congrus à l'inclusion standard $\mathbb{S}^n \subset \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$. Pour les autres lignes, l'écriture $X \times_{\mathbb{Z}_2} \mathbb{S}^1$ signifie que l'on considère le quotient du produit $X \times \mathbb{S}^1$ par $\mathbb{Z}_2$ et que l'action est donnée par la multiplication par -1 sur les deux facteurs. Dans la dernière ligne, G_2 est le groupe de Lie exceptionnel de dimension 14. Les exemples déjà connus étaient $\mathbb{S}^4 \longrightarrow \mathbb{S}^9$ et $(\mathbb{S}^{\frac{n-1}{2}} \times \mathbb{S}^{\frac{n-1}{2}}) \times_{\mathbb{Z}_2} \mathbb{S}^1 \longrightarrow \mathbb{S}^{2n+1}$, n impair [52].

Seconde série d'exemples. –

$$SU(2n)/Sp(n) \longrightarrow \mathbb{S}^{4n^2-2n-1}, \quad SU(n)/SO(n) \longrightarrow \mathbb{S}^{n^2+n-1}, \quad E_6/F_4 \longrightarrow \mathbb{S}^{53}.$$

Dans cette série E_6 et F_4 sont les groupes de Lie exceptionnels de dimension respective 78 et 52. Curieusement l'article [21] de B. Cheng présente les exemples E_6/F_4 et $SU(6)/Sp(3)$ mais «oublie» $SU(2n)/Sp(n)$ pour $n \neq 3$.

Un exemple isolé. – $\mathbb{S}^3/\mathbb{Z}_3 \longrightarrow \mathbb{S}^7$.

La construction de la première série d'exemples repose sur un fait bien connu : si $V \subset \mathbb{R}^n$ est une sous-variété quelconque, alors que l'application :

$$\begin{array}{rccc} \Phi : & NV & \longrightarrow & \mathbb{R}^n \oplus i\mathbb{R}^n \\ & \nu_x & \longmapsto & (x, i\nu) \end{array}$$

réalise un plongement lagrangien du fibré normal de V dans $\mathbb{C}^n$. L'image $\Phi(NV)$ est une sous-variété minimale si et seulement si V est une sous-variété *austère* de $\mathbb{R}^n$. Cette notion, introduite par R. Harvey et H. B. Lawson, signifie que pour tout vecteur normal unitaire ν, l'ensemble des valeurs propres de l'opérateur de forme dans la direction ν, est invariant par la multiplication par -1.

Supposons maintenant que l'on choisisse pour variété V, le cône épointé engendré par une sous-variété M de $\mathbb{S}^{n-1}$. On note $V = C^*(M)$. Alors $\Phi(NC^*(M))$ est une sous-variété lagrangienne minimale de $\mathbb{C}^n$, c'est donc un minimiseur. De plus, c'est également un cône sur

$$\widetilde{M} = \Phi(NC^*(M)) \cap \mathbb{S}^{2n-1}.$$

La variété $\widetilde{M}$ n'est cependant pas compacte, on peut bien sûr chercher à la compactifier mais rien ne garantit que l'on ne rajoute pas ainsi des points singuliers. C'est pourquoi on est amené à considérer une variante de cette construction.

On note NM le fibré normal de $M \subset \mathbb{S}^{n-1}$, $\widetilde{M}_+$, $\widetilde{M}_-$ deux copies de $NM^{\leq \frac{\pi}{2}} = \{\nu \in NM : |\nu| \leq \pi/2\}$ et $\widetilde{M}$ le double de $NM^{\leq \frac{\pi}{2}}$ i. e.

$$\widetilde{M} = \widetilde{M}_+ \cup \widetilde{M}_- / \partial\widetilde{M}_+ \sim \partial\widetilde{M}_-.$$

En particulier, si M est une hypersurface de $\mathbb{S}^{n-1}$ alors $\widetilde{M}$ est difféomorphe au produit $M \times \mathbb{S}^1$. On définit $\Psi : \widetilde{M} \to \mathbb{S}^{2n-1}$ par :

$$\Psi(\nu_x) = \begin{cases} (\epsilon \cos|\nu_x|\, x, \sin|\nu_x|\, \frac{\nu_x}{|\nu_x|}) & \text{if } \nu_x \neq 0, \\ (\epsilon x, 0) & \text{if } \nu_x = 0, \end{cases}$$

où $\epsilon = \pm 1$ selon que $x \in \widetilde{M}_\pm$. On montre alors que :

Proposition. – *Si pour tout vecteur normal unitaire ν, 0 n'est pas valeur propre de l'opérateur de Weingarten dans la direction ν, alors $\Psi : \widetilde{M} \longrightarrow \mathbb{S}^{2n+1}$ est une immersion legendrienne. Si de plus, $M \subset \mathbb{S}^{n-1}$ est austère, alors Ψ est minimale.*

On construit la première série d'exemples à partir de cette proposition en choisissant pour M des sous-variétés isoparamétriques de la sphère $\mathbb{S}^{n-1}$.

Les sous-variétés legendriennes minimales de la seconde série s'obtiennent à partir des orbites de la représentation isotropique de certains espaces symétriques hermitiens. Etant donnés un espace symétrique hermitien $X \simeq G/K$, $\rho : K \to U(\mathfrak{p})$ sa représentation isotropique, on note

$$K(p) = \{\rho(g)(p) \mid g \in K\} \subset \mathfrak{p} \simeq \mathbb{C}^{\dim X}$$

l'orbite de ρ passant par $p \in \mathfrak{p} \simeq T_{[id]}X$. En choisissant convenablement p et pour certains espaces symétriques on peut montrer que $K(p)$ est une sous-variété lagrangienne. On obtient des sous-variétés minimales legendriennes de $\mathbb{S}^{(\dim X)-1}$ en remplaçant K par un sous-groupe $K' \subset K$ qui vérifie en particulier la propriété :

$$\rho(K') \subset SU(\mathfrak{p}).$$

Cette dernière condition assure que la forme de Maslov est nulle et donc que la sous-variété est minimale.

L'article [Th.1] est un travail en commun avec B.-Y. Chen et Jean-Marie Morvan, il s'intéresse à la relation entre la courbure moyenne H et la courbure scalaire τ d'une sous-variété lagrangienne de $\mathbb{C}^n$. La recherche d'inégalités entre courbure extrinsèque et courbure intrinsèque est une des grandes activités de la théorie des sous-variétés, car on déduit de ces inégalités des obstructions à l'existence de certaines immersions isométriques. Depuis le milieu des années 90 le sujet a été revivifié avec la découverte par B.-Y. Chen d'une nouvelle famille d'inégalités comparant la courbure moyenne avec des avatars de la courbure scalaire [18]. La plus simple d'entre elles[8] s'énonce ainsi : en tout point d'une sous-variété de dimension n d'un espace euclidien, on a :

$$H^2 \geq \frac{2}{n}\tau.$$

Le cas d'égalité a lieu si et seulement si la sous-variété est un ouvert d'un n-plan. Notons que cette inégalité est indépendante de la codimension ; une application célèbre, et très connue, est la suivante : si M admet une immersion isométrique minimale dans un espace euclidien alors sa courbure scalaire est négative ou nulle.

Il s'avère que cette inégalité peut être améliorée si la sous-variété est lagrangienne.

Théorème. – *Soit $L \subset \mathbb{C}^n$ une sous-variété lagrangienne, en tout point de L on a :*

$$H^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau.$$

Le cas d'égalité permet de distinguer, outre les n-plans lagrangiens, les sphères de Whitney, c'est-à-dire les sous-variétés lagrangiennes qui, a un déplacement près, coïncident avec l'image de la sphère standard de $\mathbb{R}^{n+1}$ par l'application $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{C}^n$:

$$(x_0, x) \longmapsto \frac{1}{1+x_0^2}(x, x_0 x).$$

[8] qui était bien sûr connue avant les travaux de B.-Y. Chen...

Si n=1, l'image de l'application f n'est autre que la lemniscate de Bernouilli (équation polaire $r^2 = \cos 2\theta$), si $n > 1$, l'image de cette application est l'ensemble des points de $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ obtenus en faisant agir diagonalement $SO(n)$ sur les points $(y_0, y_1) \in \mathbb{R} \times \mathbb{R}$ de la lemniscate de Bernouilli. L'orbite d'un point $(y_0, y_1) \neq (0, 0)$ est isométrique à $SO(n)/SO(n-1) = \mathbb{S}^{n-1}$, en revanche, le point $(0, 0)$ correspondant à l'image des pôles Nord et Sud de $\mathbb{S}^n$ reste fixe.

Notons que le théorème ci-dessus avait été établi indépendamment par I. Castro et F. Urbano dans le cas de la dimension deux [23].

L'inégalité entre la courbure moyenne et la courbure scalaire d'une immersion lagrangienne découle d'une propriété remarquable des sous-variétés lagrangiennes, le 3-tenseur

$$t(X, Y, Z) =< h(X, Y), JZ >$$

est trisymétrique. Il se décompose donc orthogonalement en $t = t_0 + t_1$ où t_0 est à trace nulle, $t_1 = \frac{1}{n+2}(tr\ t).< .,. >$ et où l'écriture $\alpha.g$ désigne le 3-tenseur symétrique :

$$\alpha.g(X, Y, Z) = \alpha(X)g(X, Y) + \alpha(Y)g(Z, X) + \alpha(Z)g(X, Y).$$

Un calcul immédiat permet alors d'écrire :

$$\|h\|^2 = \|t\|^2 = \|t_0\|^2 + \frac{3n^2}{n+2}H^2$$

ce qui en remplaçant dans l'équation de Gauss : $2\tau = n^2H^2 - \|h\|^2$, donne le résultat. La caractérisation du cas d'égalité relève des techniques classiques en théorie des sous-variétés i. e. l'étude conjointe des équations de Gauss et de Codazzi. Il se trouve que le problème se ramène alors à une EDO du second ordre que l'on peut résoudre presque explicitement.

Après les résultats de [Th.1], B.-Y. Chen a traduit, dans le cas lagrangien, l'ensemble des inégalités qu'il avait découvert dans [18], cela lui a permis de mettre à jour de nouvelles obstructions à l'existence d'immersion lagrangienne minimale [19].

Le résultat de [Th.1] a également été généralisé dans le cas «contact» par D. Blair et A. Carriazo [5].

3. – Energie et volume des champs de vecteurs

Un champ de vecteurs V sur une variété M définit une immersion de la base M dans l'espace tangent TM. Si l'espace TM est muni d'une métrique riemannienne il est naturel de rechercher les champs de vecteurs pour lesquels cette immersion est minimale, et si M est elle-même une variété riemannienne, de rechercher les champs pour lesquels l'application V est harmonique. Les applications harmoniques (resp. les immersions minimales) étant critiques pour la fonctionnelle énergie (resp. la fonctionnelle volume), on est par conséquent amené à considérer ces deux fonctionnelles ainsi que leur restriction à l'espace des champs de vecteurs unitaires. Il existe de nombreuses situations en géométrie riemannienne où un champ de vecteurs particulier apparaît naturellement, par exemple, le champ de vecteurs unitaires tangents au flot géodésique, le champ radial dans le voisinage normal d'un point, le champ caractéristique d'une variété sasakienne, etc. On s'attend bien sûr à ce que ce type de champs aient des propriétés particulières relativement à l'énergie ou au volume. Ce n'est pas toujours le cas et des intuitions «évidentes» se révèlent parfois fausses : certains champs de Killing ne sont pas critiques pour l'énergie [39], le champ radial sur $\mathbb{C}P^n$ a une énergie infinie [14],... D'autre part de nombreuses questions élémentaires s'avèrent très délicates, par exemple on ne sait pas décrire[9] les champs de vecteurs de la sphère ronde standard qui sont critiques pour le volume ou pour l'énergie. Ceci explique probablement pourquoi le sujet connaît depuis les années 2000, un engouement certain (*cf.* [35],[36] pour un panorama actuel sur le sujet).

L'énergie. – Soit (M, g) une variété riemannienne compacte et orientée, l'énergie d'une application C^∞ $\varphi : (M, g) \longrightarrow (N, h)$ est l'intégrale :

$$E(\varphi) = \frac{1}{2}\int_M \mathrm{tr}(L_\varphi) vol_g$$

où $L_\varphi \in \Gamma(T^*M \otimes TM)$ est le champ d'endomorphismes défini par la relation :

$$(\varphi^*h)(X, Y) = g(L_\varphi(X), Y).$$

L'*énergie d'un champ de vecteur V C^∞* est l'énergie de l'application :

$$V : (M, g) \longrightarrow (TM, g^S)$$

où g^S est la métrique de Sasaki de TM. Cette métrique est définie par la relation :

$$g^S(\tilde{X}, \tilde{Y}) = g(d\pi(\tilde{X}), d\pi(\tilde{Y})) + g(K(\tilde{X}), K(\tilde{Y}))$$

où $\tilde{X}$ et $\tilde{Y}$ sont des vecteurs de TTM, $\pi : TM \to M$ est la projection et $K : TTM \to TM$ le connecteur associé à la connexion de Levi-Civita ∇ de g. Par conséquent, pour cette métrique le vertical et l'horizontal déterminés par la connexion ∇ sont perpendiculaires. Un calcul direct montre que, si n est la dimension de M, alors :

$$E(V) = \frac{1}{2}\int_M (n + \|\nabla V\|^2) dvol_g$$

où $dvol_g$ est l'élément de volume et $\|\nabla V\|^2 = \sum_{i=1}^n \|\nabla_{E_i} V\|^2$, $(E_1, ..., E_n)$ étant une base orthonormée locale. En particulier :

$$E(V) \geq \frac{n}{2} Vol(M),$$

avec l'égalité si et seulement si le champ V est parallèle ($\nabla V = 0$). D'autre part, on vérifie facilement que les champs critiques pour la fonctionnelle énergie sont précisément les champs parallèles, autrement dit, tous les champs critiques sont des minimiseurs.

[9] autrement qu'en disant qu'ils sont solutions d'une équation d'Euler-Lagrange

Il s'avère plus intéressant de considérer l'énergie, non plus des champs de vecteurs en général, mais des champs de vecteurs *unitaires*. Ceci suppose que la caractéristique d'Euler $\chi(M)$ de la variété M est nulle mais en contrepartie, on obtient une théorie plus riche. On constate, entre autres, que :

1) les champs de Killing unitaires ne sont pas nécessairement critiques,
2) il existe des champs de vecteurs unitaires critiques qui ne sont pas des champs de Killing [39].

Que se passe-t-il si l'on s'intéresse aux (éventuels) minimiseurs de l'énergie ? Pour les variétés[10] de dimension 2, la condition $\chi(M) = 0$ impose $M = \mathbb{T}^2$, et si l'on munit le tore d'une métrique plate, les minimiseurs de l'énergie sont alors les champs parallèles. Pour les variétés de dimension 3, le premier candidat est la sphère standard $\mathbb{S}^3 \subset \mathbb{R}^4$. Les champs de Hopf, c'est-à-dire les champs de vecteurs unitaires tangents aux fibres d'une fibration de Hopf :

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2$$

se révèlent être les minimiseurs :

Théorème (Brito, 2000, [12]). – *Dans $\mathbb{S}^3$, les champs de vecteurs unitaires C^∞ ayant la plus petite énergie sont les champs de Hopf et eux seuls.*

Pour les sphères $\mathbb{S}^{2m+1}$, $m > 1$, on dispose également de champs de Hopf (i. e. provenant d'une fibration de Hopf : $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{C}P^m$) mais ils *ne* réalisent *pas* le minimum de l'énergie. On montre en effet qu'ils sont critiques mais instables :

Théorème (Wood, 1997, [95] et [37] p. 536). – *Dans $\mathbb{S}^{2m+1}$, $m > 1$, les champs de Hopf sont des champs critiques instables de la fonctionnelle Energie.*

L'article [Bo.5] est un travail commun avec F. Brito (Université de São Paulo) et O. Gil-Medrano (Université de Valencia), il répond à la question de l'infimum de l'énergie pour les champs de vecteurs unitaires sur les sphères $\mathbb{S}^{2m+1}$, $m > 1$. Soit R le champ radial sur la sphère, c'est-à-dire le champ obtenu en prenant les vecteurs tangents unitaires aux méridiens. Bien que le champ R ait deux singularités, l'énergie d'un tel champ converge vers une valeur finie :

$$E(R) = (\frac{2m+1}{2} + \frac{m}{2m-1}) \text{ vol } (\mathbb{S}^{2m+1}).$$

Théorème. – *L'infimum de l'énergie pour les champs de vecteurs unitaires C^∞ de la sphère $\mathbb{S}^{2m+1}$ ($m \geq 2$) est $E(R)$. Cette valeur n'est atteinte par aucun champ de vecteurs unitaires sans singularité.*

La démonstration de ce théorème s'appuie sur un résultat de F. Brito et P. Walczak [16] qui énonce que si V est un champ de vecteurs unitaires C^∞ sans singularité alors

[10] connexes, compactes, orientables et sans bord

$E(V) > E(R)$. Il suffit donc de construire une famille $(R_\epsilon)_{\epsilon\in]0,1[}$ de champs de vecteurs unitaires C^∞ telle que :

$$\lim_{\epsilon\to 0} E(R_\epsilon) = E(R).$$

Pour cela, l'idée est de modifier R le long d'un voisinage tubulaire T_ϵ d'un segment géodésique joignant les deux points singuliers. Hors de T_ϵ, R reste inchangé ; à l'intérieur de T_ϵ, on transforme R de façon à éliminer les singularités. C'est topologiquement possible car, sur une sphère de dimension impaire, les indices des singularités sont opposés. La difficulté est de réaliser ceci avec un coût nul en énergie quand l'épaisseur ϵ du voisinage tubulaire tend vers zéro. Pour ce qui concerne la variation d'un tel champ, on aimerait construire le champ R_ϵ tel que, dans T_ϵ :

$$\|\nabla R_\epsilon\|^2 \sim \frac{1}{\epsilon^2}.$$

L'argument intuitif est le suivant : les lignes intégrales de R_ϵ auront, dans ce cas, une courbure de l'ordre du carré de l'inverse du rayon ϵ du voisinage tubulaire T_ϵ. Le coût en énergie d'une telle construction serait :

$$\int_{T_\epsilon} \|\nabla R_\epsilon\|^2 dvolg \sim Const \times \frac{1}{\epsilon^2} \times \mathrm{long}(N,S) \times \epsilon^{n-1}$$

où l'on a appelé N et S les deux points singuliers. Pour une simple raison dimensionnelle, si $n > 3$, ce coût tend vers zéro quand ϵ tend vers zéro. C'est, à quelques détails près, ce programme qui est réalisé dans [Bo.5] (dans cet article, le champ R_ϵ est maladroitement nommé T_R).

La technique d'élimination des singularités, mise au point dans le cas de la sphère, peut s'appliquer pour n'importe quel champ unitaire du moment que les singularités ont des indices deux à deux opposés. De façon plus générale, on peut proposer une voie d'attaque pour les problèmes de minimisation de l'énergie des champs de vecteurs :

1) définir un champ radial R *via* l'exponentiel, il aura forcément des singularités,

2) montrer qu'il minimise l'énergie des champs de vecteurs unitaires sans singularité,

3) construire une famille $(R_\epsilon)_{\epsilon\in\mathbb{R}_+^*}$ de champs sans singularité en «désingularisant» le champ radial et montrer qu'elle converge en énergie vers le champ radial : $\lim_{\epsilon\to 0} E(R_\epsilon) = E(R)$.

Après les sphères, les candidats les plus naturels qui semblent maintenant «atteignables» sont les espaces projectifs $\mathbb{R}P^{2m+1}$. Pour ces espaces, une partie de la difficulté réside dans la désingularisation : l'ensemble des points singuliers est un espace projectif $\mathbb{R}P^{2m}$ totalement géodésique dans $\mathbb{R}P^{2m+1}$ et la technique d'élimination développée pour les sphères ne s'applique pas. On pourrait alors préférer traiter le cas des espaces projectifs complexes $\mathbb{C}P^m$ car l'ensemble des points singuliers est alors $\mathbb{C}P^{m-1} \subset \mathbb{C}P^m$ qui est de codimension 2, mais un résultat surprenant, déjà cité, de E. Boeckx, J. C. González-Dávila et L. Vanhecke [14] montre que l'énergie du champ radial dans $\mathbb{C}P^m$ est infinie...

Le volume. – Le volume d'un champ de vecteurs C^∞ $V : (M,g) \to (TM, g^S)$ est le volume de la sous-variété $V(M)$ dans TM. L'intérêt pour cette quantité débute avec le résultat d'H. Gluck et W. Ziller :

Théorème (Gluck, Ziller, 1986, [40]). – *Dans $\mathbb{S}^3$, les champs de vecteurs unitaires C^∞ qui ont le plus petit volume sont les champs de Hopf et eux seuls.*

Pour obtenir ce résultat, H. Gluck et W. Ziller utilisent la théorie des calibrations. Ils montrent que si H est un champ de Hopf, la sous-variété $H(M)$ est une φ-sous-variété pour une certaine calibration φ. Leur démonstration ne se généralise pas au cas des sphères de dimension impaire, les φ-sous-variétés de plus petit volume n'étant plus des sections de $T^1M \to M$. Il est néanmoins tentant de faire la conjecture suivante, dite de Gluck et Ziller : *dans $\mathbb{S}^{2m+1}$, $m > 1$, les champs de vecteurs unitaires qui ont le plus petit volume sont les champs de Hopf et eux seuls.* Cette conjecture se révèle inexacte, D. Jonhson montre en 1988 [61] que les champs de Hopf de $\mathbb{S}^5$ sont des points critiques instables de la fonctionnelle Volume et ce résultat est ensuite étendu aux spheres de dimension plus grande par O. Gil-Medrano et E. Llinares-Fuster en 2001 [37].

Le parallèle de cette situation avec celle de l'énergie est frappant. Il est donc naturel d'adopter une démarche similaire à celle du chapitre précédent. Dans [13] il est montré que si V est un champ sans singularité alors :

$$Vol(V) > Vol(R)$$

où R est le champ radial sur la sphère. La famille (R_ϵ) construite pour approcher le champ radial pourrait, par conséquent, donner également la réponse au problème de l'infimum de la fonctionnelle Volume. Mais les choses ne sont pas aussi simples, un calcul direct montre que :

$$\lim_{\epsilon \to 0} Vol(R_\epsilon) = Vol(R) + \mathrm{long}(N, S) \times Vol(\mathbb{S}^{2m}(1)),$$

ce qui signifie que la famille (R_ϵ) ne converge pas en volume vers R, il y a un terme correctif à prendre en compte.

L'article [Bo.6] est un travail commun avec O. Gil-Medrano. L'idée principale de ce travail est de considérer la fonctionnelle Volume sur des sphères de rayon r quelconque et non plus de rayon 1. Si V est un champ de vecteurs unitaires sur $\mathbb{S}^{2m+1}(1)$, on note V^r le champ de vecteurs unitaires correspondant sur $\mathbb{S}^{2m+1}(r)$. On a alors :

$$Vol(V^r) = \int_{\mathbb{S}^{2m+1}(1)} \sqrt{r^{2n} + r^{2n-2}\sigma_1(M) + ... + r^2\sigma_{2m}(M)}\, dvol$$

où $n = 2m+1$, $M = {}^T\nabla V \circ \nabla V$ et les σ_i sont les polynômes symétriques élémentaires des valeurs propres de M. Un développement asymptotique montre alors immédiatement que lorque r est grand, l'application $r \mapsto Vol(V^r)$ est controlée par le *bending* [11] :

$$B(V) = \int_{\mathbb{S}^{2m+1}(1)} \sigma_1(M)\, dvol = \int_{\mathbb{S}^{2m+1}(1)} \|\nabla V\|^2 dvol$$

[11]Le terme *bending* – que l'on peut traduire par *infléchissement* – apparaît pour la première fois dans l'article de G. Wiegmink [93]. Je ne connais cependant pas sa traduction usuelle en mathématiques «francophones», ce terme figure donc en anglais dans le texte.

et que, quand r est petit, cette application est controlée par :

$$Tw(V) = \int_{\mathbb{S}^n(1)} \sqrt{\sigma_{n-1}(\,{}^T\nabla V \circ \nabla V)}\, dvol,$$

que l'on appelle dans [Bo.6], le *twisting* – ou encore l'*entortillement* – du champ de vecteurs V.

Théorème de l'entortillement. – *On note H un champ de Hopf sur $\mathbb{S}^n(1)$.*

1) Pour tout champ de vecteurs unitaires V de $\mathbb{S}^n(1)$, $Tw(V) \geq Tw(H)$.

2) Si $Tw(V) > Tw(H)$ alors il existe $r_0 > 0$ tel que pour tout $r < r_0$, on ait : $Vol(V^r) > Vol(H^r)$.

Notons que le *bending* est, à une constante près, l'énergie. Les résultats de [Bo.5], combinés à ceux de [16], permettent ainsi d'établir un théorème analogue pour le *bending*.

Théorème du *bending*.– *On note R un champ radial sur $\mathbb{S}^n(1) \setminus \{$ deux points antipodaux $\}$ et (R_ϵ) la famille construite dans [Bo.5]. Soit V un champ de vecteurs unitaires de $\mathbb{S}^n(1)$, (n impair) alors :*

1) $B(V) > \lim_{\epsilon \to 0} B(R_\epsilon) = B(R)$,

2) Il existe $\epsilon > 0$ et $r_0 > 0$ tels que, pour tout $r > r_0$, $Vol(V^r) > Vol(R_\epsilon^r)$.

De manière abusive, mais suggestive, on peut résumer ces deux théorèmes de la façon suivante : «Les champs de Hopf sont des minimiseurs du volume pour $r = 0$, de même, le champ radial est un minimiseur pour $r = \infty$». Il paraît alors naturel de s'interroger sur la stabilité des champs de Hopf, non plus pour $r = 1$, mais pour toute valeur de r.

Théorème de la stabilité. – *Soit $n \geq 5$. Le champ de Hopf sur $\mathbb{S}^n(r)$ est stable si et seulement si :*

$$r \leq \frac{1}{\sqrt{n-4}}.$$

Le théorème de l'entortillement et le théorème de la stabilité donnent deux arguments en faveur d'une conjecture de Gluck et Ziller mais pour de petits rayons. Il se pourrait que les champs de Hopf fournissent des minimiseurs du volume à condition que r soit suffisamment petit.

La démonstration du théorème de l'entortillement fait intervenir des inégalités ponctuelles et une inégalité globale due à F. Brito, R. Langevin et H. Rosenberg [15]. La démonstration du théorème de la stabilité est plus délicate. L'idée de départ est de s'inspirer de la démonstration de H. B. Lawson établissant la stabilité des sous-variétés complexes d'une variété kählérienne $(\overline{M}, J)$, [66]. Si A est un champ normal à M sous-variété complexe de $\overline{M}$, alors la hessienne pour le volume s'écrit :

$$\text{Hessienne du Volume}\,(A,A) = \frac{1}{2}\int_M \|D^{\mathbb{C}}A\|^2 dvol$$

où $D^{\mathbb{C}}_X A = (\nabla_{JX} - J\nabla_X)A$. Par conséquent, elle est positive et nulle si A est holomorphe. On écrit donc la hessienne du volume pour les champs de vecteurs sous une forme faisant apparaître $D^{\mathbb{C}}A$, *grosso modo* :

$$Hess_H Vol(A,A) = (\ \ast\ ) + \int \|D^{\mathbb{C}}A\|^2 \qquad (1).$$

Il s'avère que cette écriture ne permet pas de conclure à la positivité de la hessienne. On peut alors décider de privilégier $\overline{D}^{\mathbb{C}}_X A = (\nabla_{JX} + J\nabla_X)A$. Mais la nouvelle expression :

$$Hess_H Vol(A,A) = (\ \ast\ast\ ) + \int \|\overline{D}^{\mathbb{C}}A\|^2 \qquad (2)$$

ne donne pas non plus la stabilité. Néanmoins la fibration : $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{C}P^m$ permet de décomposer A en série de Fourier sur la fibre :

$$A = \sum_{l\in\mathbb{Z}} A_l.$$

On montre alors que :

$$Hess_H Vol(A,A) = \sum_l Hess_H Vol(A_l, A_l).$$

On termine la démonstration en constatant que pour certaines valeurs de l, l'expression (1) montre que $Hess_H Vol(A_l, A_l)$ est positive (si r petit), pour les autres valeurs, c'est l'expression (2) qui donne la positivité.

Pour chaque sphère de dimension impaire plus grande ou égale à 5, il existe donc un rayon «critique» à partir duquel le champ de Hopf devient instable. Que se passe-t-il pour d'autres variétés ? Par exemple, si (M, g, ξ) est une variété de sasaki, le champ caractéristique ξ est critique pour la fonctionnelle Volume et il reste critique si l'on change g en $\lambda^2 g$, $\lambda \in \mathbb{R}^*_+$. On peut alors se demander quelle est la stabilité de ξ quand λ varie. Il se pourrait que les choses se comportent de manière assez similaire au cas de la sphère. Les variétés de Brieskorn – qui englobent toutes les sphères exotiques de dimension impaire bordant une variété parallélisable compacte et orientable – pourrait fournir un panel d'exemples intéressants.

Au delà des variétés sasakiennes, on peut également penser aux variétés lorentziennes ; on a en effet sur de tels objets un champ de vecteurs unitaires privilégié qui est critique pour une généralisation, dans ce cadre, de la fonctionnelle Volume.

Néanmoins, tout ceci ne doit pas faire oublier que la question de l'infimum du volume des champs de vecteurs sur la sphère reste ouverte...

Les articles

On trouvera dans les pages qui suivent les textes des articles [Bo.1] à [Bo.6] tels qui sont parus (ou devraient paraître pour [Bo.4] et [Bo.6]).

Les rapporteurs m'ont suggéré de nombreuses remarques et/ou corrections. J'ai répertorié ci-dessous celles se rapportant directement aux articles, elles aideront certainement le lecteur aux endroits où ma rédaction a été trop rapide ou fautive.

1.– Dans [Bo.2] section 2.2, la phrase «Since $JTM^n \oplus TM^n = \mathbb{C}^n$, every manifold admitting a totally real embedding necessarily has a zero Euler-Poincaré characteristic» peut porter à confusion. Je rappelle ici l'argument complet (*cf.* [3] p 594 et 597). On se donne une variété compacte connexe orientée sans bord M^n. Si la dimension de M est impaire, il n'y a rien à dire. Supposons donc que la dimension soit paire. Pour toute immersion $f : M^n \longrightarrow \mathbb{R}^{2n}$, on a la formule suivante :

$$d(f) = -\frac{1}{2} < e(Nf), [M^n] >$$

où $d(f)$ est le nombre de point doubles algébriques d'une approximation de f à croisement normaux et $e(Nf)$ la classe d'Euler du fibré normal. Si l'immersion est totalement réelle, la condition $JTM^n \oplus TM^n = \mathbb{C}^n$ implique qu'il existe un isomorphisme entre TM et Nf, celui-ci multiplie l'orientation par $(-1)^{\frac{n(n-1)}{2}}$, donc si $n = 2k$:

$$d(f) = (-1)^{k+1}\frac{\chi(M)}{2}.$$

Par conséquent, si M admet un plongement totalement réel alors la caractéristique d'Euler-Poincaré $\chi(M)$ doit être nulle.

2.– Toujours dans [Bo.2], après le lemme B, il est déduit du théorème de J. W. Gray que les formes de contact $f_n^*\xi$ et ξ sont isotopes. Cela est inexact, seules les structures de contact sous-jacentes sont isotopes. Pour obtenir l'isotopie des structures de contact, il faut, qu'en plus, les orbites des champs de Reeb soient isotopes (ou les noyaux des differentielles). C'est bien ce qui se passe ici, car $df_n^*\xi = d\xi$ (*cf.* les premières lignes de la démonstration du lemme B).

3.– Dans [Bo.3], après le corollaire 2.1, un «raccourci» du même ordre que celui relevé dans [Bo.2] section 2.2, peut porter à confusion. Quand il est affirmé que $I^0_{TR} = I_{TR}$, on suppose bien sûr que M^n admet un plongement totalement réel dans $\mathbb{C}^n$.

4.– Dans [Bo.3], après le lemme B, il est affirmé que le groupe fondamental de l'espace des immersions $I = I(M^n, \mathbb{R}^{2n})$ est abélien mais aucune référence n'est donnée. Ceci peut se déduire des travaux de J.-P. Dax [25] sur l'homotopie des espaces de plongements. En fait, J.-P. Dax montre que le groupe $\pi_2(H, I)$ où $H = Map(M^n, W^N)$ et $I = I(M^n, W^N)$, est isomorphe à un certain groupe de cobordisme (p.374) or ce dernier groupe est abélien (p.308-309) sous certaines hypothèses sur n et N. Si $N = 2n$, ces hypothèses sont vérifiées pour $n \geq 3$ ce qui est précisément ce dont on a besoin dans [Bo.3]. Si $W^N = \mathbb{R}^N$, la suite exacte longue d'homotopie de la paire (H, I) montre que le $\pi_1(I)$ et le $\pi_2(H, I)$ sont isomorphes.

Linking class of Lagrangian or totally real embeddings

Ann. Global. Anal. Geom. 17 (1999), 371-384.

Abstract. –Given a totally real embedding j of the 2-torus $\mathbb{T}^2$ into $\mathbb{C}^2$, one defines a 1-class σ_1 - its *linking class* - which is a tool to detect arcwise connected components of the space of totally real embeddings $EmbTR(\mathbb{T}^2, \mathbb{C}^2)$. We generalize the construction of the linking class to any totally real embedding j of a connected, oriented, compact manifold without boundary M^n into $\mathbb{C}^n$. We obtain a $(n-1)$-class σ_{n-1} which is still an invariant for isotopy classes of totally real embeddings. We show that this class is non trivial by computing it for some families of totally real embeddings. We then study the relationship between isotopy classes of ordinary embeddings and the linking class. With additional assumptions on M^n ($n \geq 4$ and M^n parallelizable) we obtain the following : Two totally real embeddings of M^n into $\mathbb{C}^n$ which belong to the same isotopy class of totally real immersion, belong to the same isotopy class of ordinary embedding if and only if

1) their linking classes are the same (if n odd)

2) the images of their linking classes by the coefficient homomorphism $\mu : H^{n-1}(M^n, \mathbb{Z}) \rightarrow H^{n-1}(M^n, \mathbb{Z}_2)$ are the same (if n even).

Key words . – Embedding, Linking, Totally real embedding, Lagrangian embedding.

MSC 1991. – 57R40 (53C15, 58F05).

1.1 Introduction and main results.

Given any totally real embedding j of the torus $\mathbb{T}^2$ into $\mathbb{C}^2$ and any non-vanishing vector field v in $\mathbb{T}^2$, one can construct a 1-class $\sigma_1(j, v) \in H^1(\mathbb{T}^2, \mathbb{Z})$ called *the linking class* of j in the direction v : if $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{T}^2$ is a loop which represents a generator of $H_1(\mathbb{T}^2, \mathbb{Z})$, one defines a differential map h_ϵ by :

$$\begin{array}{rccc} h_\epsilon : & \mathbb{S}^1 & \longrightarrow & \mathbb{C}^2 \\ & t & \longmapsto & j \circ \gamma(t) + J\epsilon v_{\gamma(t)} \end{array}$$

where J is the complex structure of $\mathbb{C}^2$. For sufficiently small ϵ, this map is injective and $h_\epsilon(\mathbb{S}^1) \cap j(\mathbb{T}^2) = \emptyset$. Thus, one can speak about the linking of h_ϵ and j. One puts $\sigma_1(j, v)([\gamma]) = lk(j, h_\epsilon)$ (cf. [28] for instance). This class is an effective tool to detect totally real isotopy classes of $\mathbb{T}^2$ which are not detected by the Maslov class. Nevertheless, if j is a *Lagrangian* embedding and τ is a translation field on $\mathbb{T}^2$, one has, according to Y. Eliashberg and L. Polterovich, $\sigma_1(j, \tau) = 0$ ([28]). This result does not generalize to totally real embeddings ([30],[79]).

In the first part of this article, we generalize the construction of the linking class of a torus to any (compact, oriented, without boundary) manifold M^n admitting a totally real embedding j in $\mathbb{C}^n$. We associate to every totally real embedding $j : M^n \longrightarrow \mathbb{C}^n$ and to every non vanishing vector field v on M^n an element $\sigma_{n-1}(j,v)$ of $H^{n-1}(M^n,\mathbb{Z})$: its linking class. As in the case of the torus $\mathbb{T}^2$, the linking class σ_{n-1} allows us to detect arcwise connected components of the space of totally real embeddings $EmbTR(M^n,\mathbb{C}^n)$.

In the second part of this article, we show the non triviality of this linking class by building families of totally real embeddings into $\mathbb{C}^n$ of (topological) product manifolds of the form $M^{n-1}\times\mathbb{S}^1$. We get the following

Theorem 1-2. – *Let M^{n-1} be a compact, oriented manifold without boundary. Let v be a non-vanishing vector field of $M^{n-1}\times\mathbb{S}^1$ tangent to the factor $\mathbb{S}^1$ of $M^{n-1}\times\mathbb{S}^1$.*

1. *If n is even and if M^{n-1} admits a totally real immersion into $\mathbb{C}^{n-1}$ with an algebraic self-intersection number equals to 0 (resp. equals to 1), then, for all $p\in\mathbb{Z}$, there exists a totally real embedding $j : M^{n-1}\times\mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that*

$$<\mu(j),[\mathbb{S}^1]>=2 \text{ and } <\sigma_{n-1}(j,v),[M^{n-1}]>=2p \quad (\text{resp. } 2p+1),$$

   ($\mu(j)$ denotes the Maslov class of j).

2. *If n is odd and if M^{n-1} is an hypersurface of $\mathbb{R}^n$ such that $\frac{\chi(M^{n-1})}{2}\equiv 1 \bmod 2$, then, there exists two totally real embeddings $j_0,j_1 : M^{n-1}\times\mathbb{S}^1\longrightarrow\mathbb{C}^n$ such that :*

$$<\sigma_{n-1}(j_0,v),[M^{n-1}]>=0,$$

$$<\sigma_{n-1}(j_1,v),[M^{n-1}]>\neq 0.$$

As an illustration of this theorem, we obtain

Corollary 1.– *Let M^n be one of the following manifolds :*
1) $\mathbb{S}^1\times\mathbb{S}^{2q-1}$,
2) the unitary group $U(2q)$,
3) the oriented Lagrangian Grassmannian $\Lambda(k)$ with $k=4q$ or $k=4q+3$,
Then, the space $EmbTR(M^n,\mathbb{C}^n)$ has a countable family of arcwise connected components which are not detected by the Maslov class.

These results broaden those of L. Polterovich in [79]. In the third part of this article, we study the relationship between isotopy classes of *ordinary* embeddings and the linking class. In the case where M^n is parallelizable, we obtain the following

Theorem 3.– *Let M^n ($n\geq 4$) be a compact, parallelizable manifold without boundary. Let j_0,j_1 be two totally real (or Lagrangian) embeddings which belong to the same regular homotopy class of totally real (or Lagrangian) immersions. Then, the following propositions are equivalent :*

1. *j_0 and j_1 belong to the same isotopy class of ordinary embeddings,*

2. *there exists a non-vanishing vector field v on M^n such that :*
$$\begin{cases} \sigma_{n-1}(j_0, v) & = & \sigma_{n-1}(j_1, v) & \text{if } n \text{ is odd,} \\ \mu^*(\sigma_{n-1}(j_0, v)) & = & \mu^*(\sigma_{n-1}(j_1, v)) & \text{if } n \text{ is even,} \end{cases}$$

($\mu^* : H^{n-1}(M^n, \mathbb{Z}) \longrightarrow H^{n-1}(M^n, \mathbb{Z}_2)$ denotes the coefficients homomorphism).

Remark 1. – In fact, if $\sigma_{n-1}(j_0, v) = \sigma_{n-1}(j_1, v)$ for one non-vanishing vector field v, then $\sigma_{n-1}(j_0, w) = \sigma_{n-1}(j_1, w)$ for every non-vanishing vector field w.

This article is an expands upon the last part of my thesis. I wish to express my gratitude to my adviser, Jean-Marie Morvan, for the encouragement and help he gave me throughout this period. I would also like to thank Francois Laudenbach for reading a preliminary version of this manuscript and for suggesting several corrections.

1.2 The linking class σ_{n-1}.

1.2.1 Notations and definitions.

In what follows, we denote by M^n a compact, oriented manifold without boundary and by T^1M^n its unitary bundle (for some metric g which plays no role). We also denote by Ω the symplectic form of $\mathbb{C}^n$ and by J the canonical complex structure. An immersion $j : M^n \longrightarrow \mathbb{C}^n$ is said to be *Lagrangian* if $j^*\Omega = 0$. It is said to be *totally real* if :

$$\forall p \in M^n, T_{j(p)}j(M^n) \oplus JT_{j(p)}j(M^n) = \mathbb{C}^n.$$

We denote by $Imm(M^n, \mathbb{R}^{2n})$ the space of immersions of M^n in $\mathbb{R}^{2n}$, by $Emb(M^n, \mathbb{R}^{2n})$ the space of embeddings, by $ImmTR(M^n, \mathbb{C}^n)$ the space of totally real immersions, by $ImmLag(M^n, \mathbb{C}^n)$ the space of Lagrangian immersions, etc.

1.2.2 Construction of σ_{n-1}.

Let $j : M^n \longrightarrow \mathbb{R}^{2n} \simeq \mathbb{C}^n$ be a totally real embedding and v a section of T^1M^n. (Such a section exists because the Euler-Poincaré characteristic $\chi(M^n)$ of a manifold admitting a totally real embeddings is 0).We build, from j, a family of differentiable maps j_ϵ :

$$\begin{array}{rccc} j_\epsilon : & M^n & \longrightarrow & \mathbb{C}^n \\ & b & \longmapsto & j(b) + \epsilon J dj_b(v). \end{array}$$

Since j is a totally real embedding, $j_\epsilon(M^n) \cap j(M^n) = \emptyset$ for small ϵ. The (oriented) manifold $j_\epsilon(M^n)$ represents a n-cycle of $\mathbb{C}^n \setminus j(M^n)$. The exact sequence of the pair $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ and the Alexander duality allow us to make the identifications :

$$H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H_{n+1}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

Definition. – The *linking class* associated to the totally real embedding j and to the non-vanishing vector field v is the class $\sigma_{n-1}(j, v) \in H^{n-1}(M^n, \mathbb{Z})$ represented in $H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z})$ by the n-cycle $[j_\epsilon(M^n)]$.

The class σ_{n-1} is obviously invariant for isotopy classes of totally real embeddings. Indeed, it is easy to check that if $j_1, j_2 : M^n \longrightarrow \mathbb{C}^n$ are two totally real embeddings and if v is an non-vanishing vector field on M^n such that $\sigma_{n-1}(j_1, v) \neq \sigma_{n-1}(j_2, v)$, then j_1 and j_2 are not isotopic through totally real embeddings.

Remark 2. – One can broaden this construction to obtain other invariants for isotopy classes of totally real embeddings. Let $0 < k < n-1$. Assume that T^1M^n admits a $(n-k-1)$-sphere subbundle whose total space $S^{n-k-1}M^n$ is oriented. The manifold $S^{n-k-1}M^n$ is connected, compact, without boundary and $(2n-k-1)$ dimensional. Denote by $[S^{n-k-1}M^n]$ the element of $H_{2n-k-1}(T^1M^n, \mathbb{Z})$ represented by $S^{n-k-1}M^n$. One can build a family of differential maps :

$$\begin{array}{cccc} j_\epsilon : & S^{n-k-1}M^n & \longrightarrow & \mathbb{C}^n \\ & (b, v) & \longmapsto & j(b) + \epsilon J dj_b(v). \end{array}$$

The (oriented) manifold $j_\epsilon(S^{n-k-1}M^n)$ represents a $(2n-k-1)$-cycle of $\mathbb{C}^n \setminus j(M^n)$ (for small ϵ).The exact sequence of the pair $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ and the Alexander duality allow us to make the identifications :

$$H_{2n-k-1}(\mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H_{2n-k}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H^k(M^n, \mathbb{Z}).$$

One can define the k-th *linking class* associated to the totally real embedding j and to the oriented $(n-k-1)$-sphere bundle $S^{n-k-1}M^n$ to be the class $\sigma_k(j, S^{n-k-1}M^n) \in H^k(M^n, \mathbb{Z})$ represented in $H_{2n-k-1}(\mathbb{C}^n \setminus j(M^n), \mathbb{Z})$ by the $(2n-k-1)$-cycle $[j_\epsilon(S^{n-k-1}M^n)]$.

1.3 Non triviality of σ_{n-1}.

1.3.1 Preliminaries.

Let $j : M^n \longrightarrow \mathbb{R}^{2n}$ be an immersion such that all self-intersection points are isolated, double and transversal. We denote by $I_1, ..., I_N$ each of the double points of $j(M^n)$ and by $p_1, q_1, ..., p_N, q_N$ the $2N$ points of M^n such that $j(p_k) = j(q_k) = I_k$. We also denote by $(e_1, ..., e_{2n}, e_{2n+1})$ the canonical basis of $\mathbb{R}^{2n} \oplus \mathbb{R}$. We want to alter the immersion j into an embedding of M^n in $\mathbb{R}^{2n+1}$ by "pushing out" $j(p_k)$ and $j(q_k)$ along the line spanned by e_{2n+1}, but in opposite directions. For each couple (p_k, q_k), there are two ways to do that. Either, one deforms a neighbourhood of p_k in the direction $+e_{2n+1}$ and a neighbourhood of q_k in the direction $-e_{2n+1}$, or one does the opposite. Actually there are 2^N different ways to alter j into an embedding by this process. Let θ be a sequence of N signs $+$ or $-$. We define a function $\alpha_\theta : M^n \longrightarrow \mathbb{R}$ such that if the k-th term of the sequence θ is a $+$, α_θ is a positive bump function on a small disk $D^n(p_k)$ of M^n centered in p_k and a negative bump function on a small disk $D^n(q_k)$ centered in q_k. If the k-th term is a $-$, it is the opposite. Out of these $2N$ disks, we put : $\alpha_\theta = 0$. We define an embedding j_θ of M^n in $\mathbb{R}^{2n+1}$ by

$$\begin{array}{cccc} j_\theta : & M^n & \longrightarrow & \mathbb{R}^{2n} \oplus \mathbb{R} \\ & x & \longmapsto & (j(x), \alpha_\theta(x)), \end{array}$$

and, for every $\epsilon > 0$, an immersion $j_{\theta,\epsilon}$ by

$$\begin{array}{rccc} j_{\theta,\epsilon}: & M^n & \longrightarrow & \mathbb{R}^{2n} \oplus \mathbb{R} \\ & x & \longmapsto & (j(x), \alpha_\theta + \epsilon). \end{array}$$

If $\|\alpha_\theta\|_1$ and ϵ are sufficiently small, the field e_{2n+1} is everywhere transversal to j_θ. Hence, $j_{\theta,\epsilon}$ is an embedding and moreover $j_\theta(M^n) \cap j_{\theta,\epsilon}(M^n) = \emptyset$.

Lemma 1. – *Let Lk be the following map :*

$$\begin{array}{rccc} Lk: & \{0,1,...,2^N-1\} & \longrightarrow & \mathbb{Z} \\ & \theta & \longmapsto & Linking(j_\theta, j_{\theta,\epsilon}). \end{array}$$

One has :

	N *even*	N *odd*
n *even*	$Im\ Lk = \{2k\}$ *with* $-N \leq 2k \leq N$	$Im\ Lk = \{2k+1\}$ *with* $-N \leq 2k+1 \leq N$
n *odd*	$Im\ Lk = \{-N, ..., -2, 0, 2, ..., N\}$	$Im\ Lk = \{-N, ..., -1, 1, ..., N\}$

In particular, $Linking(j_\theta, j_{\theta,\epsilon})$ has the same parity as N. If N is odd, it can not be zero.

Proof of lemma 1. – We give here the main arguments of the proof. One puts $M^n_\theta = j_\theta(M^n_\theta)$ and $M^n_{\theta,\epsilon} = j_{\theta,\epsilon}(M^n_{\theta,\epsilon})$. We first assume that there exists a submanifold W^{n+1} of $\mathbb{R}^{2n} \oplus \mathbb{R}$ such that $\partial W^{n+1} = M^n_\theta$. The following formula is well-known :

$$Lk(M^n_\theta, M^n_{\theta,\epsilon}) = (-1)^n W^{n+1}.M^n_{\theta,\epsilon}. \qquad (*)$$

(Here $W^{n+1}.M^n_{\theta,\epsilon}$ denotes the algebraic intersection number of W^{n+1} and $M^n_{\theta,\epsilon}$). We assume besides that there are exactly N intersection points and that all of them are transversal. When deforming $M^n_{\theta,\epsilon}$ around an intersection point from the direction $+e_{2n+1}$ to the direction $-e_{2n+1}$, the sign of the index of the intersection point changes if n is odd but remains unchanged if n is even. This explains the values that appear in the array.
In general, there does not exist submanifold W^{n+1} of $\mathbb{R}^{2n} \oplus \mathbb{R}$ such that $\partial W^{n+1} = M^n_\theta$. Nevertheless, one can always build a submanifold W^{n+1} with the following properties :

1) $\partial W^{n+1} = M^n_\theta \cup V^n_\theta$ where V^n_θ is a submanifold of $\mathbb{R}^{2n} \oplus \mathbb{R}$, such that $Lk(V^n_\theta, M^n_{\theta,\epsilon}) = 0$.

2) there are exactly N intersection points of W^{n+1} with $M^n_{\theta,\epsilon}$ and all of them are transversal.

The condition on V^n_θ allows us to use equality $(*)$. Hence, the previous arguments apply. □

1.3.2 Self-intersection of totally real immersions.

A Lagrangian model

Let

$$\begin{array}{cccc} \gamma: & I=[0,1] & \longrightarrow & \mathbb{R}\oplus i\mathbb{R} \\ & t & \longmapsto & (x(t),y(t)) \end{array}$$

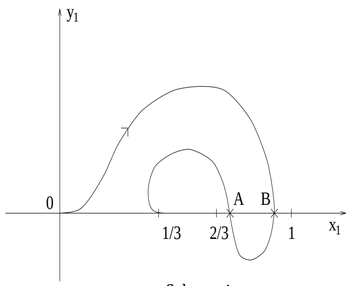
be any curve of $T^*\mathbb{R}$ such that :

$$\begin{array}{llll} 1) & \forall t\in[0,\epsilon]\ :\ \gamma(t)=(t,0), & 2) & \gamma([0,1])\subset\{x\geq 0\}, \\ 3) & \gamma([1-\epsilon,1])\subset\{x>0,y=0\}, & 4) & \|\frac{\partial\gamma}{\partial t}(t)\|>0 \text{ for all } t\in[0,1]. \end{array}$$

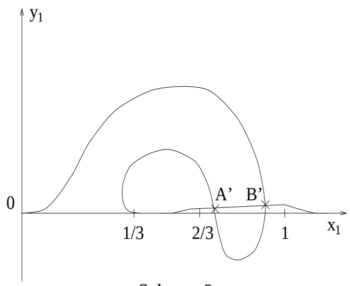
One denotes by D^{n-1} the $(n-1)$-dimensional closed disk of $\mathbb{R}^{n-1}\oplus i\{0\}$ with radius 1 and centered at 0 and by $(t,u)\in\mathbb{R}^*_+\times\mathbb{S}^{n-2}$ the spherical coordinates in $\mathbb{R}^{n-1}\setminus\{0\}$. We define a Lagrangian immersion f_γ of D^{n-1} into $\mathbb{R}^{n-1}\oplus i\mathbb{R}^{n-1}$ (with boundary lying inside $\mathbb{R}^{n-1}\oplus i\{0\}$) by putting :

$$\begin{cases} f_\gamma(t,u)=(x(t)u,y(t)u) & \text{if } (t,u)\in D^{n-1}\setminus\{0\}, \\ f_\gamma(0)=0. \end{cases}$$

Let Π be the Lagrangian n-plane $\{y_1=0,...,y_n=0\}$ and γ be one suitable parametrization of the following curve (scheme 1) :



Scheme 1



Scheme 2

The intersection set between f_γ and Π contains two $(n-2)$-spheres : $\mathcal{C}_A=\{uA/u\in\mathbb{S}^{n-2}\}$ and $\mathcal{C}_B=\{uB/u\in\mathbb{S}^{n-2}\}$. One can perturb a little this situation to obtain four transversal double points instead of these two $(n-2)$-spheres. (For example, change the n-plane Π to $\Pi_\epsilon=\{y_1=\epsilon x_1, y_2=\epsilon,...,y_n=\epsilon\}$ in the region $\{2/3\leq\|x\|\leq 1\}$ and smooth the corners, *cf.* scheme 2).

Adding self-intersection points

Let $j: M^{n-1}\longrightarrow\mathbb{C}^{n-1}$ be a Lagrangian immersion having $N(j)$ double points. From j, it is not difficult to build a totally real immersion h having more self-intersection points :

1) We apply the theorem of the symplectic tubular neighbourhood of Weinstein ([91]) to glu a copy of the Lagrangian model f_γ in the neighborhood of a regular point of j. We call j_1 the resulting immersion.

2) We pertub j_1 a little to obtain a new (totally real) immersion h which has only

transversal self-intersection points. According to §1.3.2 there exists (at least) one perturbation which add four intersection points.

1.3.4 The class σ_{n-1} of some embeddings of $M^{n-1} \times \mathbb{S}^1$ with n even.

Let $f : M^n \longrightarrow \mathbb{C}^n$ be an immersion. We denote by $I(f)$ the *algebraic* number of double points of any approximation of f with transversal double points. If n is even (and M^n orientable), $I(f) \in \mathbb{Z}$, otherwise, $I(f) \in \mathbb{Z}_2$.

Theorem 1. – *Let M^{n-1} be a compact, oriented odd dimensional manifold without boundary. If M^{n-1} admits a totally real immersion j into $\mathbb{C}^{n-1}$ such that $I(j) = 0$ (resp. such that $I(j) = 1$), then, for all $p \in \mathbb{Z}$, there exists a totally real embedding $h : M^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that*

$$< \mu(h), [\mathbb{S}^1] >= 2 \ \textit{and} \ < \sigma_{n-1}(h, v), [M^{n-1}] >= 2p \quad (\textit{resp.} \ \ 2p + 1),$$

($\mu(h)$ denotes the Maslov class of h and v a non-vanishing vector field of $M^{n-1} \times \mathbb{S}^1$ tangent to the factor $\mathbb{S}^1$).

Proof of theorem 1. – Let $j : M^{n-1} \longrightarrow \mathbb{C}^{n-1}$ be a totally real immersion and let N be the number of self-intersection points of j. As in §3.1, we associate to j the embedding :

$$\begin{array}{llll} j_\theta : & M^{n-1} & \longrightarrow & \mathbb{R}^{2n-2} \oplus \mathbb{R} \subset \mathbb{C}^{n-1} \times \mathbb{C} \\ & x & \longmapsto & (j(x), 1 + \alpha_\theta(x)). \end{array}$$

Without loss of generality, we can always assume that $j_\theta(M^{n-1})$ lies inside $\mathbb{R}^{2n-2}_+ \oplus \mathbb{R} = \{x_1 > 0, ..., x_{2n-2} > 0\}$. We put

$$\begin{array}{llll} h_\theta : & M^{n-1} \times \mathbb{S}^1 & \longrightarrow & \mathbb{C}^{n-1} \times \mathbb{C} \\ & (x, \varphi) & \longmapsto & (j(x), (1 + \alpha_\theta(x))e^{i\varphi}). \end{array}$$

If $\|\alpha_\theta\|_1$ is sufficiently small, h_θ is a totally real embedding. We want to compute $< \sigma_{n-1}(h_\theta, v), [M^{n-1}] >$, where v is the non-vanishing vector field given by $v(x, \varphi) = -ie^{i\varphi}$. We choose the element $M^{n-1}_\theta = j_\theta(M^{n-1})$ to represent $[M^{n-1}]$. The computation of $< \sigma_{n-1}(h_\theta), [M^{n-1}] >$ reduces the problem to find the linking number between h_θ and $j_{\theta,\epsilon}$ where

$$\begin{array}{llll} j_{\theta,\epsilon} : & M^{n-1} & \longrightarrow & \mathbb{C}^{n-1} \times \mathbb{C} \\ & x & \longmapsto & h_\theta(x, 0) + J\epsilon v(x, 0) = (j(x), 1 + \alpha_\theta(x) + \epsilon). \end{array}$$

A computation shows that : $< \sigma_{n-1}(h_\theta, v), [M^{n-1}] >= (-1)^n Lk(j_\theta, j_{\theta,\epsilon})$. Since the dimension of M^{n-1} is odd, lemma 1 implies :

1) if N is even, $Lk(j_\theta, j_{\theta,\epsilon})$ takes all the even values between $-N$ and N when θ varies,

2) if N is odd, $Lk(j_\theta, j_{\theta,\epsilon})$ takes all the odd values between $-N$ and N. The hypothesis $I(j) = 0$ (resp. $I(j) = 1$) implies N even (resp. N odd). Therefore : $< \sigma_{n-1}(h_\theta, v), [M^{n-1}] >= 2p$ (resp. $2p + 1$). According to §3.1, we can take N as large as we want. Indeed, due to the theorem of Gromov-Lees, if M^n admits a totally real immersion j into $\mathbb{C}^n$ with

$I(j) = 0$ (resp. $I(j) = 1$), then it admits a Lagrangian immersion $\tilde{\jmath}$ with $I(\tilde{\jmath}) = 0$ (resp. $I(\tilde{\jmath}) = 1$). One can begin with $\tilde{\jmath}$ and use §3.1 to add self-intersection points.

Regarding the Maslov class of h_θ on $[\mathbb{S}^1]$, an easy computation shows that $< \mu(h_\theta), [\mathbb{S}^1] >= 2$. $\square$

Remark 3. – When the dimension of M^{n-1} is even, the same approach shows that the integer $< \sigma_{n-1}(h_\theta, v), [M^{n-1}] >$ can not be zero if N is odd.

Corollary 1. – *Let M^n be one of the following manifolds :*

1) $\mathbb{S}^1 \times \mathbb{S}^{2q-1}$,

2) the unitary group $U(2q)$,

3) the oriented Lagrangian Grassmannian $\Lambda(k)$ with $k = 4q$ or $k = 4q + 3$,

Then, the space $EmbTR(M^n, \mathbb{C}^n)$ has a countable family of arcwise connected components which are not detected by the Maslov class.

Proof of corollary 1. – Since topologically, $U(2q) \simeq \mathbb{S}^1 \times SU(2q)$ and $\Lambda(k) = \mathbb{S}^1 \times SU(k)/SO(k)$, it is enough to show that $\mathbb{S}^{2q-1}$, $SU(2q)$ and $SU(k)/SO(k)$ admit a totally real immersion. This is well known for $\mathbb{S}^{2q-1}$ and $SU(2q)$. For $SU(k)/SO(k)$, one has to notice that the oriented Lagrangian Grassmannian $\Lambda(k)$ admits a Lagrangian embedding in $\mathbb{C}^{\frac{k(k+1)}{2}}$ (*cf.* [73], for instance). Thus $T\Lambda(k) \otimes \mathbb{C}$ is trivial and so is $T(SU(k)/SO(k)) \otimes \mathbb{C}$. Using the Gromov-Lees theorem, we deduce that $SU(k)/SO(k)$ admits a Lagrangian immersion in $\mathbb{C}^{\frac{k(k+1)}{2}-1}$. The values $k = 4q$ and $k = 4q + 3$ are those for which $SU(k)/SO(k)$ is odd dimensional. $\square$

1.3.3 The class σ_{n-1} of some embeddings of $M^{n-1} \times \mathbb{S}^1$ with n odd.

Theorem 2. – *Let M^{n-1} be a compact, oriented, even dimensional hypersurface without boundary embedded in $\mathbb{R}^n$ such that $\frac{\chi(M^{n-1})}{2} \equiv 1 \mod 2$. Then, there exists two totally real embeddings $j_0, j_1 : M^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that :*

$$< \sigma_{n-1}(j_0, v), [M^{n-1}] >= 0,$$

$$< \sigma_{n-1}(j_1, v), [M^{n-1}] > \neq 0,$$

(v denotes a non-vanishing vector field tangent to the factor $\mathbb{S}^1$ of $M^{n-1} \times \mathbb{S}^1$).

Proof of theorem 2. – An oriented hypersurface is stably parallelizable, thus it admits a Lagrangian immersion h. The algebraic number $I(h)$ of self-intersection points is given by the formula :

$$I(h) = (-1)^{\frac{n+1}{2}} \frac{\chi(M^{n-1})}{2}.$$

Therefore the number $N(h)$ of self-intersection points is odd. We deduce from remark 3, the existence of a totally real embedding $j_1 : M^{n-1} \times \mathbb{S}^1 \longrightarrow \mathbb{C}^n$ such that $< \sigma_{n-1}(j_1), [M^{n-1}] > \neq$

0.

Let $f : M^{n-1} \longrightarrow \mathbb{R}^n$ denote the inclusion $M^{n-1} \subset \mathbb{R}^n$ and n_x the normal vector of $f(M^{n-1})$ at $f(x)$. In [62], T. Kawashima shows that, for small η, the map

$$\begin{array}{rccc} j_0 : & M^{n-1} \times \mathbb{S}^1 & \longrightarrow & \mathbb{C}^n \\ & (x, \varphi) & \longmapsto & f(x) + \eta e^{i\varphi} n_x \end{array}$$

is a Lagrangian embedding of $M^{n-1} \times \mathbb{S}^1$ in $\mathbb{C}^n$. As in lemma 1, the computation of $< \sigma_{n-1}(j_0, v), [M^{n-1}] >$ reduces the problem to a computation of intersection points. It is easy so see that this number is zero. We conclude :

$$< \sigma_{n-1}(j_0, v), [M^{n-1}] >= 0. \quad \square$$

1.4 The linking class σ_{n-1} and ordinary embeddings

The first two sections of this part are devoted to the results of A. Haefliger and M. Hirsch about the classification of immersions and embeddings. These results are needed for the two last sections in which we prove theorem 3.

1.4.1 Classification of immersions of M^n into $\mathbb{R}^{2n}$ ([55],[46]).

Let M^n be a compact, oriented manifold without boundary. One denotes by FM^n the frame bundle of M^n and $V_{2n,n}$ the Stiefel-Whitney manifold of n-frames in $\mathbb{R}^{2n}$. The group $Gl(n, \mathbb{R})$ acts on FM^n and on $V_{2n,n}$ by right multiplication. One denotes by $T_{2n,n}(M^n)$ the associated bundle to FM^n with fibre $V_{2n,n}$.

If $f : M^n \longrightarrow \mathbb{R}^{2n}$ is an immersion, its derivative induces a map :

$$\begin{array}{rccc} df : & FM^n & \longrightarrow & V_{2n,n} \\ & (X_1, ..., X_n) & \longmapsto & (df(X_1), ..., df(X_n)). \end{array}$$

The map df is equivariant for the action of $Gl(n, \mathbb{R})$, i.e. $df(XA) = df(X)A$. Therefore, it induces a section $s(f)$ of $T_{2n,n}(M^n)$. Let $\Gamma(T_{2n,n}(M^n))$ be the space of all sections of $T_{2n,n}(M^n)$ and $\pi_0(Imm(M^n, \mathbb{R}^{2n}))$ be the set of regular homotopy classes of immersions of M^n in $\mathbb{R}^{2n}$. The central result is the

Immersion classification theorem (M. Hirsch).
The following map is a bijection :

$$\begin{array}{ccc} \pi_0(Imm(M^n, \mathbb{R}^{2n})) & \longrightarrow & \pi_0(\Gamma(T_{2n,n}(M^n))) \\ [f] & \longmapsto & [s(f)]. \end{array}$$

1.4.2 Classification of embeddings of M^n into $\mathbb{R}^{2n}$ ([50]).

Let M^n be a compact, oriented manifold without boundary. Let M^n_0 denotes M^n minus a point p. If $f : M^n \longrightarrow \mathbb{R}^{2n}$ is an embedding, we denote by $NM^n_0(f)$ the normal bundle of

$f(M_0^n)$ in $\mathbb{R}^{2n}$. It is well known that two embeddings of M_0^n into $\mathbb{R}^{2n}$ are isotopic. Thus, one can take off f in the notation of the normal bundle NM_0^n. Let ν be a section of the unit normal bundle $N^1M_0^n$. From f and ν, one can define a new embedding $f_\nu : M^n \longrightarrow \mathbb{R}^{2n}$ by putting :

$$f_\nu(x) = \begin{cases} f(x) + \lambda(x)\nu(x) & \text{si } x \in M^n \setminus D_1, \\ f(x) & \text{si } x \in D_1, \end{cases}$$

where D_1 is a disk of (small) radius r of M^n centered at p and $\lambda : M^n \longrightarrow \mathbb{R}$ is a C^∞ function equal to 0 on D_1 and to ϵ (small) on $M^n \setminus D_2$, D_2 beeing a concentric disk of radius $2r$. One has

Proposition (A. Haefliger, M. Hirsch). – *Let $f : M^n \longrightarrow \mathbb{R}^{2n}$ be an embedding. There exists a section ν of $N^1M_0^n$ such that $f_\nu(M^n)$ is homologous to zero in $\mathbb{R}^{2n} \setminus f(M^n \setminus D_2)$. Moreover, this section is unique, up to homotopy (between sections of $N^1M_0^n$).*

Definition. – Such a section of $N^1M_0^n$ is said to be *associated* to the embedding f. One denotes by $\nu(f)$ any associated section to f.

Let $V_{2n,n+1}$ be the Stiefel-Whitney manifold of $(n+1)$-frames in $\mathbb{R}^{2n}$ and let $T_{2n,n+1}(M^n)$ be the associated bundle to FM^n with fibre $V_{2n,n+1}$ (the group $Gl(n,\mathbb{R})$ acting on the first n vectors of a $(n+1)$-frame in $\mathbb{R}^{2n}$). Every embedding $f : M^n \longrightarrow \mathbb{R}^{2n}$, induces a section $\tilde{s}(f) = (s(f), \nu(f))$ of $T_{2n,n+1}(M_0^n)$ ($s(f)$ is the section built in 4.1). One denotes by $\Gamma(T_{2n,n+1}(M_0^n))$ the space of sections of $T_{2n,n+1}(M_0^n)$. The central result is the

Embedding classification theorem (A. Haefliger, M. Hirsch).
If $n \geq 4$, then the following map is a bijection :

$$\begin{array}{ccc} \pi_0(Emb(M^n, \mathbb{R}^{2n})) & \longrightarrow & \pi_0(\Gamma(T_{2n,n+1}(M_0^n))) \\ [f] & \longmapsto & [\tilde{s}(f)]. \end{array}$$

1.4.3 Some properties of the linking class.

Existence of a field w such that $\sigma_{n-1}(j, w) = 0$.

Let G be a n-plane of the Euclidian space $\mathbb{E}^{2n} \simeq \mathbb{C}^n$. We denote by $\pi_G : \mathbb{E}^{2n} \to G$ the orthogonal projection and we define a linear map $F_G : G^\perp \to G$ by putting $F_G = \pi_G \circ J$. Note that F_G is invertible if and only if G is a totally real n-plane. If j is a totally real embedding of M^n into the Euclidian space $\mathbb{E}^{2n} \simeq \mathbb{C}^n$ we denote by $F_{j(m)}$ (or simply F) the map $F_{T_{j(m)}j(M^n)}$.
Let $\nu(j)$ be a section of $N^1M_0^n$ associated to the embedding j. This section can be extended to a section $\tilde{\nu}(j)$ of N^1M^n, (because $\chi(NM^n) = \chi(TM^n) = 0$). According to the definition of both $\nu(j)$ and σ_{n-1}, it is easy to check that $\sigma_{n-1}(j, F\nu(j)) = 0$. We have obtained

Proposition 1. – *Let $j : M^n \longrightarrow \mathbb{C}^n$ be a totally real embedding, $\nu(j)$ a section of $N^1M_0^n$ associated to the embedding j and $\tilde{\nu}(j)$ an extension of $\nu(j)$ on M^n (such an extension always exists). Then, one has :*

$$\sigma_{n-1}(j, F\nu(j)) = 0.$$

A formula connecting $\sigma_{n-1}(j,v)$ and $\sigma_{n-1}(j,v')$

Proposition 2. – *Let v, v' be two sections of the unitary tangent bundle T^1M^n. Then, one has :*

$$\sigma_{n-1}(j,v) - \sigma_{n-1}(j,v') = d(v,v'),$$

where the difference class $d(v,v') \in H^{n-1}(M^n,\mathbb{Z})$ is the first obstruction to build a homotopy from the section v to the section v'.

The proof of this proposition is left to the reader (it is only a reformulation of an argument appearing in the proof 4.3 of [50]). When M^n is parallelizable, we have an explicit expression for $d(v,v')$. Let $R = (e_1, ..., e_n)$ denote a direct frame field on M^n and v a non vanishing section of TM^n. We define a function $f_{R,v} : M^n \longrightarrow \mathbb{S}^{n-1} \subset \mathbb{E}^n$ by associating to each point p of M^n, the (normalized) coordinates of v_p in R_p. Proposition 2 gives :

$$\sigma_{n-1}(j,v) = \sigma_{n-1}(j,e_1) + (-1)^{n(n+1)/2} f^*_{R,v}u.$$

where u is the generator of $H^{n-1}(\mathbb{S}^{n-1},\mathbb{Z})$ such that $< u, [\mathbb{S}^{n-1}] >= 1$.

1.4.4 Relationship between σ_{n-1} and ordinary embeddings.

Theorem 3. – *Let M^n $(n \geq 4)$ be a compact, parallelizable manifold without boundary. Let j_0, j_1 be two totally real (or Lagrangian) embeddings which belong to the same regular homotopy class of totally real (or Lagrangian) immersions. Then, the following propositions are equivalent :*

1. *j_0 and j_1 belong to the same isotopy class of ordinary embeddings,*
2. *there exists a non-vanishing vector field v on M^n such that :*
$$\begin{cases} \sigma_{n-1}(j_0,v) = \sigma_{n-1}(j_1,v) & \textit{if } n \textit{ is odd,} \\ \mu^*(\sigma_{n-1}(j_0,v)) = \mu^*(\sigma_{n-1}(j_1,v)) & \textit{if } n \textit{ is even,} \end{cases}$$

($\mu^* : H^{n-1}(M^n,\mathbb{Z}) \longrightarrow H^{n-1}(M^n,\mathbb{Z}_2)$ denotes the coefficients homomorphism).

Proof of theorem 3.– Since M^n is parallelizable, the fibre bundles $T_{2n,n}$ (M^n_0) and $T_{2n,n+1}(M^n_0)$ are trivial and $\pi_0(Emb(M^n,\mathbb{C}^n)) \simeq [M^n_0, V_{2n,n+1}]$. Let $V_{TR}(n)$ (resp. $V_{Lag}(n)$) be the submanifold of $V_{2n,n}$ which is defined as the set of n-frames such that the n-planes they span are totally real (resp. Lagrangian). Since j_0 and j_1 belong to the same regular homotopy class of totally real (resp. Lagrangian) immersions, there exists a path s_t among the sections of $V_{TR}(n)$ (resp. $V_{Lag}(n)$) such that :

$$s_0 = s(j_0), \quad s_1 = s(j_1).$$

The manifold $V_{2n,n+1}$ is a bundle over $V_{2n,n}$. The section s_t lifts to a section $\tilde{s}_t = (s_t, \nu_t)$ such that $\tilde{s}_0 = (s_0,\nu_0) = (s(j_0), \nu(j_0))$.

$$\begin{tikzcd} & V_{2n,n+1} \arrow[d] \\ M^n_0 \arrow[ur, "\tilde{s}_t"] \arrow[r, "s_t"'] & V_{2n,n} \end{tikzcd}$$

Therefore, we get two sections of $T_{2n,n+1}(M_0^n)$ over $s(j_1)$:

$$\tilde{s}_1 = (s(j_1), \nu_1), \quad \tilde{s}(j_1) = (s(j_1), \nu(j_1)).$$

The manifold $V_{2n,n}$ is $(n-1)$-connected, thus : $[M_0^n, V_{2n,n}] \simeq \{0\}$. There exists a homotopy σ_t among the sections of $V_{2n,n}$ such that :

$$\sigma_0 = s(j_1), \quad \sigma_1 = \text{ constant section.}$$

One can lift this homotopy to a section $M_0^n \longrightarrow V_{2n,n+1}$ in to ways : either to a homotopy $\tilde{\sigma}_t^1 = (\sigma_t, n_t^1)$ where $n_0^1 = \nu_1$, or to an other homotopy $\tilde{\sigma}_t^2 = (\sigma_t, n_t^2)$ where $n_0^2 = \nu(j_1)$. Similarly, the manifold $V_{2n,2n} \simeq Gl(2n, \mathbb{R})$ being a fibration over $V_{2n,n}$, the homotopy σ_t also lifts to a homotopy $\Sigma_t = (\sigma_t, \mathcal{R}_t)$ of $V_{2n,2n}$. One can always take $\mathcal{R}_0 = F_{j_1}^{-1}\sigma_0$.

The first non vanishing homotopy group of $V_{2n,n+1}$ is $\pi_{n-1}(V_{2n,n+1})$. It is $\mathbb{Z}$ if n is odd and $\mathbb{Z}_2$ if not. Thus :

$$[M_0^n, V_{2n,n+1}] \simeq H^{n-1}(M^n, \mathbb{Z} \text{ or } \mathbb{Z}_2).$$

A generator of $\pi_{n-1}(V_{2n,n+1})$ is given by the map :

$$\begin{array}{rccc} \alpha : & \mathbb{S}^{n-1} \subset \mathbb{R}^n & \longrightarrow & V_{2n,n+1} \\ & x & \longmapsto & (e_1, ..., e_n, x) \end{array}$$

where $(e_1, ..., e_n)$ is a fixed n-frame in $\mathbb{R}^{2n}$ and x is a vector of the unitary sphere of the orthogonal n-plane spanned by $(e_1, ..., e_n)$ (*cf.* [85], p. 132). Therefore the sections $\tilde{\sigma}_1^1$ and $\tilde{\sigma}_1^2$ of $V_{2n,n+1}$ are homotopic if and only if :

$$\begin{cases} f^*_{\mathcal{R}_1, n_1^1} u & = & f^*_{\mathcal{R}_1, n_1^2} u & \text{if } n \text{ is odd,} \\ \mu^*(f^*_{\mathcal{R}_1, n_1^1} u) & = & \mu^*(f^*_{\mathcal{R}_1, n_1^2} u) & \text{if } n \text{ is even.} \end{cases}$$

(The notations $f^*_{\mathcal{R}_t, n_t^i} u$ are obvious generalizations of those introduced in §1.4.3.2). Thus, $\tilde{\sigma}_0^1 = \tilde{s}_1$ and $\tilde{\sigma}_0^2 = \tilde{s}(j_1)$ are homotopic if and only if :

$$\begin{cases} f^*_{\mathcal{R}_0, \nu_1} u & = & f^*_{\mathcal{R}_0, \nu(j_1)} u & (n \text{ odd }), \\ \mu^*(f^*_{\mathcal{R}_0, \nu_1} u) & = & \mu^*(f^*_{\mathcal{R}_0, \nu(j_1)} u) & (n \text{ even }). \end{cases}$$

Since $\tilde{s}(j_0)$ and $\tilde{s}_1$ are homotopic, this equality tells us when $[\tilde{s}(j_0)] = [\tilde{s}(j_1)]$. In an other hand, according to propositions 1 and 2 and using the identification $H^{n-1}(M_0^n) \simeq H^{n-1}(M^n)$, we get :

$$\sigma_{n-1}(j_0, e_1) + (-1)^{n(n+1)/2} f^*_{s_0, F_{j_0}\nu(j_0)} u = 0, \quad (1)$$
$$\sigma_{n-1}(j_1, e_1) + (-1)^{n(n+1)/2} f^*_{s_1, F_{j_1}\nu(j_1)} u = 0. \quad (2)$$

As s_t lies inside $V_{TR}(n)$ (resp. $V_{Lag}(n)$), one can write the equality (1) for each $t \in [0, 1]$, i.e.

$$\sigma_{n-1}(j_0, e_1) + (-1)^{n(n+1)/2} f^*_{s_t, F_{Span(s_t)}\nu_t} u = 0.$$

In particular :

$$\sigma_{n-1}(j_0,e_1)+(-1)^{n(n+1)/2}f^*_{s_1,F_{j_1}\nu_1}u=0.$$

As $s_1=F_{j_1}\mathcal{R}_0$, one finally obtains :

$$[\tilde{s}(j_0)]{=}[\tilde{s}(j_1)]\Longleftrightarrow\begin{cases}\sigma_{n-1}(j_0,e_1) & = & \sigma_{n-1}(j_1,e_1) & (n \text{ odd })\\ \mu^*(\sigma_{n-1}(j_0,e_1)) & = & \mu^*(\sigma_{n-1}(j_1,e_1)) & (n \text{ even }).\end{cases}$$

This finishes the proof of theorem 3. $\square$

Remark 4. – Theorem 3 can be seen as a first step to understand the map :

$$\pi_0(EmbTR(M^n,\mathbb{C}^n))\longrightarrow\pi_0(Emb(M^n,\mathbb{C}^n)).$$

In the immersions case, the map :

$$\pi_0(ImmLag(M^n,\mathbb{C}^n))\ \text{(or }\pi_0(ImmTR(M^n,\mathbb{C}^n)))\longrightarrow\pi_0(Imm(M^n,\mathbb{C}^n))$$

has been intensively studied by M. Audin. The interested reader can refer to [3].

New examples of Lagrangian rigidity

Israel J. of Math. 125 (2001), 221-235.

Abstract. – Let $n = 4$ or 8. We prove that any Lagrangian embedding of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ into $\mathbb{C}^n$ has a trivial linking class. We deduce that every embedding of $\mathbb{S}^3 \times \mathbb{S}^1$ into $\mathbb{C}^4$ is isotopic to a Lagrangian embedding. This is false if $n = 8$.

2.1 Introduction

Given a totally real embedding j of a compact, oriented, manifold M^n without boundary into $\mathbb{C}^n$ and a nowhere vanishing tangent vector field v, one can build a cohomology class $\sigma_{n-1}(j,v) \in H^{n-1}(M^n,\mathbb{Z})$ called the **linking class** of j associated to v. This class is an invariant of isotopy classes of totally real embeddings. If $j_1, j_2\colon M^n \longrightarrow \mathbb{C}^n$ are two totally real embeddings and if there exists a non vanishing vector field v such that $\sigma_{n-1}(j_1,v) \neq \sigma_{n-1}(j_2,v)$, then j_1 and j_2 are not isotopic through totally real embeddings.

If M^n is the 2-torus and v the translation field on $\mathbb{T}^2$, it has been shown in [79] that for every $\sigma \in H^1(\mathbb{T}^2,\mathbb{Z})$ there exists a totally real embedding $j\colon \mathbb{T}^2 \longrightarrow \mathbb{C}^2$ such that $\sigma_1(j,v) = \sigma$. Nevertheless if one looks for Lagrangian embeddings, then the situation is completely different. In 1994, Y. Eliashberg and L. Polterovich proved that for every Lagrangian embedding $j\colon \mathbb{T}^2 \longrightarrow \mathbb{C}^2$, one has : $\sigma_1(j,v) = 0$ ([28]).

Here we consider the product manifolds $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^7$. It is easily checked that these manifolds admit a Lagrangian embedding into $\mathbb{C}^n$, for n =4 or 8. This follows from the Gromov-Lees theorem on Lagrangian immersions [67] combined with a result of M. Audin (see [3] p. 617-618 and also remark B below). Moreover, it has been shown in [9] that these manifolds admit a family of totally real embeddings such that $\sigma_{n-1}(j,v)$ takes a countable number of different values. Using the techniques developped by Y. Eliashberg and L. Polterovich, we prove the following rigidity results.

Theorem 1. – *Let $n = 4$ or 8 and $j\colon \mathbb{S}^1 \times \mathbb{S}^{n-1} \longrightarrow \mathbb{C}^n$ be a Lagrangian embedding. Let v be a nonvanishing vector field on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ tangent to the factor $\mathbb{S}^1$. Then : $\sigma_{n-1}(j,v) = 0$.*

By a slight modification of the proof of theorem 1, we also obtain similar results for some quotients of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $n = 4$ or 8 (see section 4).

Let $E(M^n,\mathbb{C}^n)$ be the space of embeddings of M^n into $\mathbb{C}^n$ and $E_{Lag}(M^n,\mathbb{C}^n)$ be the subspace of Lagrangian embeddings (all those spaces are endowed with the compact-open topology). We apply the preceding results to study the natural inclusion $i\colon E_{Lag}(M^n,\mathbb{C}^n) \longrightarrow E(M^n,\mathbb{C}^n)$ with $M^n = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ and $n = 4$ or 8.

Theorem 2. – *The map*

$$i_\sharp\colon \pi_0(E_{Lag}(\mathbb{S}^1 \times \mathbb{S}^{n-1},\mathbb{C}^n)) \longrightarrow \pi_0(E(\mathbb{S}^1 \times \mathbb{S}^{n-1},\mathbb{C}^n)) \simeq \mathbb{Z}_2$$

$$is \begin{cases} \textit{onto} & \textit{if } n = 4 \\ \textit{a constant map} & \textit{if } n = 8. \end{cases}$$

The group $Diff(M^n)$ acts on $E(M^n, \mathbb{C}^n)$ in an obvious way by reparametrizations of M^n. We shall see that $Diff(\mathbb{S}^1 \times \mathbb{S}^3)$ acts trivially on $\pi_0(E(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{C}^4))$. Theorem 2 implies that there exist two Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C}^4$ such that the images of these maps are not isotopic in $\mathbb{C}^4$ as non-parametrized submanifolds.

Given two Lagrangian immersions $j_0, j_1 \colon M^n \longrightarrow \mathbb{C}^n$ we define a cohomology class $\Delta(j_0, j_1) \in H^3(M^n, \mathbb{Z})$ which is an obstruction to the existence of a Lagrangian regular homotopy joining j_0 and j_1. If j_0 and j_1 are two Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^3$, it turns out that the modulo 2 reduction of this class - denoted by $\epsilon(j_0, j_1)$ - is the only obstruction to the existence of a smooth isotopy between j_0 and j_1. Of course, $\epsilon(j_0, j_1) \neq 0$ also implies that j_0 and j_1 belong to different components of the space $I_{Lag}(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{C}^4)$ of Lagrangian immersions. In particular, two Lagrangian embeddings which correspond to different elements of $\pi_0(E(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{C}^4))$ are not only non isotopic as smooth embeddings but also not regular homotopic as Lagrangian immersions.

It is also worth noticing that all Lagrangian immersions, and in particular, all Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ into $\mathbb{C}^n$ (n =4 or 8) are regular homotopic as smooth immersions. This is due to M. Audin, see [3] theorem 0.1.

Acknowledgement. – L. Polterovich suggested to me the problem of the linking class while he was visiting the Université de Lyon. It is a pleasure to thank him for many fruitful discussions.

2.2 Definitions and notations

Let M^n be a compact, connected, oriented, n-manifold without boundary. Let $< \cdot, \cdot >$ denote the Euclidean scalar product of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, J the canonical complex structure and $\omega =< J\cdot, \cdot >$ the symplectic form. An embedding $j \colon M^n \longrightarrow \mathbb{C}^n$ is called **totally real** if : $T_{j(p)}j(M^n) \oplus JT_{j(p)}j(M^n) = \mathbb{C}^n$ for every $p \in M^n$. It is called **Lagrangian** if : $JT_{j(p)}j(M^n) = T^{\perp}_{j(p)}j(M^n)$ for every $p \in M^n$, or equivalently if : $j^*\omega \equiv 0$. Since $JTM^n \oplus TM^n = \mathbb{C}^n$, every manifold admitting a totally real embedding necessarily has a zero Euler-Poincaré characteristic. Thus, there exists a nowhere vanishing vector field v on M^n. One can stretch the submanifold $j(M^n)$ in the direction $Jdj(v)$ by a small distance ϵ. We denote by $Jv(M^n)$ the resulting submanifold. If ϵ is small enough, $Jv(M^n)$ is disjoint from $j(M^n)$. It represents a n-cycle in $\mathbb{C}^n \setminus j(M^n)$. Using the exact sequence of the pair $(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n))$ and Alexander duality, one obtains the following identifications :

$$H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H_{n+1}(\mathbb{C}^n, \mathbb{C}^n \setminus j(M^n), \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

Definition. – (cf. [9]) The **linking class** associated to the totally real embedding j and to the vector field v is the class

$$\sigma_{n-1}(j, v) \in H^{n-1}(M^n, \mathbb{Z})$$

represented in $H_n(\mathbb{C}^n \setminus j(M^n), \mathbb{Z})$ by the n-cycle $[Jv(M^n)]$.

2.3 Proof of theorem 1

2.3.1 A model for $\mathbb{S}^1 \times \mathbb{S}^3$ and $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$.

Let $\mathbb{H}$ denote the quaternionic field and let $x = x_0 + x_1 i + x_2 j + x_3 k$ be any element of $\mathbb{H}$. Consider the free action of $\mathbb{Z}$ on $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$, where the generator of $\mathbb{Z}$ acts by multiplication by 2. The quotient $\mathbb{H}^*/\mathbb{Z}$ can be identified with $\mathbb{S}^1 \times \mathbb{S}^3$. Let $<,>$ denote the canonical scalar product on $\mathbb{H}^*$ ($< x, x' >= Re(x\overline{x'})$). A basis of $T^*_x(\mathbb{S}^1 \times \mathbb{S}^3)$ is given by :

$$\theta_0 = \frac{1}{|x|^2} < x, \cdot >, \quad \theta_1 = \frac{1}{|x|^2} < xi, \cdot >,$$
$$\theta_2 = \frac{1}{|x|^2} < xj, \cdot >, \quad \theta_3 = \frac{1}{|x|^2} < xk, \cdot >,$$

and the Liouville form at a point (x, y) in $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$ will be given by :

$$\lambda_{(x,y)} = y_0\theta_0 + y_1\theta_1 + y_2\theta_2 + y_3\theta_3 = \frac{1}{|x|^2} < xy, \cdot >= \frac{1}{|x|^2} Re(xyd\overline{x}).$$

Let $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$ be the unit cotangent bundle of $\mathbb{S}^1 \times \mathbb{S}^3$, and let ξ be the restriction of the Liouville form on $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$. Then ξ is the canonical contact form of $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$. Latter on, we shall need the following lemma.

Lemma A. – *Let*

$$\begin{aligned} \Xi_1 &= d(\tfrac{1}{|x|^2}) \wedge Re(xyd\overline{x}) + \tfrac{1}{|x|^2} Re(dx.y \wedge d\overline{x}), \\ \Xi_2 &= \tfrac{1}{|x|^2} Re(xdy \wedge d\overline{x}). \end{aligned}$$

One has : $d\xi = \Xi_1 + \Xi_2$ *and* $(d\xi)^3 = \Xi_2^3$.

Proof of lemma A. – The decomposition $d\xi = \Xi_1 + \Xi_2$ is obvious. Since $(d\xi)^3 = \Xi_2^3 + 3\Xi_2^2\Xi_1$, it suffices to prove that $\Xi_2^2\Xi_1 = 0$ to conclude. Let $G_4(\xi) = \{A \in SO(4) : f^*\xi = \xi$ where $f(x, y) = (Ax, y)\}$. It is easily seen that $G_4(\xi) = Sp(1) = \mathbb{S}^3$. Thus, one can work out computations at points of the form $(x, y) = (x_0, 0, 0, 0, y_0, y_1, y_2, y_3)$. One gets :

$$\begin{aligned} \Xi_1 &= \tfrac{2}{x_0^2}(-y_1 dx_2 \wedge dx_3 + y_2 dx_1 \wedge dx_3 - y_3 dx_1 \wedge dx_2), \\ \Xi_2 &= \tfrac{1}{x_0} \textstyle\sum_{i=0}^3 dy_i \wedge dx_i. \end{aligned}$$

A direct computation shows that :

$$\Xi_2^2\Xi_1 = 4dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_0 \wedge (y_1dy_1 + y_2dy_2 + y_3dy_3),$$

and since $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$, it follows that $\Xi_2^2\Xi_1 = 0$. □

Remark A. – Let $\Delta_4(\xi) = \{A \in SO(4) : f^*\xi = \xi$ where $f(x, y) = (Ax, Ay)\}$. Since : $\Delta_4(\xi) = \{\phi_x : y \mapsto xyx^{-1}$ where $x \in \mathbb{S}^3\} = SO(Im\ \mathbb{H}) \simeq SO(3)$, actually one could reduce the computations of lemma A starting with a point of the form $(x, y) = (x_0, 0, 0, 0, y_0, y_1, 0, 0)$.

2.3.2 Symplectic surgery

Using the symplectic tubular neighbourhood theorem of A. Weinstein, one gets that any Lagrangian embedding $j: \mathbb{S}^1 \times \mathbb{S}^3 \longrightarrow \mathbb{C}^4$ can be extended to a symplectic diffeomorphism (still denoted by j) of a tubular neighbourhood of the zero section of $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$ into a tubular neighbourhood of $j(\mathbb{S}^1 \times \mathbb{S}^3)$ in $\mathbb{C}^4$. Let $N_0^8 = \{|y| \leq \epsilon\}$ denote a closed tubular neighbourhood of the zero section of $T^*(\mathbb{S}^1 \times \mathbb{S}^3)$ and let $N^8 = j(N_0^8)$. We also set $\Sigma_0^7 = \partial N_0^8 = \{|y| = \epsilon\} \simeq \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3$, $\Sigma^7 = j(\Sigma_0^7)$ and $K^8 = \mathbb{C}^4 \setminus Int(N^8)$. Let f_n be the diffeomorphisms of $T^1\mathbb{H}^*$ given by :

$$\begin{array}{rccc} f_n\colon & \mathbb{H}^* \times \mathbb{S}^3 & \longrightarrow & \mathbb{H}^* \times \mathbb{S}^3 \\ & (x, y) & \longmapsto & (xy^n, y). \end{array}$$

Each f_n gives rise to a diffeomorphism of Σ_0^7 (still denoted by f_n). Moreover, for every $n \neq m$ the maps f_n and f_m belong to different arcwise connected components of $Diff(\mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3)$, because they induce different maps in homology. Let V_n^8 be the manifold obtained by gluing K^8 and N^8 along their boundaries via the diffeomorphism $\tilde{f}_n = j \circ f_n \circ j^{-1}$:

$$V_n^8 = K^8 \cup_{\tilde{f}_n : \Sigma^7 \longrightarrow \Sigma^7} N^8.$$

This surgery can be made symplectic. Indeed, if $Z = (X, Y)$ is a tangent vector at $T_{(x,y)}\Sigma_0^7 \simeq T_x(\mathbb{S}^1 \times \mathbb{S}^3) \times T_y(\mathbb{S}^3)$, a straightforward computation shows that $(f_n^*\xi)_{(x,y)}(Z) = \xi_{(x,y)}(Z) + n < y^2, Y >$.

Lemma B. – *The forms $f_n^*\xi$ and ξ are homotopic through contact forms.*

Proof of lemma B. – Let $\alpha_{(x,y)}(Z) =< y^2, Y >$ and let $\xi_t = \xi + tn\alpha$. The form α is exact, since

$$\alpha =< y^2, dy >=< 1, \frac{dy}{y^2} >= - < 1, d(y^{-1}) >= - < 1, d\overline{y} >= -dy_0.$$

Thus $(d\xi_t)^3 = (d\xi)^3$ and, by lemma A, $(d\xi)^3 = \Xi_2^3$. Moreover, we have $\alpha \wedge \Xi_2^3 = 0$, so :

$$\xi_t \wedge (d\xi_t)^3 = (\xi + tn\alpha) \wedge \Xi_2^3 = \xi \wedge \Xi_2^3 = \xi \wedge (d\xi)^3.$$

For each t, ξ_t is a contact form. □

By the theorem of J. W. Gray (cf. [41]), we then deduce that $f_n^*\xi$ and ξ are isotopic. Therefore, the manifold V_n^8 admits a symplectic structure which coincides outside of a large ball with the standard symplectic structure of $\mathbb{C}^4$. Moreover, it is easy to check that $H_2(V_n^8, \mathbb{Z}) = 0$. It then follows that V_n^8 is diffeomorphic to $\mathbb{C}^4$ from a result of Y. Eliashberg, A. Floer, M. Gromov and D. McDuff (see [68], p. 653 theorem 1.5 and [26] for a detailed account of their personal contributions).

2.3.3 Proof of theorem 1 for $n = 4$

We mimic what is done in [28]. The idea is to show that $V_n^8 \simeq \mathbb{C}^4$ implies $\sigma_3(j,v) = 0$. We first define explicit generators of each homology group involved in the proof. Let :

$$\begin{array}{lcll} H_3(\mathbb{S}^3 \times \mathbb{S}^1) & = & \mathbb{Z} < \alpha_0 >, & H_3(K^8) = \mathbb{Z} < H >, \\ H_3(\Sigma_0^7) & = & \mathbb{Z} < \alpha > \oplus\, \mathbb{Z} < \eta >, & H_3(N^8) = \mathbb{Z} < A >, \\ H_3(\Sigma^7) & = & \mathbb{Z} < a > \oplus\, \mathbb{Z} < h > . & \end{array}$$

Here $\mathbb{Z} < g >$ denotes the infinite cyclic group generated by g, α_0 is the 3-cycle $\{1\} \times [\mathbb{S}^3]$ of $\mathbb{S}^3 \times \mathbb{S}^1$, α is $\{1\} \times [\mathbb{S}^3] \times \{1\}$ in Σ_0^7, η is $\{1\} \times \{1\} \times [\mathbb{S}^3]$, $a = j_*\alpha$, $h = j_*\eta$, $A = j_*\alpha_0$ and H is such that $lk(j(\mathbb{S}^1 \times \mathbb{S}^3), H) = 1$. Let $i\colon \Sigma^7 \longrightarrow K^8$ and $\iota\colon \Sigma^7 \longrightarrow N^8$ denote the natural inclusions and let k be the composition $\iota \circ \tilde{f}_n$. The maps i and k induce two homomorphisms :

$$i_*\colon H_3(\Sigma^7) \longrightarrow H_3(K^8) \ \text{ and } \ k_*\colon H_3(\Sigma^7) \longrightarrow H_3(N^8).$$

It is obvious that :

$$\begin{array}{ll} k_*(h) = nA, & k_*(a) = A, \\ i_*(h) = H, & i_*(a) = lk(j(\mathbb{S}^1 \times \mathbb{S}^3), a). \end{array}$$

By the choice of the unitary tangent field v on $\mathbb{S}^1 \times \mathbb{S}^3$, we get $\alpha = \ Jv(\alpha_0)$ (with obvious notation). Since j is a symplectomorphism, it is easy to see that :

$$lk(j(\mathbb{S}^1 \times \mathbb{S}^3), a) = lk(j(\mathbb{S}^1 \times \mathbb{S}^3), Jv(\{1\} \times \mathbb{S}^3)) =< \sigma_3(j,v), \alpha_0 > .$$

Hence, with respect to the basis (a,h) and (A,H), the matrix of the map $i_* \oplus (-k_*)$ is given by :

$$M = \begin{pmatrix} 1 & -n \\ < \sigma_3(j,v), \alpha_0 > & -1 \end{pmatrix}$$

On the other hand, from the Mayer-Vietoris sequence

$$0 \simeq H_4(V_n^8) \longrightarrow H_3(\Sigma^7) \stackrel{i_* \oplus (-k_*)}{\longrightarrow} H_3(K^8) \oplus H_3(N^8) \longrightarrow H_3(V_n^8) \simeq 0$$

one deduces that the map $i_* \oplus (-k_*)$ is an isomorphism. The determinant of the matrix M must be ± 1. Since this determinant is equal to $-1 + n < \sigma_3(j,v), \alpha_0 >$, it follows that $< \sigma_3(j,v), \alpha_0 >= 0$. □

2.3.4 Proof of theorem 1 for $n = 8$

It suffices to consider the Cayley algebra $\mathbb{O}$ instead of $\mathbb{H}$ and to mimic the above proof. Nevertheless, some points of this proof deserve more explanations since the non-associativity of $\mathbb{O}$ introduces some new difficulties.

Recall that the associator $[x,y,z]$ of a triple of elements of $\mathbb{O}$ is the difference $(xy)z - x(yz)$. In $\mathbb{O}$, a weak form of associativity holds : namely, the associator is a trilinear map. Moreover : $[y, y^{-1}, z] = 0$. We need this property to prove that the map

$$\begin{array}{rccc} f_1\colon & \mathbb{O}^* \times \mathbb{S}^7 & \longrightarrow & \mathbb{O}^* \times \mathbb{S}^7 \\ & (x,y) & \longmapsto & (xy, y). \end{array}$$

is a diffeomorphism. Indeed,

$$\begin{aligned} x_1y = x_2y &\implies (x_1y)y^{-1} = (x_2y)y^{-1} \\ &\implies x_1(yy^{-1}) = x_2(yy^{-1}), \end{aligned}$$

since $[y, y^{-1}, z] = 0$. As $f_n = f_1 \circ ... \circ f_1$, f_n is also a diffeomorphism.

Difficulties arise when one wants to establish an analogue of lemma A. This is due to the fact that the group $G_8(\xi) = \{A \in SO(8)\colon f^*\xi = \xi \text{ where } f(x,y) = (Ax, y)\}$ is very small : it contains only $\pm Id$! (Here $\xi_{(x,y)} = \frac{1}{|x|^2} Re((xy)d\overline{x})$, with $(x,y) \in \mathbb{O}^* \times \mathbb{S}^7$).

Lemma C. – *Let*

$$\begin{aligned} \Xi_1 &= d(\tfrac{1}{|x|^2}) \wedge Re(xyd\overline{x}) + \tfrac{1}{|x|^2} Re(dx.y \wedge d\overline{x}), \\ \Xi_2 &= \tfrac{1}{|x|^2} Re(xdy \wedge d\overline{x}). \end{aligned}$$

One has : $d\xi = \Xi_1 + \Xi_2$ *and* $(d\xi)^7 = \Xi_2^7$.

Proof of lemma C. – Similarly to lemma A, the nontrivial step is to prove that $\Xi_2^6 \Xi_1 = 0$. Let $\Delta_8(\xi) = \{A \in SO(8)\colon f^*\xi = \xi \text{ where } f(x,y) = (Ax, Ay)\}$. It turns out that $\Delta_8(\xi)$ is the exceptional Lie group G_2 (see [52] p. 114). This Lie group can be seen as a Lie subgroup of $SO(Im\ \mathbb{O}) \simeq SO(7)$. Moreover, it is well known that $G_2/SU(3) \simeq \mathbb{S}^6$. Thus, one can work out computations at points (x,y) with $x = (x_0, x_1, 0, 0, 0, 0, 0, 0)$. Let Vol_x denote $dx_0 \wedge ... \wedge dx_7$ and let $Vol_{i,j,k}$ denote the form

$$dy_0 \wedge ... \wedge \widehat{dy_i} \wedge ... \wedge \widehat{dy_j} \wedge ... \wedge \widehat{dy_k} \wedge ... \wedge dy_7$$

where dy_i, dy_j and dy_k are missing and set $\beta_{i,j,k} = Vol_{i,j,k} \wedge (y_i dy_i + y_j dy_j + y_k dy_k)$. A direct computation shows :

$$\begin{aligned} \Xi_2^6 \Xi_1 = 1440\ (&\tfrac{1}{(x_0^2+x_1^2)^4} && Vol_x \wedge [\beta_{1,2,3} - \beta_{1,4,5} - \beta_{1,6,7}] \\ +&\tfrac{x_0^2 - x_1^2}{(x_0^2+x_1^2)^5} && Vol_x \wedge [\beta_{2,5,7} - \beta_{2,4,6} - \beta_{3,5,6} - \beta_{3,4,7}] \\ +&\tfrac{2x_0x_1}{(x_0^2+x_1^2)^5} && Vol_x \wedge [\beta_{2,5,6} + \beta_{2,4,7} + \beta_{3,5,7} - \beta_{3,4,6}]). \end{aligned}$$

Then, since $\sum_{i=0}^{7} y_i^2 = 1$, all the $\beta_{i,j,k}$ vanish. $\square$

The rest of the proof of theorem 1 runs exactly as in 3.3.

2.4 Some generalizations

2.4.1 On Lagrangian embeddings of some quotients of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$

Let Γ be a finite multiplicative subgroup of $\mathbb{S}^3$. The quotient $\mathbb{S}^3/\Gamma$ is an orientable Riemannian homogeneous 3-dimensional manifold of positive constant curvature (every finite subgroup of $\mathbb{S}^3 \simeq SU(2)$ is fixed point free, see [94]). Let j be a Lagrangian embedding of $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ into $\mathbb{C}^4$. As $H^3(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma) \simeq \mathbb{Z} \oplus \Gamma'$ where $\Gamma' = \Gamma/[\Gamma, \Gamma]$ is the abelianization of Γ, the linking class $\sigma_3(j,v)$ splits in a linear part and a torsion one.

Proposition. – *1) Let* $j\colon \mathbb{S}^1 \times \mathbb{S}^3/\Gamma \longrightarrow \mathbb{C}^4$ *be any Lagrangian embedding and let* v *be a nonvanishing vector field on* $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ *tangent to the factor* $\mathbb{S}^1$*. Then, the linear part of* $\sigma_3(j,v)$ *vanishes.*
2) Let $j\colon \mathbb{S}^1 \times \mathbb{R}P^7 \longrightarrow \mathbb{C}^8$ *be any Lagrangian embedding and let* v *be a nonvanishing vector field on* $\mathbb{S}^1 \times \mathbb{R}P^7$ *tangent to the factor* $\mathbb{S}^1$*. Then, the linear part of* $\sigma_7(j,v)$ *vanishes.*

Remark B. – If M^n admits a Lagrangian immersion in $\mathbb{C}^n$, then $\mathbb{S}^1 \times M^n$ admits a Lagrangian embedding into $\mathbb{C}^{n+1}$ (see [3], for instance). As any oriented 3-dimensional manifold is parallelizable, it follows from Gromov-Lees theorem that such 3-manifolds admit a Lagrangian immersion. Thus $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ admits a Lagrangian embedding. Similarly, since $\mathbb{R}P^7$ is parallelizable, it admits a Lagrangian immersion and $\mathbb{S}^1 \times \mathbb{R}P^7$ admits a Lagrangian embedding.

Remark C. – There is only one (proper) finite multiplicative subgroup Γ of $\mathbb{S}^3 \subset \mathbb{H}$ such that $\Gamma' = 0$ (and thus σ_3 has no torsion part). This subgroup is the lift of the dodecahedral subgroup of $SO(3)$ to the double covering $\mathbb{S}^3 \longrightarrow SO(3)$. The quotient $\Sigma^3 = \mathbb{S}^3/\Gamma$ is the Poincaré sphere.

2.4.2 Proof of the proposition

Proof of 1). – We first assume that Γ acts by left multiplication on $\mathbb{S}^3$. The diffeomorphism

$$\begin{array}{cccc} f_n\colon & \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3 & \longrightarrow & \mathbb{S}^1 \times \mathbb{S}^3 \times \mathbb{S}^3 \\ & (x,y) & \longmapsto & (xy^n, y). \end{array}$$

gives a diffeomorphism on the quotient $\mathbb{S}^1 \times (\mathbb{S}^3/\Gamma) \times \mathbb{S}^3$. Since $G_4(\xi) = \mathbb{S}^3$, the contact structure ξ on $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3)$ also induces a contact structure on $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma)$, still denoted by ξ. The pull-back of ξ by f_n is $(f_n^*\xi)_{(x,y)}(Z) = \xi_{(x,y)}(Z) + n < y^2, Y >$, where Z is a tangent vector at a point (x,y) of $T^{1*}(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma)$. As in section 3.2, one can perform a symplectic surgery on $\mathbb{C}^4$ and denote the resulting manifold by V_n^8. It is easy to check that $H_2(V_n^8) \simeq 0$ and thus V_n^8 is diffeomorphic to $\mathbb{C}^4$. Repeating the arguments of theorem 1, one gets :

$$< \sigma_3(j,v), \alpha_0 >= 0,$$

where α_0 is the 3-cycle of $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ represented by $\{1\} \times [\mathbb{S}^3/\Gamma]$. Note that the third cohomology group of $\mathbb{S}^1 \times \mathbb{S}^3/\Gamma$ has some torsion. Namely, $H^3(\mathbb{S}^1 \times \mathbb{S}^3/\Gamma, \mathbb{Z}) \simeq \mathbb{Z} \oplus \Gamma'$. Since Γ is finite, Γ' is a torsion group.

If the group Γ acts by right multiplication, one has to consider the contact form $\xi_{(x,y)} = \frac{1}{|x|^2} Re(yxd\overline{x})$ and the diffeomorphisms $(x,y) \longmapsto (y^n x, y)$.

Proof of 2). – The proof is similar to the proof of point 1). But since $G_8(\xi) \simeq \mathbb{Z}_2$, it only applies to the quotient $\mathbb{S}^7/\mathbb{Z}_2 = \mathbb{R}P^7$. $\square$

2.5 Isotopy classes of embeddings

2.5.1 General theory of A. Haefliger and M. Hirsch [50]

Let M^n be a n-dimensional ($n \geq 4$) compact connected orientable manifold without boundary and let $j\colon M^n \longrightarrow \mathbb{C}^n \simeq \mathbb{R}^{2n}$ be an embedding. Let M_0^n denote the manifold $M^n \backslash \{x_0\}$ where x_0 is any point of M^n. Let $D_1 \subset D_2$ be two embedded open disks centered at x_0 and let $\lambda\colon M^n \longrightarrow [0,1]$ be a C^∞ map such that $\lambda \equiv 1$ on $M^n \setminus D_2$ and $\lambda \equiv 0$ on D_1. Given a nonvanishing normal vector field ν on M_0, let us set :

$$j_\nu(x) = \begin{cases} j(x) & \text{if } x \in D_1 \\ j(x) + \lambda(x)\nu_x & \text{if } x \in M^n \setminus D_1. \end{cases}$$

Finally, let $X = \mathbb{C}^n \setminus j(M^n \setminus D_2)$. There exists an unique (up to homotopy) normal vector field ν on M_0^n such that :

$$[j_\nu(M^n)] = 0 \quad \text{in } H_n(X, \mathbb{Z}) \simeq H^{n-1}(M^n \setminus D_2, \mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}).$$

We call this vector field the **associated vector field** $\nu(j)$ to the embedding j. If ν and ν' are two nonvanishing normal vector fields, the following simple relation holds :

$$[j_{\nu'}(M^n)] - [j_\nu(M^n)] = d(\nu', \nu)$$

where $d(\nu', \nu)$ is the difference class, the first obstruction to homotopy between ν' to ν.

Let $T_{2n,n+1}(M_0^n)$ be the associated bundle to the frame bundle of M_0^n whose fiber is the Stiefel manifold $V_{2n,n+1}$ of $(n+1)$-frames of $\mathbb{R}^{2n}$, with $Gl(n, \mathbb{R})$ acting in a natural way on the first n vectors of a frame. Every embedding j gives rise to a natural section σ_j of $T_{2n,n+1}(M_0^n)$ given by :

$$\begin{array}{rccc} \sigma_j\colon & M_0^n & \longrightarrow & T_{2n,n+1}(M_0^n) \\ & x & \longmapsto & [(dj_x(\mathcal{R}), \nu_x(j))], \end{array}$$

where $\mathcal{R}$ is any frame in $T_x M_0^n$. Let $\Gamma(T_{2n,n+1}(M_0^n))$ be the space of (continuous) sections of $T_{2n,n+1}(M_0^n)$ and let $E(M^n, \mathbb{R}^{2n})$ be the space of embeddings of M^n into $\mathbb{R}^{2n}$.

Theorem (A. Haefliger, M. Hirsch). – *The map $j \longrightarrow \sigma_j$ induces a bijection on the π_0-level between the space of embeddings $E(M^n, \mathbb{R}^{2n})$ and the space of sections $\Gamma(T_{2n,n+1}(M_0^n))$.*

There is a 1-1 correspondence between $\pi_0(\Gamma(T_{2n,n+1}(M_0^n)))$ and the cohomology group $H^{n-1}(M_0^n, \pi_{n-1}(V_{2n,n+1}))$. Thus $\pi_0(E(M^n, \mathbb{R}^{2n}))$ is in bijection with $H^{n-1}(M^n, \mathbb{Z}_2)$ if n is even, and with $H^{n-1}(M^n, \mathbb{Z})$ if n is odd.

2.5.2 The case $M^n = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ with $n = 4$ or 8

Throughout this section n will be 4 or 8. For short, we put $E_n = E(\mathbb{S}^1 \times \mathbb{S}^{n-1}, \mathbb{R}^{2n})$. According to the preceding subsection, $\pi_0(E_n)$ is in bijection with $\mathbb{Z}_2$.

Let v be a nonvanishing vector field on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ tangent to the factor $\mathbb{S}^1$. By using the quaternionic or the octonian structure (we see $\mathbb{O}$ as $\mathbb{H} \oplus e\mathbb{H}$), we can define a global n-frame $\mathcal{R} = \{e_1, e_2, ..., e_n\}$ on M^n as :

$$\begin{array}{rcll} (e_1, ..., e_4) & = & (v, iv, jv, kv) & \text{if } n = 4, \\ (e_1, ..., e_8) & = & (v, iv, jv, kv, ev, eiv, ejv, ekv) & \text{if } n = 8. \end{array}$$

The map $j \mapsto \sigma_j$ can be identified with :

$$\begin{array}{rccl} \Phi\colon & E_n & \longrightarrow & \Gamma(T_{2n,n+1}(M_0^n)) \\ & j & \longmapsto & \sigma_j \colon x \mapsto (dj_x(\mathcal{R}), \nu_x(j)). \end{array}$$

The n-frame $\mathcal{R}$ gives a trivialization of $T_{2n,n+1}(M_0^n)$. There are obvious identifications :

$$\begin{array}{rclcl} \pi_0(\Gamma(T_{2n,n+1}(M_0^n))) & \simeq & [M_0^n, V_{2n,n+1}] & \simeq & [\mathbb{S}^1 \vee \mathbb{S}^{n-1}, V_{2n,n+1}] \\ & \simeq & [\mathbb{S}^1, V_{2n,n+1}] & \times & [\mathbb{S}^{n-1}, V_{2n,n+1}] \\ & \simeq & \pi_{n-1}(V_{2n,n+1}). & & \end{array}$$

As usual $\mathbb{S}^1 \vee \mathbb{S}^{n-1}$ denotes the one-point union of $\mathbb{S}^1$ and $\mathbb{S}^{n-1}$. Let θ be any point of $\mathbb{S}^1$. The map $\Phi_\sharp \colon \pi_0(E_n) \longrightarrow \mathbb{Z}_2$ is just given by the class of the map :

$$\begin{array}{ccc} \{\theta\} \times \mathbb{S}^{n-1} & \longrightarrow & V_{2n,n+1} \\ x & \longmapsto & (dj_x(\mathcal{R}), \nu_x(j)) \end{array}$$

in $\pi_{n-1}(V_{2n,n+1}) \simeq \mathbb{Z}_2$. We call this class $\epsilon_{\mathcal{R}}(j)$. This modulo 2 number tells us to which component of E_n the map j belongs.

2.5.3 Proof of theorem 2

Here again, n is 4 or 8. If j is Lagrangian, it is natural to introduce another map :

$$\begin{array}{rccc} \overline{T}_j\colon & \{\theta\} \times \mathbb{S}^{n-1} & \longrightarrow & SO(2n) \\ & x & \longmapsto & (dj_x(\mathcal{R}), J dj_x(\mathcal{R})). \end{array}$$

Let F be the forgetting map $SO(2n) \longrightarrow V_{2n,n+1}$. The composition yields a new map :

$$\begin{array}{rcc} F \circ \overline{T}_j \colon \{\theta\} \times \mathbb{S}^{n-1} & \longrightarrow & V_{2n,n+1} \\ x & \longmapsto & (dj_x(\mathcal{R}), J dj_x(v)). \end{array}$$

By theorem 1, the nonvanishing vector field $Jdj(v)$ is exactly $\nu(j)$ (on M_0^n). Thus, the class $[F \circ \overline{T}_j]$ of this map in $\pi_{n-1}(V_{2n,n+1})$ is $\epsilon_{\mathcal{R}}(j)$. The map F induces the modulo 2 reduction at the π_{n-1}-level. Indeed :

$$\begin{array}{ccccc} \pi_{n-1}(SO(2n)) & \xrightarrow{F_\sharp} & \pi_{n-1}(V_{2n,n+1}) & \longrightarrow & \pi_{n-2}(SO(n-1)) \\ || & & || & & || \\ \mathbb{Z} & \xrightarrow{\text{mod } 2} & \mathbb{Z}_2 & \longrightarrow & 0. \end{array}$$

The relation : $[F \circ \overline{T}_j] = F_\sharp \cdot [\overline{T}_j]$ implies that $\epsilon_{\mathcal{R}}(j)$ is the modulo 2 reduction of the class $[\overline{T}_j] \in \pi_{n-1}(SO(2n))$. Let $i \colon U(n) \longrightarrow SO(2n)$ be the natural inclusion, by construction the map $\overline{T}_j$ factors through $U(n)$ i. e. $\overline{T}_j = i \circ T_j$ for some $T_j\colon \{\theta\} \times \mathbb{S}^{n-1} \longrightarrow U(n)$.

IF N=8. – The study of the long homotopy exact sequence associated to the fibration $U(8) \longrightarrow SO(16) \longrightarrow SO(16)/U(8)$ shows that the map $i_\sharp\colon \mathbb{Z} = \pi_7(U(8)) \longrightarrow \pi_7(SO(16)) = \mathbb{Z}$ induced by i is the multiplication by 2. Thus, the modulo 2 reduction of the class $[\overline{T}_j]$ in $\pi_7(SO(16))$ is zero i. e. $\epsilon_{\mathcal{R}}(j) = 0$.

IF N=4. – Again by the use of the homotopy sequence of the fibration $U(4) \longrightarrow SO(8) \longrightarrow SO(8)/U(4)$, one shows that the map $i_\sharp\colon \mathbb{Z} = \pi_3(U(4)) \longrightarrow \pi_3(SO(8)) = \mathbb{Z}$ is an isomorphism. Thus, *a priori*, $\epsilon_{\mathcal{R}}(j)$ can be any of the two elements of $\mathbb{Z}_2$. Let us see that both values are actually taken.

Let $\mathbb{S}^3$ be the unit sphere of $\mathbb{H}$, v the outward unit normal field of $\mathbb{S}^3$ and R the 3-frame (iv, jv, kv). Any Lagrangian immersion g of $\mathbb{S}^3$ into $\mathbb{C}^3$ gives rise to a map :

$$\begin{array}{rccl} dg^{\mathbb{C}}\colon & \mathbb{S}^3 & \longrightarrow & U(3) \subset SO(6) \\ & x & \longmapsto & (dg_x(R), Jdg_x(R)). \end{array}$$

Let g_0 and g_1 be two Lagrangian immersions of $\mathbb{S}^3$ into $\mathbb{C}^3$ such that $[dg_0^{\mathbb{C}}]$ is zero in $\pi_3(U(3))$ and $[dg_1^{\mathbb{C}}]$ is a generator of $\pi_3(U(3))$. Then $f_i = h \times g_i$, $(i = 0, 1)$ are two Lagrangian immersions of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C} \times \mathbb{C}^3$ (h denotes the natural (Lagrangian) inclusion $\mathbb{S}^1 \subset \mathbb{C}$). Of course : $df_i^{\mathbb{C}} = dh^{\mathbb{C}} \times dg_i^{\mathbb{C}}$. A result of [3] claims that there exist two Lagrangian embeddings j_0 and j_1 of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C}^4$ such that $dj_i^{\mathbb{C}}$ is homotopic to $df_i^{\mathbb{C}}$ (as maps from $\mathbb{S}^1 \times \mathbb{S}^3$ to $U(4)$). Thus the maps T_{j_i} are homotopic to T_{f_i}. Let θ be a point of $\mathbb{S}^1$ and l be the inclusion :

$$\begin{array}{rccl} l\colon & U(3) & \longrightarrow & U(4) \\ & A & \longmapsto & \begin{pmatrix} dh_\theta^{\mathbb{C}}(v) & 0 \\ 0 & A \end{pmatrix} \end{array}$$

Then $T_{f_i} = l \circ dg_i^{\mathbb{C}}$. As l induces an isomorphism at the π_3-level, T_{f_i} represents a generator of $\pi_3(U(4))$ if $i = 1$ and the zero class if $i = 0$. Therefore, $\epsilon_{\mathcal{R}}(j_i) = i$.

2.5.4 The obstructions $\Delta(\cdot,\cdot)$ and $\epsilon(\cdot,\cdot)$.

We first need a preparatory lemma. Let $n \geq 2$ and let Ψ be the classical isomorphism between $U(n)$ and $\mathbb{S}^1 \times SU(n)$. We denote by $\iota_{SU(n)} \in H^3(SU(n), \mathbb{Z})$ the characteristic class of $SU(n)$ and by a the class $\Psi^* \iota_{SU(n)}$. The following lemma is well known (see theorem 7.16 p. 146 of [92]).

Lemma D. – *The "degree map" :* $[M^n, U(n)] \xrightarrow{deg} H^3(M^n, \mathbb{Z})$, $f \longmapsto f^* a$ *is a group homomorphism.*

Let M^n be a manifold admitting a Lagrangian immersion into $\mathbb{C}^n$, s_0 an unitary trivialization of $TM^n \otimes \mathbb{C}$ and let $(e_1, ..., e_n)$ be an unitary basis of $\mathbb{C}^n$. The theorem of Gromov-Lees asserts that the map

$$\begin{array}{ccc} I_{Lag}(M^n, \mathbb{C}^n) & \longrightarrow & C^0(M^n, U(n)) \\ j & \longmapsto & dj^{\mathbb{C}}(s_0) \end{array}$$

is a weak homotopy equivalence. By using this theorem, it is easy to build invariants for Lagrangian immersions. For instance, the class $\delta(j, s_0) = (dj^{\mathbb{C}}(s_0))^* a$ is an obvious one. (Lemma D shows that $\delta(j, s_0)$ does not depend on the choice of the unitary basis of $\mathbb{C}^n$). If s_1 is another trivialization, then there exists a map $f\colon M^n \longrightarrow U(n)$ such that $s_1 = s_0 f$. By Lemma D :

$$\delta(j, s_1) = (dj^{\mathbb{C}}(s_1))^* a = (dj^{\mathbb{C}}(s_0) \cdot f)^* a = \delta(j, s_0) + f^* a.$$

Therefore, if j_0 and j_1 are two Lagrangian immersions, the class

$$\Delta(j_0, j_1) = \delta(j_1, s_0) - \delta(j_0, s_0)$$

does not depend on the choice of the trivialization. By construction, this class is an obstruction to the existence of a Lagrangian regular homotopy between j_0 and j_1. Let $\epsilon(j_0, j_1)$ be the modulo 2 reduction of $\Delta(j_0, j_1)$.

Lemma E. – *Let j_0, j_1 be two Lagrangian embeddings of $\mathbb{S}^1 \times \mathbb{S}^3$ into $\mathbb{C}^4$. Then $\epsilon(j_0, j_1)$ is the only obstruction to the existence of a smooth isotopy between j_0 and j_1.*

Proof of lemma E. – Let k be the inclusion $\{0\} \times \mathbb{S}^3 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^3$. One has the following commutative diagram :

$$\begin{array}{ccccc}
\pi_3(U(4)) \simeq [\mathbb{S}^3, U(4)] & \xrightarrow{deg} & H^3(\mathbb{S}^3, \mathbb{Z}) & \xrightarrow{(k^*)^{-1}} & H^3(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{Z}) \\
F \circ i \ \downarrow & & \downarrow \rho_2 & & \downarrow \rho_2 \\
\pi_3(V_{8,5}) \simeq [\mathbb{S}^3, V_{8,5}] & \xrightarrow{deg} & H^3(\mathbb{S}^3, \mathbb{Z}_2) & \xrightarrow{(k^*)^{-1}} & H^3(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{Z}_2)
\end{array}$$

where k^* and the degree maps are isomorphisms. (The second degree map is just $f \mapsto f^* \iota_{V_{8,5}}$ where $\iota_{V_{8,5}} \in H^3(V_{8,5}, \mathbb{Z}_2)$ is the characteristic class of $V_{8,5}$). The first vertical maps between the cohomology groups is induced by the coefficient homomorphism :

$$\mathbb{Z} = \pi_3(U(4)) \stackrel{mod\ 2}{\longrightarrow} \pi_3(V_{8,5}) = \mathbb{Z}_2$$

(see for instance [4] pp. 275-278 for this point). Let s_0 be the trivialization of $TM^n \otimes \mathbb{C}$ induced by the global n-frame $\mathcal{R}$. It is readily seen that : $(k^*)^{-1} deg\ [T_j] = \delta(j, s_0)$ and that $(k^*)^{-1} deg\ \epsilon_{\mathcal{R}} = (k^*)^{-1} deg\ [F \circ i \circ T_j] = \rho_2 \delta(j, s_0)$. Hence :

$$\epsilon(j_0, j_1) = (k^*)^{-1} deg\ (\epsilon_{\mathcal{R}}(j_0) - \epsilon_{\mathcal{R}}(j_1)).$$

$\square$

2.5.5 The action of $Diff(M^n)$ on $\pi_0(E(M^n, \mathbb{C}^n))$.

We first recall an alternative description of the solution to the embeddings classification problem. Any embedding $j\colon M^n \longrightarrow \mathbb{R}^{2n}$ gives rise to a $\mathbb{Z}_2$-equivariant map :

$$\begin{array}{ccc}
M^n \times M^n - \Delta M^n & \longrightarrow & \mathbb{S}^{2n-1} \\
(x, y) & \longmapsto & \frac{j(x) - j(y)}{\|j(x) - j(y)\|}
\end{array}$$

(ΔM^n denotes the diagonal of $M^n \times M^n$). Let $\mathcal{S}$ be the space of $\mathbb{Z}_2$-equivariant maps of $M^n \times M^n - \Delta M^n$ into $\mathbb{S}^{2n-1}$. A celebrated result of A. Haefliger [47] states that, if $n \geq 4$, there is a one-to-one correspondence between $\pi_0(E(M^n, \mathbb{R}^{2n}))$ and the set $\pi_0(\mathcal{S})$. The computation of $\pi_0(\mathcal{S})$ is a classical problem in homotopy theory. Let $M^* = (M^n \times M^n - \Delta M^n)/\mathbb{Z}_2$ be the reduced symmetric product of M^n and P be the bundle $(M^n \times M^n - \Delta M^n) \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow M^*$. There is a natural bijection between $\mathcal{S}$ and the space of sections of P, thus $\pi_0(\mathcal{S}) = \pi_0(\Gamma(P))$. Since the fiber of P is the sphere $\mathbb{S}^{2n-1}$, one has : $\pi_0(E(M^n, \mathbb{R}^{2n})) \simeq \pi_0(\Gamma(P)) \simeq H^{2n-1}(M^*, \mathbb{Z})$.

Lemma G. – *The action of* $Diff(\mathbb{S}^1 \times \mathbb{S}^3)$ *on* $\pi_0(E(\mathbb{S}^1 \times \mathbb{S}^3, \mathbb{C}^4))$ *is trivial.*

Proof of lemma G. – A diffeomorphism ϕ of M^n induces a $\mathbb{Z}_2$-equivariant diffeomorphism of $M^n \times M^n - \Delta M^n$ and thus a diffeomorphism Φ of M^*. The action of ϕ on $\pi_0(E(M^n, \mathbb{R}^{2n}))$ is given by the induced isomorphism Φ^* on $H^{2n-1}(M^*, \mathbb{Z})$. It is well-known that for an orientable even dimensional manifold M^n, the cohomology group $H^{2n-1}(M^*, \mathbb{Z})$ is isomorphic to $H^{n-1}(M^n, \mathbb{Z}_2)$. In our case $M^n = \mathbb{S}^1 \times \mathbb{S}^3$ and one gets $H^7(M^*, \mathbb{Z}) = \mathbb{Z}_2$; Φ^* is simply the identity. $\square$

On Totally Real Isotopy Classes

Int. Math. Res. Not. (2002), 89-109.

Abstract. – Given two totally real embeddings $j_0, j_1 : M^n \to \mathbb{C}^n$ is it possible to find an isotopy joining j_0 and j_1 and such that j_t is totally real for every t? A positive answer obviously implies that j_0 and j_1 belong to the same arcwise connected component of the space $I_{TR} = I_{TR}(M^n, \mathbb{C}^n)$ of totally real immersions and of the space E of ordinary embeddings. Let $E_{TR} = I_{TR} \cap E$ be the space of totally real embeddings and $\epsilon \in \pi_0(E)$, $\iota \in \pi_0(I_{TR})$. In what follows we give an upper bound to the cardinality of $\pi_0(E_{TR} \cap \epsilon \cap \iota)$. In some cases - for instance if the dimension n is 3 modulo 4 - this upper bound is one and so we deduce the computation of $\pi_0(E_{TR})$.

3.1 Introduction and main results.

Let M^n be a n-dimensional manifold without boundary and (W^{2n}, J) be a $2n$-dimensional almost complex manifold. An immersion $j : M^n \longrightarrow W^{2n}$ is said to be *totally real* if for every point p of M^n the complexified tangential map $dj_p^{\mathbb{C}} : T_pM^n \otimes \mathbb{C} \longrightarrow T_{j(p)}W^{2n}$ is of maximal rank. We denote by $I_{TR}(M^n, W^{2n})$ - or simply I_{TR} - the space of totally real immersions of M^n into W^{2n}, by E_{TR} the space of totally real embeddings, by I and E the spaces of ordinary immersions and ordinary embeddings (all these spaces are provided with the compact-open topology). Since the works of M. Gromov [44] and J. A. Lees [67], the space I_{TR} is well understood, so the next step is the study of the space E_{TR}. In [3], M. Audin gives a topological characterization of the manifolds M^n which admit a totally real embeddings into $\mathbb{C}^n$ (at least for $n \not\equiv 3$ [4]). Here we consider the following question.

Q : Given two totally real embeddings j_0 and j_1, is it possible to find an isotopy j_t joining j_0 and j_1 and such that j_t is totally real for every t?

We call such an isotopy, a totally real isotopy. One trivial necessary condition for a positive answer is that the two embeddings must belong at once to the same component of E and of I_{TR}. Recall that $\pi_0(E)$ has been computed (when $n \geq 4$) by A. Haefliger and M. Hirsch [46],[49], and $\pi_0(I_{TR})$ by M. Gromov and J. A. Lees. So one can ask a more precise question :

Q' : Given two totally real embeddings j_0 and j_1 that belong to the same component of E and of I_{TR}, is it possible to find a totally real isotopy joining j_0 and j_1?

In this article, we partially answer to this question. We mainly use a parametric version of a h-principle due to M. Gromov combined with the study of the pairs (I, I_{TR}) and (I, E).

Let $\epsilon \in \pi_0(E)$ and $\iota \in \pi_0(I_{TR})$, we put $E_{TR}(\epsilon, \iota) = E_{TR} \cap \epsilon \cap \iota$. Of course $E_{TR} = \cup_{\epsilon,\iota} E_{TR}(\epsilon, \iota)$ and some of the $E_{TR}(\epsilon, \iota)$ may be empty. We denote by $N(\epsilon, \iota)$ the cardinality of $\pi_0(E_{TR}(\epsilon, \iota))$. If $N(\epsilon, \iota) = 1$ then the answer to the question Q' is positive for every couple of totally real embeddings such that $j_0, j_1 \in \epsilon \cap \iota$. Let $I^0_{TR} = \{j \in I_{TR}$: there exists a path inside I joining j to an ordinary embedding }. If $\iota \notin \pi_0(I^0_{TR})$ then $E_{TR}(\epsilon,\iota) = \emptyset$ but the converse is still true. Let $\iota \in \pi_0(I^0_{TR})$, then for every ϵ in $\pi_0(E)$, one has $E_{TR}(\epsilon,\iota) \neq \emptyset$ i. e. $N(\epsilon,\iota) \geq 1$. This is a consequence of a theorem of M. Gromov ([44], p. 192). Using the h-principle for totally real embeddings and a combinatoric argument we obtain an upper bound for $N(\epsilon,\iota)$.

Theorem 1. – *Let M^n be a closed, connected, n-dimensional manifold admitting a totally real embedding in j into W^{2n}. Let $\epsilon = [j] \in \pi_0(E)$ and $\iota = [j] \in \pi_0(I^0_{TR})$. If $|\pi_1(I,j)|$ is finite, then : $N(\epsilon,\iota) \leq |\pi_1(I,j)|$.*

When $W^{2n} = \mathbb{C}^n$, we can improve theorem 1 and obtain the answer to the question Q' at least for some dimensions.

Theorem 2. – *Let M^n $(n \geq 3)$ be a closed, connected, n-dimensional manifold admitting a totally real embedding j into $\mathbb{C}^n$. Let $\epsilon = [j] \in \pi_0(E)$ and $\iota = [j] \in \pi_0(I^0_{TR})$, then*

n	M^n *orientable*	M^n *non orientable*
$n \equiv 0$ [2]	$N(\epsilon,\iota)$ *is finite under condition* C_1	$N(\epsilon,\iota) \leq 2$
$n \equiv 1$ [4]	$N(\epsilon,\iota) \leq 2$ $N(\epsilon,\iota) = 1$ *under condition* C_2	$N(\epsilon,\iota) \leq 4$
$n \equiv 3$ [4]	$N(\epsilon,\iota) = 1$	$N(\epsilon,\iota) \leq 2$

where the C_i are the following conditions :

(C_1) $H_1(M^n,\mathbb{Z})$ is finite,

(C_2) Coker $Sq^2 \circ \rho_2 = \{0\}$ where $\rho_2 : H^{n-2}(M^n,\mathbb{Z}) \to H^{n-2}(M^n,\mathbb{Z}_2)$ is the reduction modulo 2 and $Sq^2 : H^{n-2}(M^n,\mathbb{Z}_2) \longrightarrow H^n(M^n,\mathbb{Z}_2)$ the Steenrod squaring operation.

One deduces :

Corollary 2.1 – *Let M^n $(n \geq 3)$ be a closed, connected, orientable manifold admitting a totally real embedding j into $\mathbb{C}^n$.*

1) If $n \equiv 3$ [4],

2) or if $n \equiv 1$ [4] and M^n satisfies the condition C_2,

then :

$$\pi_0(E_{TR}) \simeq \pi_0(E) \times \pi_0(I^0_{TR})$$

and the bijection is given by the induced map of the diagonal map $E_{TR} \longrightarrow E \times I^0_{TR}$.

When $W^{2n} = \mathbb{C}^n$ the space I^0_{TR} has a very simple interpretation in terms of the algebraic self-intersection number $\mathcal{I}_a$.

1) If n is odd, there is a natural splitting of I_{TR} into $I^0_{TR} \cup I^1_{TR}$ given by $\mathcal{I}_a \in \mathbb{Z}_2$. Namely, a totally real immersion is in I^0_{TR} (resp. in I^1_{TR}) if $\mathcal{I}_a = 0$ (resp. $\mathcal{I}_a = 1$). M. Audin proves that $I_{TR} = I^0_{TR}$ if dim $M^n \neq 2^q - 1$ ([3], th. 0.1) or if $M^n$is parallelizable and $n \neq 1, 3$ ([3], prop. 0.3). In an other hand, she shows that, for every q, there exists a

manifold M^{2^q-1} (admitting a totally real embedding into $\mathbb{C}^n$) such that $I^1_{TR} \neq \emptyset$.

2) If n is even then $I^0_{TR} = I_{TR}$ since the algebraic self-intersection number must be equal (up to the sign) to half the Euler-Poincaré caracteristic $\chi(M^n)$ and this latter is zero because the tangent space and the normal space are isomorphic via J.

Thus in many case $I^0_{TR} = I_{TR}$ and according to [44],[67] (see also [31]) :

$$\pi_0(I_{TR}) = [M^n, U(n)] = \tilde{K}^{-1}(M^n),$$

where $\tilde{K}^{-1}(M^n)$ denote the reduced K-theory group of the suspension of M^n. Let us do, as examples, some explicite computations of $\pi_0(E_{TR})$.

THE SPHERE $\mathbb{S}^3$. – In [45] M. Gromov shows that $\mathbb{S}^n$ admits a totally real embedding into $\mathbb{C}^n$ if and only if $n = 1$ or 3. An explicit example of a totally real embedding of $\mathbb{S}^3$ into $\mathbb{C}^3$ is built in [1]. As we shall see, there is in fact many "different" totally real embeddings of $\mathbb{S}^3$ into $\mathbb{C}^3$. Let $V_{6,3}$ be the Stiefel manifold of 3-frames of $\mathbb{R}^6$. According to [55] and [45], one has :

$$\begin{array}{lclclcl} \pi_0(I_{TR}) & \simeq & [\mathbb{S}^3, U(3)] & \simeq & \pi_3(U(3)) & \simeq & \mathbb{Z} \\ \pi_0(I) & \simeq & [\mathbb{S}^3, V_{6,3}] & \simeq & \pi_3(V_{6,3}) & \simeq & \mathbb{Z}_2. \end{array}$$

With consistent identifications, the map $\pi_0(I_{TR}) \longrightarrow \pi_0(I)$ induced by the inclusion $I_{TR} \subset I$ is the same as the map $l_\sharp : \mathbb{Z} \simeq \pi_3(U(3)) \longrightarrow \pi_3(V_{6,3}) \simeq \mathbb{Z}_2$ induced by the inclusion $l : U(3) \to V_{6,3}$. This last map is the reduction modulo 2 (see the proof of theorem 2, $n = 3$). It is then clear that : $\pi_0(I_{TR}) \simeq ker\ l_\sharp \simeq 2\mathbb{Z}$. According to [48], one has $\pi_0(E(\mathbb{S}^3, \mathbb{C}^3)) \simeq \mathbb{Z}$, so : $\pi_0(E_{TR}(\mathbb{S}^3, \mathbb{C}^3)) \simeq \mathbb{Z} \oplus 2\mathbb{Z}$.

THE PROJECTIVE SPACES $\mathbb{R}P^3$ AND $\mathbb{R}P^7$. – The projective spaces $\mathbb{R}P^n$ that admit a totally real embedding into $\mathbb{C}^n$ are $\mathbb{R}P^1$, $\mathbb{R}P^3$ and $\mathbb{R}P^7$ (see. [3]). One can find an explicit totally real embedding of $\mathbb{R}P^3$ into $\mathbb{C}^3$ in [32]. Similarly to the case of $\mathbb{S}^3$ we get : $\pi_0(E_{TR}(\mathbb{R}P^3, \mathbb{C}^3)) \simeq \pi_0(E(\mathbb{R}P^3, \mathbb{C}^3)) \oplus 2\mathbb{Z}$. For $\mathbb{R}P^7$, $\pi_0(E(\mathbb{R}P^7, \mathbb{C}^7)) \simeq H_1(\mathbb{R}P^7, \mathbb{Z}) = \mathbb{Z}_2$ and $I^0_{TR} = I_{TR} = \tilde{K}^{-1}(\mathbb{R}P^7) = \mathbb{Z}$ (see [2]) so : $\pi_0(E_{TR}(\mathbb{R}P^7, \mathbb{C}^7)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$.

THE PRODUCT OF SPHERES $\mathbb{S}^3 \times \mathbb{S}^{n-3}$ ($n = 3 + 4k, k \geq 1$). – It was first shown in [83] that $M^n = \mathbb{S}^3 \times \mathbb{S}^{n-3}$ admits a totally real emdedding into $\mathbb{C}^n$. This manifold is parallelizable (thus $I^0_{TR} = I_{TR}$) and simply connected (thus $\pi_0(E) = \{0\}$). Therefore : $\pi_0(E_{TR}) = \pi_0(I^0_{TR}) = \pi_0(I_{TR}) = \tilde{K}^{-1}(M^n)$ i. e. $\pi_0(E_{TR}(\mathbb{S}^3 \times \mathbb{S}^{n-3}, \mathbb{C}^n)) = \{0\} \oplus \mathbb{Z}^2$.

THE TORUS $\mathbb{T}^n$ ($n = 3 + 4k,\ k \geq 0$). – If $k \geq 1$, $I^0_{TR} = I_{TR}$ and we immediatly get : $\pi_0(E_{TR}(\mathbb{T}^n, \mathbb{C}^n)) = \mathbb{Z}^n \oplus \mathbb{Z}^{2^{n-1}}$. If $k = 0$, $\pi_0(I_{TR}) \simeq [\mathbb{T}^3, U(3)] \simeq [\mathbb{T}^3, \mathbb{S}^1] \times [\mathbb{T}^3, SU(3)] \simeq \mathbb{Z}^3 \times \mathbb{Z}$. With consistent identifications : $\pi_0(E_{TR}(\mathbb{T}^3, \mathbb{C}^3)) \simeq \pi_0(E(\mathbb{T}^3, \mathbb{C}^3)) \oplus (\mathbb{Z}^3 \times 2\mathbb{Z})$.

The cardinal $N(\epsilon, \iota)$ can be strictly greater than 1. When $n = 2$, such phenomenon was first noticed by T. Fiedler [30] and by L. Polterovich [79].

Theorem 3 . – *Let $M^n = \mathbb{T}^n$ with $n \geq 4$ even and let ι, ϵ be the components of I^0_{TR} and E containing the Lagrangian natural inclusion $\mathbb{T}^n \subset \mathbb{C}^n$. Then $N(\epsilon, \iota)$ is not finite.*

Acknowledgements. – I would like to thank D. Spring for fruitful discussions ; F. Havet, S. Thomassé for their help in Lemma A and T. Yasui for his interest and for giving me simpler proofs in §3.

3.2 Reduction to a problem of homotopy theory

3.2.1 A theorem from Convex Integration Theory

If E is a bundle, we denote by $\Gamma(E)$ the space of sections of E. Let X be the product $M^n \times W^{2n}$, $X^{(1)}$ be the 1-jet space of the bundle $X \to M^n$ and $\mathcal{R} \subset X^{(1)}$ be the total reality condition.

Theorem. – *Let $f_t : M^n \to W^{2n}$ be an isotopy of embeddings joining two totally real embeddings f_0 and f_1, and let $H_{t,s} : TM^n \to TW^{2n}$ be a homotopy of bundle momomorphisms such that :*

$$\begin{array}{ll} H_{t,0} = df_t & \forall\, t \in [0,1] \\ H_{0,s} = df_0 & \forall\, s \in [0,1] \\ H_{1,s} = df_1 & \forall\, s \in [0,1] \\ H_{t,1} \in \Gamma(\mathcal{R}) & \forall\, t \in [0,1]. \end{array}$$

Then there exists an isotopy of embeddings $F_{t,s}$ such that :

$$\begin{array}{ll} F_{t,0} = f_t & \forall\, t \in [0,1] \\ F_{0,s} = f_0 & \forall\, s \in [0,1] \\ F_{1,s} = f_1 & \forall\, s \in [0,1] \end{array}$$

and $F_{t,1}$ is a totally real isotopy.

This theorem is a parametric version of a theorem of M. Gromov ([44], p. 192, see also [84], p. 164). The transition from a h-principle to a parametric h-principle is classical in convex integration theory. (See the remark p. 17 of [44] or [84] p. 139-144).

3.2.2 Sketch of the proofs

Let j_0 and j_1 be two totally real embeddings of $E_{TR}(\epsilon, \iota)$, f_t an (ordinary) isotopy joining j_0 and j_1 and g_t a regular totally real homotopy joining j_0 and j_1. One can build with f_t and g_t a loop $\gamma(f,g)$ in I by putting :

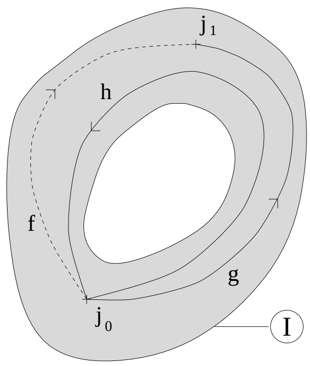
$$\gamma(f,g)(t) = \begin{cases} f_{2t} & \text{if } t \leq 1/2, \\ g_{2(1-t)} & \text{if } t \geq 1/2. \end{cases}$$

According to the theorem of the preceding section, if $[\gamma(f,g)]$ is zero in $\pi_1(I, j_0)$ then there exists a totally real isotopy j_t joining j_0 and j_1. This is trivially the case when $\pi_1(I, j_0) = 0$.

Suppose now that $[\gamma(f,g)]$ is not zero but there exists $[h] \in \pi_1(I_{TR}, j_0)$ such that $l_\sharp([h]) = [\gamma(f,g)]^{-1}$ ($l_\sharp$ is the homomorphism induced at the π_1 level by the inclusion $I_{TR} \subset I$). Replacing g by $\tilde{g}$:

$$\tilde{g}(t) = \begin{cases} h_{2t} & \text{if } t \leq 1/2, \\ g_{2t-1} & \text{if } t \geq 1/2, \end{cases}$$

one obtains a new element $[\gamma(f,\tilde{g})]$ which is null-homotopic.



Similarly, one can perform the same trick if there exists $[h] \in \pi_1(E, j_0)$ such that $i_\sharp([h]) = [\gamma(f,g)]$ (this time $i_\sharp$ is the homomorphism induced at the π_1 level by the inclusion $E \subset I$). This technique leads to theorems 1 and 2.

3.2.3 Proof of theorem 1.

Assume that $|\pi_1(I,j)| = n$ and that $N(\epsilon,\iota) > n$. Let $x_0 = [j], x_1[j_1], ..., x_n = [j_n]$ be $n+1$ distinct elements of $\pi_0(E_{TR}(\epsilon,\iota))$. For each couple (x_i, x_k) with $i < k$, there is a loop $\gamma_{ik} = \gamma(f_{ik}, g_{ik})$ where $f^t_{ik} \in \epsilon$ is an isotopy joining j_i and j_k and $g^t_{ik} \in \iota$ is a regular totally real homotopy joining j_i and j_k (by convention, $j_0 = j$). This loop determines an element $[\gamma_{ik}]$ in $\pi_1(I,j)$. If $i > k$, one defines γ_{ki} to be $\gamma(g_{ik}, f_{ik})$ and thus $[\gamma_{ki}] = -[\gamma_{ik}]$. By the Convex Integration Theorem, since j_i and j_k belong to two different arcwise connected components of $E_{TR}(\epsilon,\iota)$, the element $[\gamma_{ik}]$ is not zero in $\pi_1(I,j)$. We encode all these data in a complete directed $(n+1)$-graph K_{n+1} with vertex set $V(K_{n+1}) = \{x_0, ..., x_n\}$. In this way, the elements $[\gamma_{ij}]$ appear as the values of a weight function f defined on the set of directed edges $E(K_{n+1})$:

$$\begin{array}{rccc} f: & E(K_{n+1}) & \longrightarrow & G^* = \pi_1(I,j_0) \setminus \{0\} \\ & x_i x_k & \longmapsto & [\gamma_{ik}], \end{array}$$

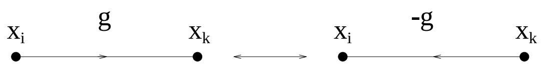
and satisfying the property (P) $\forall i \neq k, \ : \ f(x_i x_k) = -f(x_k x_i)$. We now need a purely combinatorial result.

Lemma A. – *Let G be a group of order n, K_{n+1} be the complete directed $(n+1)$-graph and $f : E(K_{n+1}) \longrightarrow G^* = G \setminus \{0\}$ be a weight function satisfying the property (P). Then there exists a triple (x_i, x_j, x_k) of distinct vertices of K_{n+1} such that : $f(x_i x_j) + f(x_j x_k) = 0$.*

We postpone the proof of this lemma. Let (x_i, x_j, x_k) be a triple satisfying $f(x_i x_j) + f(x_j x_k) = [\gamma_{ij}] + [\gamma_{jk}] = 0$ with loops $\gamma_{ij} = \gamma(f_{ij}, g_{ij})$ and $\gamma_{jk} = \gamma(f_{jk}, g_{jk})$. Then the loop

$\gamma(f_{ij} + f_{jk}, g_{ij} + g_{jk})$ is contractible and thus by the Convex Integration Theorem, j_i and j_k belong to the same arcwise connected component of $E_{TR}(\epsilon, \iota)$. This is a contradiction. $\square$

Proof of the lemma A. – One can visualize the situation with a tournament (an oriented K_{n+1}) provided with a weight function (see the figure above). Nevertheless, this representation is not unique. Indeed, the property (P) allows us to make the following change in the graph :

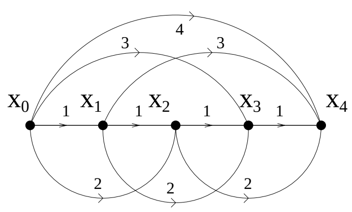


By applying several changes of this kind, one can assume that the tournament representing all the data is in fact a total order on the set $V(K_{n+1})$.
Let x be the minimal element. We can assume $x = x_0$. Since $|G^*| = n - 1$, the n-tuple $(f(x_0x_1), ..., f(x_0x_n))$ contains at least two elements $f(x_0x_p)$ and $f(x_0x_q)$ which are identical. Then (x_p, x_0, x_q) is the desired triple. $\square$

Remark. – In the preceding result, the number of vertices on the complete graph cannot be reduced even if one replaces the conclusion of the lemma A by the following weaker conclusion :

There exists a m-tuple $(x_{i_1}, ..., x_{i_m})$ such that :
1) the loop $(x_{i_1}, ..., x_{i_m})$ contains no cycle,
2) $f(x_{i_1}x_{i_2}) + ... + f(x_{i_{m-1}}x_{i_m}) = 0$.

that also suits to obtain a contradiction. Here is an example in which $G = \mathbb{Z}_5$, $f(x_ix_{i+k}) = k$ and the complete graph K_5 does not satisfy the conditions 1 and 2.



Let i be the natural inclusion $E \subset I$ and $i_\sharp$ the induced map at the π_1-level. We shall need the following result to prove theorem 2.

Lemma B. – *If $\pi_1(I, j)/i_\sharp(\pi_1(E, j))$ has a finite number of elements then $N(\epsilon, \iota) \leq |\pi_1(I, j)/i_\sharp(\pi_1(E, j))|$.*

The proof of lemma B is similar to one of lemma A. Note that $\pi_1(I, j_0)$ is Abelian and so the quotient makes sense. We shall also need the same kind of result for the natural inclusion $l : I_{TR} \to I$.

Lemma B'. – *If* $\pi_1(I,j)/l_\sharp(\pi_1(I_{TR},j))$ *has a finite number of elements then* $N(\epsilon,\iota) \leq |\pi_1(I,j)/l_\sharp(\pi_1(I_{TR},j))|$.

3.3 Proof of theorem 2 for $n \geq 4$

We first recall the results of A. Haefliger and M. Hirsch [46],[47],[49],[50],[55], on the classification of embeddings and immersions of a n-dimensional closed connected manifold M^n into $\mathbb{R}^m$. As we are only interested in the n-codimensional case, we take $m = 2n$.

Let SM be the tangent sphere bundle of M^n. Every immersion $f : M^n \longrightarrow \mathbb{R}^{2n}$ gives rise to a $\mathbb{Z}_2$-equivariant map :

$$\begin{array}{rccc} D(f): & SM & \longrightarrow & \mathbb{S}^{2n-1} \\ & U_x & \longmapsto & \frac{df_x(U_x)}{\|df_x(U_x)\|}. \end{array}$$

If $n \geq 2$, there is a one-to-one correspondence between the set of regular homotopy classes of immersions $\pi_0(I(M^n,\mathbb{R}^{2n}))$ and the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps of SM into $\mathbb{S}^{2n-1}$. Let $PM = SM/\mathbb{Z}_2$ be the tangent projective bundle. There is a natural bijection between $\mathbb{Z}_2$-equivariant maps of SM into $\mathbb{S}^{2n-1}$ and the sections of the bundle $SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow PM$. Let $[PM, SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1}]$ denote the set of homotopy classes of the sections of this bundle. As the fiber $\mathbb{S}^{2n-1}$ is $(2n-2)$-connected, one has :

$$\pi_0(I(M^n,\mathbb{R}^{2n})) \simeq [PM, SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1}] \simeq H^{2n-1}(PM,\mathbb{Z}).$$

This last cohomology group is isomorphic to $\mathbb{Z}$ if n is even and to $\mathbb{Z}_2$ if n is odd.

Let ΔM be the diagonal of $M^n \times M^n$. Every embedding $f : M^n \to \mathbb{R}^{2n}$ gives rise to a $\mathbb{Z}_2$-equivariant map :

$$\begin{array}{ccc} M^n \times M^n - \Delta M & \longrightarrow & \mathbb{S}^{2n-1} \\ (x,y) & \longmapsto & \frac{f(x)-f(y)}{\|f(x)-f(y)\|}. \end{array}$$

If $n \geq 4$, there is a one-to-one correspondence between the set of isotopy classes of embeddings $\pi_0(E(M^n,\mathbb{R}^{2n}))$ and the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps of $M^n \times M^n - \Delta M$ into $\mathbb{S}^{2n-1}$. Let $M^* = (M^n \times M^n - \Delta M)/\mathbb{Z}_2$ be the reduced symmetric product of M^n. There is a natural bijection between $\mathbb{Z}_2$-equivariant maps of $M^n \times M^n - \Delta M$ into $\mathbb{S}^{2n-1}$ and the sections of the bundle $(M^n \times M^n - \Delta M) \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow M^*$. The fiber is still $\mathbb{S}^{2n-1}$ and one has :

$$\pi_0(E(M^n,\mathbb{R}^{2n})) \simeq [M^*, P] \simeq H^{2n-1}(M^*,\mathbb{Z}),$$

where we have written P for the total space $(M^n \times M^n - \Delta M) \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1}$. The cohomology group $H^{2n-1}(M^*,\mathbb{Z})$ is isomorphic to $H^{n-1}(M^n,\mathbb{Z}_2)$ if M^n is orientable and n is even or M^n is non orientable and n is odd. It is isomorphic to $H^{n-1}(M^n,\mathbb{Z})$ if M^n is orientable and n is odd and to $\mathbb{Z} \oplus ker\ Sq^1$ if M^n is non orientable and n is even ($Sq^1 : H^{n-1}(M^n,\mathbb{Z}_2) \longrightarrow H^n(M^n,\mathbb{Z}_2)$).

The tangent sphere bundle SM is diffeomorphic to the boundary of a tubular neighborhood of the diagonal ΔM in $M^n \times M^n$. In this way, the projective tangent bundle PM can be seen as siting inside M^*. An immersion $f : M^n \longrightarrow \mathbb{R}^{2n}$ $(n \geq 3)$ is regularly homotopic to an embedding if and only if the section $\tilde{f}$ of $SM \times_{\mathbb{Z}_2} \mathbb{S}^{2n-1} \longrightarrow PM$ induced by $D(f)$ can be extended all over M^*. Assume that f is an embedding. If $n \geq 4$, there is a one-to-one correspondence between $\pi_1(I(M^n, \mathbb{R}^{2n}), E(M^n, \mathbb{R}^{2n}), f)$ and the set $[M^*, P; \tilde{f}]$ of rel. PM homotopy classes of extensions of $\tilde{f}$. Thus, one has :

$$\pi_1(I(M^n, \mathbb{R}^{2n}), E(M^n, \mathbb{R}^{2n}), f) \simeq [M^*, P; \tilde{f}].$$

Since $H^{2n}(M^*, PM, \mathbb{Z}_2) = \mathbb{Z}_2$ is non zero, the computation of $[M^*, P; \tilde{f}]$ is more complicated. Let η be the line bundle over M^* associated with $M \times M - \Delta M \longrightarrow M^*$ and $\xi = \eta \oplus ... \oplus \eta = 2n\eta$. There is a filtration

$$[M^*, P; \tilde{f}] = F_0 \supset F_1 \supset F_2 = 0$$

with

1) $F_0/F_1 = H^{2n-1}(M^*, PM, \mathbb{Z}[w_1(\xi)])$,
2) $F_1 = \operatorname{Coker}(\Theta : H^{2n-2}(M^*, PM, \mathbb{Z}[w_1(\xi)]) \longrightarrow H^{2n}(M^*, PM, \mathbb{Z}_2))$.

The map Θ is the composition $\alpha \circ \rho_2$ where the map ρ_2 is the reduction modulo 2 : $H^{2n-2}(M^*, PM, \mathbb{Z}[w_1(\xi)]) \to H^{2n}(M^*, PM, \mathbb{Z}_2)$ and the map $\alpha : H^{2n}(M^*, PM, \mathbb{Z}_2) \to H^{2n}(M^*, PM, \mathbb{Z}_2)$ is defined by $\alpha(x) = Sq^2 x + Sq^1(x) \cup w_1(\xi) + x \cup w_1(\xi)^2 + x \cup w_2(\xi)$ (see [65] for example). Let $v = w_1(\eta)$, note that $w_1(\xi) = \binom{2n}{1} v = 0$ and $w_2(\xi) = \binom{2n}{2} v^2$. Thus in our case, the cohomology is untwisted and $\Theta(x) = Sq^2 \rho_2(x) + nv^2 \rho_2(x)$.

Lemma C.– *Let $n \geq 4$. The cohomology group $H^{2n-1}(M^*, PM, \mathbb{Z})$ is isomorphic to :*

$$\begin{cases} H^{n-1}(M^n, \mathbb{Z}) \oplus \mathbb{Z}_2 & \textit{if } M^n \textit{ is orientable and } n \textit{ is even,} \\ H^{n-1}(M^n, \mathbb{Z}_2) & \textit{if } M^n \textit{ is non orientable and } n \textit{ is even,} \\ H^{n-1}(M^n, \mathbb{Z}) & \textit{if } M^n \textit{ is orientable and } n \textit{ is odd,} \\ \ker Sq^1 \oplus \mathbb{Z}_4 & \textit{if } M^n \textit{ is non orientable and } n \textit{ is odd.} \end{cases}$$

Proof of lemma C.– The proof of this lemma is a computation using [66] and [90] (see the appendix). □

Lemma D.– *Let $n \geq 4$; Coker Θ is $\{0\}$ in the following cases :*
1) $n \equiv 3$ mod 4,
2) n is even and M^n is non orientable,
3) M^n is orientable and Coker $Sq^2 \circ \rho_2 = \{0\}$ where ρ_2 is the reduction modulo 2 and $Sq^2 \circ \rho_2 : H^{n-2}(M^n, \mathbb{Z}) \longrightarrow H^n(M^n, \mathbb{Z}_2)$.

Proof of lemma D.– We suppose that the reader is familiar with the notations and results of [66] and [90] (if not, see the appendix).

1) $n \equiv 3 \bmod 4$. Let $\beta_2 : H^{2n-3}(M^*, PM, \mathbb{Z}_2) \longrightarrow H^{2n-2}(M^*, PM, \mathbb{Z})$ be the Bockstein operator. Then :

$$\begin{array}{rcl}\Theta(\beta_2(v^{n-3}\Lambda\rho_2 M)) & = & (Sq^2+v^2)\rho_2\beta_2(v^{n-3}\Lambda\rho_2 M)\\ & = & (Sq^2+v^2)Sq^1(v^{n-3}\Lambda\rho_2 M)\\ & = & (Sq^2+v^2)(v^{n-2}\Lambda\rho_2 M)\\ & = & v^n\Lambda\rho_2 M + Sq^2 Sq^{n-2}(\Lambda\rho_2 M)\\ & = & v^n\Lambda\rho_2 M + Sq^{n-1}Sq^1\Lambda\rho_2 M\\ & = & v^n\Lambda\rho_2 M \neq 0.\end{array}$$

Therefore, Coker $\Theta = \{0\}$.

2) n is even and M^n is non orientable. Let $M' \in H^{n-1}(M^n, \mathbb{Z}_2)$ such that $Sq^1 M' = \rho_2 M$. One has : $Sq^1(\Lambda M'\Lambda M') = 0$. The Bockstein sequence

$$H^{2n-2}(X,\mathbb{Z}_2) \xrightarrow{\beta_2} H^{2n-1}(X,\mathbb{Z}) \xrightarrow{\times 2} H^{2n-1}(X,\mathbb{Z}) \xrightarrow{\rho_2} H^{2n-1}(X,\mathbb{Z}_2)$$

for $X = (\Lambda^2 M, \Delta M)$ shows that ρ_2 is a monomorphism since $H^{2n-1}(X,\mathbb{Z}) \simeq H^{n-1}(M^n, \mathbb{Z}_2)$ by lemma C. Thus :

$$Sq^1(\Lambda M'\Lambda M') = \rho_2\beta_2(\Lambda M'\Lambda M') = 0 \Longrightarrow \beta_2(\Lambda M'\Lambda M') = 0$$

i.e. $\Lambda M'\Lambda M' \in Ker\ \beta_2 = Im\ (\rho_2 : H^{2n-2}(X,\mathbb{Z}) \longrightarrow H^{2n-2}(X,\mathbb{Z}_2))$. On the other hand,

$$\alpha(\Lambda M'\Lambda M') = Sq^2(\Lambda M'\Lambda M') = \Lambda Sq^1 M'\Lambda Sq^1 M' = \Lambda\rho_2 M\Lambda\rho_2 M \neq 0$$

and so Coker $\Theta = \{0\}$.

3) M^n is orientable and Coker $Sq^2 \circ \rho_2 = \{0\}$. By the assumption, there exists $x \in H^{n-2}(M,\mathbb{Z})$ such that $Sq^2\rho_2(x) = M$. Note that, since Λx and ΛM belong to the twisted cohomology groups $H^{n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ and $H^n(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$, the product $\Lambda x\Lambda M$ is inside $H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z})$. We have :

$$\begin{array}{rcl}\Theta(\Lambda x\Lambda M) & = & (Sq^2+nv^2)\rho_2(\Lambda x\Lambda M) = (Sq^2+nv^2)(\Lambda\rho_2 x\Lambda\rho_2 M)\\ & = & \Lambda Sq^2\rho_2 x\Lambda\rho_2 M \neq 0.\end{array}$$

and so Coker $\Theta = \{0\}$. □

We are now ready to prove theorem 2 for $n \geq 4$. Let $j : M^n \to \mathbb{R}^{2n}$ be a totally real embedding and let $E \xrightarrow{i} I \xrightarrow{h} (I,E)$ be the natural inclusions. Our main tool here is the lemma B, so we have to compute the order of $\pi_1(I,j)/i_\sharp(\pi_1(E,j))$. The study of the long exact homotopy sequence of the pair (I,E) :

$$... \longrightarrow \pi_1(E,j) \xrightarrow{i_\sharp} \pi_1(I,j) \xrightarrow{h_\sharp} \pi_1(I,E,j) \xrightarrow{\partial_\sharp} \pi_0(E) \xrightarrow{J} \pi_0(I).$$

leads to an upper bound of this number. Indeed, $\pi_0(E)$, $\pi_0(I)$ and $\pi_1(I,E,j)$ are known and the arrow $J : \pi_0(E) \longrightarrow \pi_0(I)$ is always the zero map except for the case n even and

M^n non orientable. In this case $J : \mathbb{Z} \oplus ker\ Sq^1 \longrightarrow \mathbb{Z}$ is given by $(a,b) \longmapsto 2a$ (see [90]). It is then straightforward to obtain the results of the array in theorem 2.

3.4 Proof of theorem 2 for $n = 3$

Let $j : M^3 \longrightarrow \mathbb{C}^3$ be a totally real embedding. By lemma B', it suffices to show that $\pi_1(I,j)/l_\sharp(\pi_1(I_{TR},j))$ has only one element if M^3 is orientable and at most two elements if M^3 is not orientable.

We first briefly recall the theory of the classification of totally real immersions. Let $V_{2n,n}(M^n)$ be the associated bundle to the frame bundle of M^n whose fiber is the Stiefel manifold $V_{2n,n}$ of n-frames of $\mathbb{R}^{2n}$ with $Gl(n,\mathbb{R})$ acting in the natural way. Any bundle monomorphism gives rise to a section of $V_{2n,n}(M^n)$ and there is a weak homotopy equivalence between the space $I(M^n,\mathbb{C}^n)$ and the space $\Gamma(V_{2n,n}(M^n))$ of the sections of $V_{2n,n}(M^n)$. Let $V_{TR} \subset V_{2n,n}$ be the submanifold of n-frames of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ such that the n-planes they span are totally real. We denote by $V_{TR}(M^n)$ the obvious subbundle of $V_{2n,n}(M^n)$ with fiber V_{TR}. (Since M^n admits a totally real embedding into $\mathbb{C}^n$, the bundle $V_{TR}(M^n)$ is trivial). According to M. Gromov and J. A. Lees [44],[67], there is a weak homotopy equivalence between the spaces I_{TR} and $\Gamma(V_{TR}(M^n))$ and one has the following commutative diagram :

$$\begin{array}{ccc} I_{TR} & \xrightarrow{l} & I \\ df \downarrow & & \downarrow df \\ \Gamma(V_{TR}(M^n)) & \xrightarrow{l} & \Gamma(V_{2n,n}(M^n)). \end{array}$$

Let $p : M^3 \times \mathbb{S}^1 \to \mathbb{S}^1$ be the projection, we denote by $V_{6,3}(M^3 \times \mathbb{S}^1)$ (resp. $V_{TR}(M^3 \times \mathbb{S}^1)$) the bundle $p^*V_{6,3}(M^3)$ (resp. $p^*V_{TR}(M^3)$) A totally real embedding $j : M^3 \longrightarrow \mathbb{C}^3$ gives rise to a section dj of $V_{TR}(M^3) = V_{TR}(M^3 \times \{0\})$ and a section $\overline{dj} = l \circ dj$ of $V_{6,3}(M^3 \times \{0\})$. We have :

$$\begin{array}{rcl} \pi_1(I,j) & = & [M^3 \times \mathbb{S}^1, V_{6,3}(M^3 \times \mathbb{S}^1); \overline{dj}] \\ \pi_1(I_{TR},j) & = & [M^3 \times \mathbb{S}^1, V_{TR}(M^3 \times \mathbb{S}^1); dj], \end{array}$$

where $[X,Y,f]$ denote the set of homotopy classes of extensions over X of a partial section $f : A \to Y$. Let u (resp. v) be the trivial extension over $M^3 \times \mathbb{S}^1$ of dj (resp. $\overline{dj}$) given by $u(x,\theta) = dj(x)$ (resp. $v(x,\theta) = \overline{dj}(x)$) for every $(x,\theta) \in M^3 \times \mathbb{S}^1$.

We now adapt the General Classification Theorem for sections to our case (see [4] p. 276). For short we put $X = M^3 \times \mathbb{S}^1$ and $A = M^3 \times \{0\}$ and we denote by E (resp. E') the total space of the bundle $V_{6,3}(M^3 \times \mathbb{S}^1)$ (resp. $V_{TR}(M^3 \times \mathbb{S}^1)$). We still write $l : E' \to E$ for the natural inclusion. Let $X^0 \subset X^1 \subset ... \subset X^4$ be the successive skeletons of X, $i_k : X_k = X^k \cup A \to X$ be the natural inclusion and E_k be the bundle i_k^*E. The restriction map $X_n \longrightarrow X_{n-1}$ induces an arrow $j_n : [X_n, E_n; v] \longrightarrow [X_{n-1}, E_{n-1}; v]$, (here, of course, $[X,E;v]$ denotes the set of homotopy classes of extensions of $v_{|A}$ over X). We get a filtration :

$$(*) \qquad [X,E;v] \xrightarrow{j_4} [X_3,E_3;v] \xrightarrow{j_3} ... \xrightarrow{j_1} [X_0,E_0;v] = \{0\}.$$

Put $v_n = v_{|X_n}$ and $J_n(X,A,v) = j_n^{-1}([v_{n-1}])$. If $n \geq 2$, the set $J_n(X,A,v)$ has an Abelian group structure with $[v_n]$ as neutral element. There exists a spectral sequence $(E_r^{**}(X,A,v), d_r(v)) = (E_r^{**}, d_r)$ such that :

1) d_r has bidegree $(r,-1)$,
2) $E_r^{p,q} = 0$ for $p < 1$ and $q < 0$,
3) $E_2^{p,q} = H^p(X_{p+q}, A, \{\pi_{p+q}(F)\})$ for $p+q \geq 2$ and $p \geq 1$,
4) $E_n^{n,0} = J_n(X,A,v)$.

This spectral sequence is natural, that is, in the following situation

$$\begin{array}{ccc} E' & \xrightarrow{l} & E \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

the map l induces a homomorphism (l^r) between the spectral sequences :

$$E_r^{'**}(X,A,u) \xrightarrow{l^r} E_r^{**}(X,A,v = l \circ u).$$

Moreover, for $r = 2$, this homomorphism is induced by the coefficient homomorphism $\pi_i(F') \longrightarrow \pi_i(F)$.

In our case, the fiber of E is $V_{6,3}$ which is a 2-connected space with $\pi_3(V_{6,3}) = \mathbb{Z}_2$ and $\pi_4(V_{6,3}) = 0$, thus the filtration $(*)$ reduces to :

$$[X,E;v] \overset{id}{\simeq} [X_3,E_3;v] \xrightarrow{j_3} \{0\},$$

and therefore : $[X,E;v] = J_3(X,A,v) = E_3^{3,0} = E_2^{3,0} = H^3(X,A;\mathbb{Z}_2)$.

The bundle E' is trivial since M^3 admit a totally real immersion and its fiber is $Gl(3,\mathbb{C})$ which is homotopy equivalent to $U(3)$. Since $\pi_4(U(3)) = 0$ and $\pi_3(U(3)) = \mathbb{Z}$ we have : $[X,E';u] \supset J_3'(X,A,u) = E_3^{'3,0} = E_2^{'3,0} = H^3(X,A;\mathbb{Z})$. We denote by $k : J_3'(X,A,u) \longrightarrow [X,E;v] = J_3(X,A,v)$ the restriction of $l_\sharp$ on $J_3'(X,A,u)$. By the naturality of the spectral sequence, this map is the induced map given by the coefficient homomorphism $h : \pi_3(U(3)) \longrightarrow \pi_3(V_{6,3})$.

Let us see what h is. The inclusion $U(3) \longrightarrow V_{6,3}$ is obtained via the composition $U(3) \to SO(6) \to SO(6)/SO(3)$. Since $\pi_3(SO(6)/SO(3)) = 0$, the arrow $\pi_3(U(3)) \to \pi_3(SO(6)) = \mathbb{Z}$ is an isomorphism and since $\pi_2(SO(3)) = 0$ the map $\pi_3(SO(6)) \to \pi_3(V_{6,3}) = \pi_3(SO(6)/SO(3)) = \mathbb{Z}_2$ is the modulo 2 reduction map. Thus the composition h is still the modulo 2 reduction map and so is k on cohomology.

By using the Universal Coefficient Theorem we get that k is onto if M^3 is orientable and $\pi_1(I,j)/Im\ k = \mathbb{Z}_2$ if not. $\square$

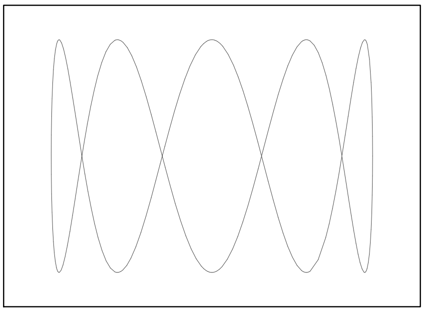
3.5 Some comments on $l_\sharp : \pi_1(I_{TR}, j) \to \pi_1(I, j)$

The reader may wonder why we do not study the map $l_\sharp : \pi_1(I_{TR}, j) \to \pi_1(I, j)$ induced by the inclusion $I_{TR} \subset I$ for $n \geq 4$. A first answer is that the author is unable to compute it completely. Nevertheless, for many important cases, the image of this map is very small. For instance if M^n is an even-dimensional parallelizable manifold, then $|Im\ l_\sharp| \leq 4$ and thus $l_\sharp$ will not give us a significant improvement of the theorem 2.

3.5 Proof of theorem 3.

We denote by $i_n : \mathbb{T}^n \subset \mathbb{C}^n$ the natural Lagrangian inclusion. By pertubing i_n we are going to build sufficiently many distinct totally real embeddings to prove theorem 3.

Let $N \geq 1$ be an odd integer and $\gamma_N : \mathbb{S}^1 \to \mathbb{C}$ be the curve $t \mapsto (\cos t, \sin Nt)$. This curve has $N-1$ double points and is regularly homotopic to the inclusion $i_1 : \mathbb{S}^1 \subset \mathbb{C}$. (See the figure below for $N = 5$).



We set :

$$\begin{array}{rccc} g_\eta : & \mathbb{T}^{n-1} & \longrightarrow & \mathbb{C}^{n-1} \\ & (t, \theta_1, ..., \theta_{n-2}) & \longmapsto & (\gamma_N(t), h(t,\theta_1)e^{i\theta_1}, ..., h(t,\theta_{n-2})e^{i\theta_{n-2}}) \end{array}$$

where $h(t, \theta) = 1 + \eta \sin t \sin\theta$ and $\eta > 0$. We take η small enough so that g_η is a C^∞-close to the Lagrangian immersion $g_0 = \gamma_N \times i_1 \times ... \times i_1$ and thus g_η is a totally real immersion. Then $s \mapsto g_{s\eta}$ is a homotopy in $I^0_{TR}(\mathbb{T}^{n-1}, \mathbb{C}^{n-1})$ joining g_η to g_0. The set of double points of g_η :

$$\{g_\eta(\pm\frac{k\pi}{N}, 0, ..., 0), g_\eta(\pm\frac{k\pi}{N}, \pi, ..., 0), ..., g_\eta(\pm\frac{k\pi}{N}, \pi, ..., \pi) \mid k = 1, .., N-1\}$$

has cardinality $P = 2^{n-2}(N-1)$. Moreover, all the double points are transversal.
Finally, we define a totally real immersion f of $\mathbb{T}^n = \mathbb{T}^{n-1} \times \mathbb{S}^1$ into $\mathbb{C}^n$ by putting $f(x, \varphi) = (g_\eta(x), e^{i\varphi})$. From f, by applying the techniques of [9] §3.1 and §3.3, we get a family

$$\{j_p : \mathbb{T}^n \to \mathbb{C}^n\}_{p \in \{-P/2, ..., P/2\}}$$

of totally real embeddings such that $< \sigma_{n-1}(j_p, v), [\mathbb{T}^{n-1}] >= 2p$, where $\sigma_{n-1} \in H^{n-1}(\mathbb{T}^n, \mathbb{Z})$ is the linking class of f associated to a non-vanishing vector field v tangent to the last factor $\mathbb{S}^1$. Since the linking class is invariant by totally real isotopy, we obtain that the j_p represent $P+1$ distinct components of E_{TR}. By construction, all the j_p are obtained by a C^∞-small pertubation of f, thus they all belong to the same component $\iota = [f]$ of I^0_{TR}. But $f = g_\eta \times i_1$ is regularly homotopic among totally immersions to $g_0 \times i_1$, which is itself regularly homotopic among totally immersions to the standard Lagrangian inclusion $i_n = i_1 \times ... \times i_1$. Thus $\iota = [i_n]$. Since $\mathbb{T}^n$ is parallelizable and n even, theorem 3 of [9] states that two totally real embeddings of $\mathbb{T}^n$ into $\mathbb{C}^n$ that belong to the same component of I^0_{TR} belong to the same component of E if and only if the reduction modulo 2 of their linking classes is the same. Let us denote by E_q the $(n-1)$-cycle $[\mathbb{S}^1 \times ... \times \{0\} \times ... \times \mathbb{S}^1]$ where the $\{0\}$ is the q-th factor. We already have $< \sigma_{n-1}(j_p, v), E_n >= 2p$ and it is not difficult to check that :

$$< \sigma_{n-1}(j_p, v), E_q >= 0, \quad \text{for every} \quad 1 \leq q \leq n-1.$$

Thus $\rho_2\sigma_{n-1}(j_p, v) = 0$ and all the j_p belong to the same component $\epsilon \subset E$. Of course $\sigma_{n-1}(i_n, v) = 0$, hence $\epsilon = [i_n]$. So, given any odd integer $N \geq 1$ we have shown that : $N(\epsilon, \iota) \geq 2^{n-2}(N-1) + 1$. □

3.7 Appendix

We include this appendix for the sake of completness. All the results of subsections 7.1 and 7.2 are from [66], [65] and [90]. Section 7.3 is devoted to the proof of lemma C.

3.7.1 Cohomology in $\mathbb{Z}_2$ of the pair (M^*, PM)

Let $\Lambda^2 M = (M \times M)/\mathbb{Z}_2$ be the 2-fold symmetric product of M and N be a tubular neighborhood of ΔM in $\Lambda^2 M$. The natural inclusions $(M^*, PM) \subset (\Lambda^2 M, N)$ and $(\Lambda^2 M, \Delta M) \subset (\Lambda^2 M, N)$ induce isomorphisms on cohomology groups. To save space, we sometimes denote by X the pair $(\Lambda^2 M, \Delta M)$. Let $\pi : (M^2, \Delta M) \longrightarrow (\Lambda^2 M, \Delta M)$ be the double covering, v be the first Stiefel-Whitney class of this covering and :

$$\to H^k(X, \mathbb{Z}_2) \xrightarrow{\pi^*} H^k(M^2, \Delta M, \mathbb{Z}_2) \xrightarrow{\theta} H^k(X, \mathbb{Z}_2) \xrightarrow{\cup v} H^{k+1}(X, \mathbb{Z}_2) \to$$

be the exact Thom-Gysin sequence of this covering. It is built in [66] a monomorphism $\Lambda : \tilde{H}^k(M, \mathbb{Z}_2) \longrightarrow H^k(\Lambda^2 M, \Delta M, \mathbb{Z}_2)$ and Λx satisfies : $\pi^*\Lambda x = x \otimes 1 + 1 \otimes x$ ($\tilde{H}^*$ denotes the reduced cohomology). Moreover, if $\Delta(x, y) = \Lambda x \Lambda y + \Lambda(xy)$, then : $\pi^*\Delta(x, y) = x \otimes y + y \otimes x$.

By [66] theorem 11, as an algebra over $\mathbb{Z}_2[v]$, $H^*(\Lambda^2 M, \Delta M, \mathbb{Z}_2)$ is generated by all Λx for $x \in \tilde{H}^*(M, \mathbb{Z}_2)$ subject to the relations :

i) $\Lambda(x+y) = \Lambda x + \Lambda y$ for all $x, y \in \tilde{H}^*(M, \mathbb{Z}_2)$,
ii) $v\Lambda x \Lambda y = 0$ for all $x, y \in \tilde{H}^*(M, \mathbb{Z}_2)$,
iii) $\Lambda x \Lambda y \Lambda z = \Lambda x \Lambda(yz) + \Lambda y \Lambda(xz) + \Lambda z \Lambda(xy)$ for all $x, y, z \in \tilde{H}^*(M, \mathbb{Z}_2)$,
iv) $(\Lambda x)^2 = \sum_{i=0}^k v^{k-i} \Lambda^i x$ for all $x \in \tilde{H}^k(M, \mathbb{Z}_2)$, $k > 0$.

It is also worth noting that :

$$Sq^k \Lambda x = \sum_{i=0}^{k} v^{k-i} \Lambda Sq^i x \text{ for } x \in H^n(M, \mathbb{Z}_2) \text{ and } k \geq 0.$$

As an immediate application, one obtains :

$$\begin{array}{rcl} H^{2n}(\Lambda^2 M, \Delta M, \mathbb{Z}_2) & = & \mathbb{Z}_2 < \Lambda\rho_2 M \Lambda\rho_2 M > \\ H^{2n-1}(\Lambda^2 M, \Delta M, \mathbb{Z}_2) & = & \quad \{\Lambda x \Lambda\rho_2 M \mid x \in H^{n-1}(M, \mathbb{Z}_2)\} \\ & & \oplus \ \mathbb{Z}_2 < v^{n-1}\Lambda\rho_2 M > \\ H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}_2) & = & \quad \{\Lambda x \Lambda\rho_2 M \mid x \in H^{n-2}(M, \mathbb{Z}_2)\} \\ & & \oplus \{\Lambda x \Lambda y \mid x, y \in H^{n-1}(M, \mathbb{Z}_2)\} \\ & & \oplus \ \mathbb{Z}_2 < v^{n-2}\Lambda\rho_2 M > . \end{array}$$

where $\rho_2 : H^n(M, \mathbb{Z}) \longrightarrow H^n(M, \mathbb{Z}_2)$ is the reduction modulo 2 and M is a generator of $H^n(M, \mathbb{Z})$ ($= \mathbb{Z}$ or $\mathbb{Z}_2$ depending on the orientability of M).

3.7.2 Cohomology in $\mathbb{Z}[v]$ of the pair (M^*, PM)

There is still a monomorphism $\Lambda : \tilde{H}^k(M, \mathbb{Z}) \longrightarrow H^k(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ and a bilinear map $\Delta : \tilde{H}^k(M, \mathbb{Z}) \times \tilde{H}^k(M, \mathbb{Z}) \longrightarrow H^k(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ and a long exact sequence

$$(*) \quad \to H^k(X, \mathbb{Z}[v]) \xrightarrow{(\pi^*)^T} H^k(M^2, \Delta M, \mathbb{Z}) \xrightarrow{\theta} H^k(X, \mathbb{Z}) \xrightarrow{\cup V} H^{k+1}(X, \mathbb{Z}[v])$$

with $(\pi^*)^T \Lambda x = x \otimes 1 - 1 \otimes x$ and $(\pi^*)^T \Delta(x, y) = x \otimes y - (-1)^{rs} y \otimes x$ ($x \in H^r(M, \mathbb{Z})$ and $y \in H^s(M, \mathbb{Z})$). Moreover $V = \beta_2^T(1)$ with $1 \in H^0(X, \mathbb{Z}_2)$ where $\beta_2^T : H^{k-1}(M, \mathbb{Z}_2) \longrightarrow H^k(M, \mathbb{Z}[v])$ is the twisted Bockstein operator. In particular, $\rho_2^T(V) = v$ ($\rho_2^T : H^k(X, \mathbb{Z}[v]) \longrightarrow H^k(X, \mathbb{Z}_2)$ is the twisted reduction modulo 2). Note also that $\rho_2^T(\Lambda x) = \Lambda(\rho_2 x)$ and $\rho_2^T(\Delta(x, y)) = \Delta(\rho_2 x, \rho_2 y)$ for $x \in H^k(M, \mathbb{Z}),\ y \in H^r(M, \mathbb{Z})$ (see [90], lemma 1.5).

In [90], the cohomology groups $H^{2n}(X, \mathbb{Z}[v])$ and $H^{2n-1}(X, \mathbb{Z}[v])$ are computed. The result depends on four cases :

(i) n is even and M is orientable,
(ii) n is even and M is non orientable,
(iii) n is odd and M is orientable,
(iv) n is odd and M is non orientable.

Precisely, $H^{2n}(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ is

$$\begin{array}{llll} (i) & \mathbb{Z}_2 < \beta_2^T(v^{n-1}\Lambda\rho_2 M) > & (ii) & \mathbb{Z}_2 < \beta_2^T(v^{n-1}\Lambda\rho_2 M) > \\ (iii) & \mathbb{Z} < \Delta(M, M) > & (iv) & \mathbb{Z}_2 < \beta_2^T(\Lambda M' \Lambda\rho_2 M) > \end{array}$$

and $H^{2n-1}(\Lambda^2 M, \Delta M, \mathbb{Z}[v])$ is

$$\begin{array}{llll} (i) & G & (ii) & K \oplus \mathbb{Z}_4 < \frac{1}{2}\beta_2^T(v^{n-1}\Lambda M') > \\ (iii) & G \oplus \mathbb{Z}_2 < \beta_2^T(v^{n-2}\Lambda\rho_2 M) > & (iv) & K \oplus \mathbb{Z}_2 < \beta_2^T(v^{n-2}\Lambda\rho_2 M) > \end{array}$$

where

$$\begin{array}{rl} G = & \{\Delta(x, M) \mid x \in H^{n-1}(M, \mathbb{Z})\} \simeq H^{n-1}(M, \mathbb{Z}), \\ K = & \{\beta_2^T(\Delta(M', \rho_2 x)) \mid x \in H^{n-1}(M, \mathbb{Z})\} \simeq Im\ \rho_2. \end{array}$$

3.7.3 Cohomology in $\mathbb{Z}$ of the pair (M^*, PM)

It is easy to see that $H^{2n}(\Lambda^2 M, \Delta M, \mathbb{Z})$ is $\mathbb{Z}$ in the case (i) and $\mathbb{Z}_2$ in the other cases. We denote by H the cohomology group $H^{2n-1}(\Lambda^2 M, \Delta M, \mathbb{Z})$. The result still depends on the four cases (i) - (iv).

(i) The exact sequence $(*)$ becomes :

$$... \longrightarrow G \xrightarrow{(\pi^*)^T} G \oplus G \longrightarrow H \xrightarrow{\cup V} \mathbb{Z}_2 \longrightarrow \mathbb{Z} \longrightarrow ...$$

where $(\pi^*)^T$ can be identified with the map $x \longmapsto (x, -x)$. Hence this sequence is equivalent to :

$$0 \longrightarrow G \longrightarrow H \xrightarrow{\cup V} \mathbb{Z}_2 < \beta_2^T(v^{n-1}\Lambda\rho_2 M) > \longrightarrow 0.$$

But

$$\begin{aligned} \rho_2^T(\beta_2(v^{n-2}\Lambda\rho_2 M) \cup V) &= \rho_2(\beta_2(v^{n-2}\Lambda\rho_2 M)) \cup \rho_2^T(V) \\ &= Sq^1(v^{n-2}\Lambda\rho_2 M) \cup v \\ &= v^n \Lambda\rho_2 M. \end{aligned}$$

This implies that $\beta_2(v^{n-2}\Lambda\rho_2 M) \cup V \neq 0$ and $\sigma : \mathbb{Z}_2 \longrightarrow H$ defined by $\sigma(\beta_2^T(v^{n-1}\Lambda\rho_2 M)) = \beta_2(v^{n-2}\Lambda\rho_2 M)$ is a monomorphism which satisfies $(\cup V) \circ \sigma = id_{\mathbb{Z}_2}$. Thus : $H = G \oplus \mathbb{Z}_2$.

(ii) The exact sequence $(*)$:

$$... \xrightarrow{(\pi^*)^T} (G \otimes \mathbb{Z}_2)^2 \oplus \mathbb{Z}_2 \longrightarrow H \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

tells that H is of the form $\mathbb{Z}_2^{P+1}$ or $\mathbb{Z}_2^P \oplus \mathbb{Z}_4$ with $P \in \mathbb{N}$. Applying the universal coefficient theorem, we get the exact sequence :

$$0 \longrightarrow H \otimes \mathbb{Z}_2 \longrightarrow H^{2n-1}(X, \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

By section 7.2, $H^{2n-1}(X, \mathbb{Z}_2) \simeq H^{n-1}(M, \mathbb{Z}_2) \oplus \mathbb{Z}_2$, applying again the universal coefficient theorem, we have :

$$H^{n-1}(M, \mathbb{Z}_2) \simeq G \otimes \mathbb{Z}_2 \oplus \mathbb{Z}_2 \simeq \mathbb{Z}_2^{N+1}$$

where $N = dim_{\mathbb{Z}_2} G \otimes \mathbb{Z}_2$. Hence $H \otimes \mathbb{Z}_2 = \mathbb{Z}_2^{N+1}$ and thus $H = \mathbb{Z}_2^{N+1}$ or $\mathbb{Z}_2^N \oplus \mathbb{Z}_4$. Assume that $H = \mathbb{Z}_2^N \oplus \mathbb{Z}_4$. The Bockstein sequence is then :

$$\begin{array}{ccccccc} H^{2n-2}(X, \mathbb{Z}_2) & \xrightarrow{\beta_2} & H^{2n-1}(X, \mathbb{Z}) & \xrightarrow{(0, \times 2)} & H^{2n-1}(X, \mathbb{Z}) & \xrightarrow{\rho_2} & H^{2n-1}(X, \mathbb{Z}_2) \\ & & \| & & \| & & \| \\ & & \mathbb{Z}_2^N \oplus \mathbb{Z}_4 & & \mathbb{Z}_2^N \oplus \mathbb{Z}_4 & & \mathbb{Z}_2^{N+2} \end{array}$$

Let g be a generator of the factor $\mathbb{Z}_4$. Then Im $\beta_2 = \mathbb{Z}_2^N \oplus \mathbb{Z}_2 < 2g >$ and Im $\rho_2 \circ \beta_2 = \mathbb{Z}_2^N$. On the other hand, $Sq^1 = \rho_2\beta_2$ and it is easy to verify that : $Sq^1(H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}_2)) \simeq \mathbb{Z}_2^{N+1}$. Contradiction. Thus, $H = \mathbb{Z}_2^{N+1}$.

(iii) One has :

$$... \longrightarrow G \oplus \mathbb{Z}_2 < a > \xrightarrow{(\pi^*)^T} G \oplus G \longrightarrow H \xrightarrow{0} \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

with $a = \beta_2^T(v^{n-2}\Lambda\rho_2 M)$. Since $(\pi^*)^T\beta_2^T(v^{n-2}\Lambda\rho_2 M) = \beta_2^T\pi^*(v^{n-2}\Lambda\rho_2 M)$ and $v^{n-2}\Lambda\rho_2 M = v \cup v^{n-3}\Lambda\rho_2 M \in Im\ (\cup v) = Ker\ \pi^*$, we get :$(\pi^*)^T(a) = 0$. Hence, $H \simeq G$.

(iv) The exact sequence $(*)$:

$$... \xrightarrow{(\pi^*)^T} (G \otimes \mathbb{Z}_2)^2 \oplus \mathbb{Z}_2 \longrightarrow H \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

tells that H is of the form $\mathbb{Z}_2^{P+1}$ or $\mathbb{Z}_2^P \oplus \mathbb{Z}_4$ with $P \in \mathbb{N}$. Similarly to the case (ii), one can easily show that $P = N = dim_{\mathbb{Z}_2} G \otimes \mathbb{Z}_2$. Assume that $H = \mathbb{Z}_2^{N+1}$. The Bockstein sequence is then :

$$\begin{array}{ccccccc} H^{2n-2}(X,\mathbb{Z}_2) & \xrightarrow{\beta_2} & H^{2n-1}(X,\mathbb{Z}) & \xrightarrow{0} & H^{2n-1}(X,\mathbb{Z}) & \xrightarrow{\rho_2} & H^{2n-1}(X,\mathbb{Z}_2) \\ & & \| & & \| & & \| \\ & & \mathbb{Z}_2^{N+1} & & \mathbb{Z}_2^{N+1} & & \mathbb{Z}_2^{N+2} \end{array}$$

One has : Im $\beta_2 = \mathbb{Z}_2^{N+1}$ and Im $\rho_2 \circ \beta_2 = \mathbb{Z}_2^{N+1}$. On the other hand :

$$Sq^1(H^{2n-2}(\Lambda^2 M, \Delta M, \mathbb{Z}_2)) \simeq \mathbb{Z}_2^N.$$

Contradiction. Thus, $H = \mathbb{Z}_2^N \oplus \mathbb{Z}_4$.

Minimal Legendrian submanifolds of $\mathbb{S}^{2n+1}$ and absolutely area-minimizing cones

A paraître dans Diff. Geom. and its App.

Abstract. – We use minimal Legendrian submanifolds in spheres to construct examples of absolutely area-minimizing cones and we prove a result about Legendrian 2-tori in S^5.

4.1 Introduction

A Legendrian submanifold of a $(2n+1)$-dimensional contact manifold is an n-dimensional integral submanifold of the contact distribution. In this paper we consider the unit sphere S^{2n+1} equipped with the contact structure inherited from its standard embedding in $\mathbb{C}^{n+1}$. To an arbitrary immersion $f : M \to S^{2n+1}$ one can associate its cone map $\hat{f} : M \times (0, +\infty) \to \mathbb{R}^{2n+2}$ where $\hat{f}(x,t) = tf(x)$ and we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$. It turns out that f is a Legendrian immersion if and only if $\hat{f}$ is a Lagrangian immersion with respect to the standard symplectic structure of $\mathbb{R}^{2n+2}$. We also note that the normal vectors to f in S^{2n+1} at x are exactly the normal vectors to $\hat{f}$ at (x,t) for $t > 0$. Furthermore, if the second fundamental form of f in S^{2n+1} with respect to a normal vector ν in x has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the second fundamental form of $\hat{f}$ with respect to ν in (t,x) has eigenvalues $t\lambda_1, \ldots, t\lambda_n, 0$. It follows that f is a minimal immersion if and only if $\hat{f}$ is a minimal immersion.

The manifold of all Lagrangian planes in $\mathbb{R}^{2n+2}$ is called the *Lagrangian Grassmannian*, and it is easy to see that this manifold is diffeomorphic to the homogeneous space $\mathbf{U}(n+1)/\mathbf{O}(n+1)$. To an arbitrary Lagrangian immersion $g : P \to \mathbb{R}^{2n+2}$ one can now naturally associate a *Gauss map* $G(g) : P \to \mathbf{U}(n+1)/\mathbf{O}(n+1)$. Consider the composition

$$P \xrightarrow{G(g)} \mathbf{U}(n+1)/\mathbf{O}(n+1) \xrightarrow{\det^2_{\mathbb{C}}} S^1. \qquad (1)$$

The pull-back of the volume form of S^1 under (1) is called the *Maslov form* $\mu(g)$ of g, and the class $[\mu(g)]$ it defines is called the *Maslov class* of g. A result of Morvan [70] states that the Maslov form of g is given by $\mu(g) = -\frac{n+1}{\pi} < JH, \cdot >$, where H is the mean curvature vector of g, J the complex structure in $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ and $< \cdot, \cdot >$ denotes the real inner product. As a corollary, a Lagrangian immersion is minimal if and only if it has a vanishing Maslov form.

Notice that a Lagrangian immersion has a vanishing Maslov form precisely when the tangent planes at any two distinct points differ by an element of $\mathbf{SU}(n+1)$. In [52], section III, Harvey and Lawson define a $(n+1)$-dimensional submanifold of $\mathbb{R}^{2n+2}$ to be *special Lagrangian* if the tangent planes at any two distinct points differ by an element of $\mathbf{SU}(n+1)$. As an application of the theory of calibrations introduced in the same paper, they show that special Lagrangian submanifolds have the property of being absolutely area-minimizing. In particular, we conclude from the above discussion that *the cone over a compact minimal Legendrian submanifold of S^{2n+1} is absolutely area-minimizing.* It is interesting to notice that the same conclusion can be drawn from Proposition 2.17 in [52], which in fact contains a restatement of Morvan's formula (namely, equation (2.19)).

In section 2 of this paper, we use this conclusion to construct two series of examples of absolutely area-minimizing cones. The first series generalizes Corollary 3.21 and Example 3.22 in [52]. The second series unifies the proofs of the absolute area-minimality of some examples in [20, 21].

In section 3, we prove a result about Legendrian 2-tori in S^5. Any minimal Legendrian 2-torus projects down via the Hopf fibration to a minimal Lagrangian 2-torus in the complex projective plane $\mathbb{C}P^2$, and there is now a growing literature on minimal Legendrian 2-tori in S^5 using this correspondence, see [23, 53, 58, 69]. The following theorem is a weak form of the converse to the fact that any minimal Legendrian 2-torus of S^5 has a vanishing Maslov class.

Theorem. – *Let $f : T^2 \to S^5$ be a Legendrian immersion. Then f is regularly homotopic through Legendrian immersions to a minimal Legendrian embedding if and only if* $[\mu(\hat{f})] = 0$.

Part of this work was completed while the first author was visiting Univ. of São Paulo, for which he wishes to thank Prof. Fabiano G. B. Brito for his hospitality and FAPESP for financial suport, and part of it was completed while the second author was visiting Univ. of Lyon, for which he wishes to thank Prof. Jean-Marie Morvan for his hospitality and the CNRS and the Alexander von Humboldt Foundation for their financial support. The authors also would like to thank the referee for many interesting suggestions and comments which improved significantly the text.

4.2 Examples of minimal Legendrian submanifolds of S^{2n+1}

In this section we present examples of compact homogeneous minimal Legendrian submanifolds of S^{2n+1}. The cones over these submanifolds are area-minimizing. Although some of the examples are already known, they are here presented from a new and unified perspective.

4.2.1 Twisted normal bundles

In [52] Harvey and Lawson defined a submanifold M of S^n to be *austere* if the set of eigenvalues of the Weingarten operator of M with respect to any normal vector is invariant under multiplication by -1. Then they showed that for any compact austere submanifold M of S^n, the cone over

$$\Phi : N^1M \times S^1 \to S^{2n+1}, \qquad \Phi(\nu_x, e^{i\theta}) = (\cos\theta\ x, \sin\theta\ \nu_x),$$

where N^1M denotes the unit normal bundle of M in S^n, is special Lagrangian. Note that the map Φ fails to be an immersion for $\sin\theta = 0$ if the codimension of M in S^{2n+1} is bigger than one, and it fails to be an immersion for $\cos\theta = 0$ if 0 is an eigenvalue of the Weingarten operator of M with respect to some unit normal. Therefore we will consider the following variation of this construction. Let M be a submanifold of the unit sphere S^n and NM be the normal bundle of M in S^n. We denote by $\widetilde{M}_+$, $\widetilde{M}_-$ two copies of $NM^{\leq\frac{\pi}{2}} = \{\nu \in NM : |\nu| \leq \pi/2\}$ and by $\widetilde{M}$ the double of $NM^{\leq\frac{\pi}{2}}$ i. e. $\widetilde{M} = \widetilde{M}_+ \cup \widetilde{M}_- / \partial\widetilde{M}_+ \sim \partial\widetilde{M}_-$. Note that if M is a closed hypersurface of S^n then $\widetilde{M}$ is diffeomorphic to the product $M \times S^1$. Define

$$\Psi : \widetilde{M} \to S^{2n+1}, \qquad \Psi(\nu_x) = \begin{cases} (\epsilon \cos|\nu_x|\, x, \sin|\nu_x|\, \frac{\nu_x}{|\nu_x|}) & \text{if } \nu_x \neq 0, \\ (\epsilon x, 0) & \text{if } \nu_x = 0, \end{cases}$$

where $\epsilon = \pm 1$ according to whether $x \in \widetilde{M}_\pm$. It is easy to see that Ψ has the same image as Φ.

Proposition. – *If* 0 *is never an eigenvalue of the Weingarten operator of* M *with respect to any unit normal, then* $\Psi : \widetilde{M} \to S^{2n+1}$ *is a Legendrian immersion. If in addition* M *is austere, then* Ψ *is minimal.*

Proof. – We will first compute the differential of Ψ at a normal vector ν_{x_0}. Near x_0 we choose an orthonormal frame field $e_1, \ldots, e_p$ and a normal frame field $\nu_1, \ldots, \nu_q$ where $p + q = n$ and $e_1, \ldots, \nu_q$ is positively oriented. Without loss of generality, we may assume that $(\nabla\nu_k)^N_{x_0} = 0$ where $()^N$ denotes the orthogonal projection onto the normal space $N_{x_0}M$. We can also take $e_1, \ldots, e_p$ to be the eigenvectors of the Weingarten operator A^{ν_1}, namely $A^{\nu_1}(e_k) = \lambda_k e_k$. Upon a linear change of coordinates we may further assume that at x_0 the vectors $x_0, e_1, \ldots, \nu_q$ give the standard basis of $\mathbb{R}^{n+1} \supset S^n \supset M$.

If $\nu_{x_0} \neq 0$ we may assume $\nu_1 = \frac{\nu_{x_0}}{|\nu_{x_0}|}$. The differential of Ψ at ν_{x_0} is then given by

$$\Psi_*(e_j) = (\epsilon \cos|\nu_{x_0}|\, e_j, -\lambda_j \sin|\nu_{x_0}|\, e_j) \quad \text{for } 1 \leq j \leq p \text{ and}$$

$$\Psi_*(\nu_j) = \begin{cases} (-\epsilon \sin|\nu_{x_0}|\, x_0, \cos|\nu_{x_0}|\, \frac{\nu_{x_0}}{|\nu_{x_0}|}) & \text{if } j = 1, \\ (0, \sin|\nu_{x_0}|\, \frac{\nu_j}{|\nu_{x_0}|}) & \text{if } 2 \leq j \leq q. \end{cases}$$

If $\nu_{x_0} = 0$ we have

$$\Psi_*(e_j) = (\epsilon e_j, 0) \quad \text{for } 1 \leq j \leq p \text{ and}$$
$$\Psi_*(\nu_j) = (0, \nu_j) \quad \text{for } 1 \leq j \leq q.$$

In these formulae the fields e_j, ν_j are evaluated at x_0.

The above calculation immediately shows that Ψ is an immersion (if $\lambda_j \neq 0$ for $1 \leq j \leq p$) and Legendrian (since $< \Psi_*(v), J\Psi(\nu_{x_0}) >=< \Psi_*(v), J\Psi_*(v') >= 0$ where each of v, v' is one of $e_1, \ldots, \nu_q$ and $J : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ is given by $J(x, y) = (-y, x)$). Next we compute the Maslov form of the cone over Ψ. For this purpose, we view $\mathbb{R}^{2n+2}$ as $\mathbb{C}^{n+1}$ and J as the multiplication by i, and we represent $G(\hat{\Psi})(\nu_{x_0})$ by a unitary matrix which we specify by its column vectors and then compute the squared determinant. If $\nu_{x_0} \neq 0$ the matrix has as columns the unit vectors

$$\epsilon \cos|\nu_{x_0}|\, x_0 + i \sin|\nu_{x_0}|\nu_1, \quad \tfrac{\epsilon \cos|\nu_{x_0}| - i\lambda_1 \sin|\nu_{x_0}|}{\cos^2|\nu_{x_0}| + \lambda_1^2 \sin^2|\nu_{x_0}|} e_1, \ldots, \tfrac{\epsilon \cos|\nu_{x_0}| - i\lambda_p \sin|\nu_{x_0}|}{\cos^2|\nu_{x_0}| + \lambda_p^2 \sin^2|\nu_{x_0}|} e_p,$$
$$-\epsilon \sin|\nu_{x_0}|\, x_0 + i \cos|\nu_{x_0}|\, \nu_1, \quad i\nu_2, \ldots, i\nu_q.$$

Therefore

$$(\det)^2 G(\hat{\Psi})(\nu_{x_0}) = \epsilon^2 i^{2q} \left[\prod_{j=1}^{p} \frac{(\epsilon \cos|\nu_{x_0}| - i\lambda_j \sin|\nu_{x_0}|)^2}{(\cos^2|\nu_{x_0}| + \lambda_j^2 \sin^2|\nu_{x_0}|)^2} \right] (\det)^2(x_0, e_1, \dots, e_p, \nu_1, \nu_2, \dots \nu_q).$$

If M is austere this expression equals $(-1)^q$ (because

$$(\epsilon \cos|\nu_{x_0}| - i\lambda \sin|\nu_{x_0}|)(\epsilon \cos|\nu_{x_0}| + i\lambda \sin|\nu_{x_0}|) = \cos^2|\nu_{x_0}| + \lambda^2 \sin^2|\nu_{x_0}|$$

for $\lambda \in \mathbb{R}$). If $\nu_{x_0} = 0$ we get :

$$(\det)^2(\epsilon x_0, \epsilon e_1, \dots, \epsilon e_p, i\nu_1, \dots, i\nu_q) = \epsilon^{2p} i^{2q} (\det)^2(x_0, e_1, \dots, e_p, \nu_1, \dots, \nu_q) = (-1)^q.$$

Thus $(\det)^2 G(\hat{\Psi})$ is a constant map. This implies that the Maslov form of $\hat{\Psi}$ vanishes, hence Ψ is minimal. □

4.2.2 Twisted normal bundles of isoparametric submanifolds

We shall now apply the construction discussed in section 4.2.1 to certain leaves of certain isoparametric foliations of spheres. An *isoparametric hypersurface* M^{n-1} of S^n is a hypersurface with constant principal curvatures. A good introduction to the theory of isoparametric hypersurfaces in spheres can be found in chapter 3 of [24], and a comprehensive survey, including generalizations, can be found in [87]. It follows from the formulae for Weingarten operators of tubes that if M is isoparametric, then so are all of its parallel hypersurfaces. A parallel submanifold of an arbitrary submanifold M is a submanifold of the same dimension obtained by exponentiating a parallel normal vector field in M, and a focal point of M is a critical value of the exponential map restricted to the normal bundle of M. The focal set of M, which is the set of all the focal points of M, need not have a manifold structure. However, in the case of an isoparametric hypersurface, the exponential map has constant rank along a parallel normal vector field, and this can be used to show that the focal set of a complete isoparametric hypersurface decomposes into a union of submanifolds (see [72]).

Let M be an isoparametric hypersurface of S^n. Münzner [71], generalizing previous work by Cartan, showed that the distinct principal curvatures $\lambda_1 > \cdots > \lambda_g$ of M satisfy $\lambda_k = \cot(\theta + (k-1)\pi/g)$ for $k = 1, \dots, g$, where $0 < \theta < \pi/g$, and the multiplicities $m_1, \dots, m_g$ satisfy $m_1 = \dots = m_g$ if g is odd, and $m_1 = m_3 = \dots = m_{g-1}$, $m_2 = m_4 = \dots = m_g$ if g is even. Münzner's computation also shows that the focal manifolds must be minimal. Beginning with these facts, he proved that the isoparametric parallel hypersurfaces of M can be represented as open subsets of level hypersurfaces in S^n of a homogeneous polynomial F of degree g on $\mathbb{R}^{n+1}$ which satisfies the Cartan-Münzner differential equations :

$$|\mathrm{grad} F(x)|^2 = g^2 |x|^{2g-2}, \quad \Delta F = \frac{g^2(m_2 - m_1)}{2} |x|^{g-2},$$

where g is the number of distinct principal curvatures and m_1, m_2 are the two (possibly equal) multiplicities of the principal curvatures. This result implies that any piece of isoparametric hypersurface can be extended to a compact isoparametric hypersurface. His construction also shows that there are only two focal manifolds, even though there may be more than two principal curvatures ; these correspond precisely to the only two singular levels of the polynomial F ; and they sit apart at a distance of π/g. Further, each element of the family of isoparametric parallel hypersurfaces divides the ambient sphere into two ball bundles over the respective focal submanifolds. From this topological information, he was able to determine the cohomology rings of the hypersurfaces and the focal submanifolds, and then by using delicate cohomological and algebraic arguments, he obtained the splendid result that the number g of distinct principal curvatures must satisfy $g = 1, 2, 3, 4$ or 6.

A large class of examples of isoparametric families of hypersurfaces in spheres is provided by the orbital foliations induced by orthogonal representations of compact Lie groups that have cohomogeneity two in Euclidean space. These examples are of course homogeneous, and, in fact, a classification of homogeneous isoparametric families follows from the classification of orthogonal representations of cohomogeneity two that is given in [56]. Examples of inhomogeneous isoparametric hypersurfaces are known only in the case $g = 4$, see [75, 76, 29]. So far no complete classification of isoparametric hypersurfaces is known, although Cartan classified the case $g \leq 3$, and in the case $g = 6$ the unsolved problem is whether there exists an inhomogeneous isoparametric hypersurface with $m_1 = m_2 = 2$. See section 1 in [87] for more details.

Let $\{M_t\}$ for $0 < t < \pi/g$ be an isoparametric family of parallel hypersurfaces of S^n with g distinct principal curvatures, and let M_0 and $M_{\pi/g}$ denote the focal manifolds ; here t is the distance from M_0 to M_t. Then the principal curvatures of M_t are $\lambda_k(t) = \cot(t+(k-1)\pi/g)$ for $k = 1, \ldots, g$, and the principal curvatures of the focal manifold M_0 (resp. $M_{\pi/g}$) with respect to any unit normal are $\lambda_k(0)$ for $k = 2, \ldots, g$ (resp. $\lambda_k(\pi/g)$ for $k = 1, \ldots, g-1$), with the multiplicities satisfying the same identities as above. As we said before, our aim is to apply the construction of section 4.2.1 to some element of $\{M_t\}$. A moment's thought shows that we can find $t \in [0, \pi/g]$ such that M_t is austere with no zero principal curvature with respect to any unit normal if and only if $m_1 = m_2$. So we make this assumption from now on, and we proceed as follows.

If g is odd, then the focal manifolds are congruent under the antipodal map of S^n. Since $m_1 = m_2$, they are compact austere submanifolds of S^n, and since they have no principal curvature equal to zero, the construction of section 4.2.1 yields an example of a minimal Legendrian embedding of $\widetilde{M}$ into S^{2n+1}. The global structure theorem of Münzner says that each compact isoparametric hypersurface divides S^n into two manifolds with boundary, each of which is a normal ball bundle over one of the two focal manifolds. It follows that $\widetilde{M}$ is diffeomorphic to the sphere S^n.

If g is even, it follows from $m_1 = m_2$ that the isoparametric hypersurface $M_{\pi/2g}$ is a compact austere submanifold of S^n all of whose principal curvatures are nonzero. Therefore the construction of section 4.2.1 yields an example of a minimal Legendrian immersion of $\widetilde{M}$ into S^{2n+1}. It is easy to use the fact that $M_{\pi/2g}$ is invariant under the antipodal map of S^n to show that this immersion induces a minimal Legendrian embedding of $\widetilde{M}/\mathbb{Z}_2 \simeq M_{\pi/2g} \times_{\mathbb{Z}_2} S^1$ into S^{2n+1} (the nontrivial element of $\mathbb{Z}_2$ acts on each factor of the product $M_{\pi/2g} \times S^1$ by the antipodal map).

Next, we consider all the known examples of isoparametric foliations of S^n that satisfy $m_1 = m_2 = m$. It turns out that these are homogeneous, namely the orbital foliation induced on S^n by an orthogonal representation ρ of a compact connected Lie group G on $\mathbb{R}^{n+1}$ of cohomogeneity two. Some of these examples are discussed in detail in section 7 of chapter 3 of [24] ; see also [88]. According to whether g is odd or even, we have possibility 1 or 2, as described above.

g=1. – Here $m = n - 1$. The foliation is by umbilic hypersurfaces, namely the hyperspheres parallel to the equator. The focal manifolds are points and $\widetilde{M}$ is a totally geodesic totally real $S^n \subset S^{2n+1}$.

g=2. – Here n is odd and $m = \frac{n-1}{2}$. The foliation is by hypersurfaces parallel to the Clifford torus $M_{\pi/4} = S^{\frac{n-1}{2}}(1/\sqrt{2}) \times S^{\frac{n-1}{2}}(1/\sqrt{2}) \subset S^n$, and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(S^{\frac{n-1}{2}} \times S^{\frac{n-1}{2}}) \times_{\mathbb{Z}_2} S^1$. In particular we get a minimal Legendrian immersion

$$S^1 \times S^{\frac{n-1}{2}} \times S^{\frac{n-1}{2}} \to S^{2n+1}.$$

This example first appears in [52], Example 3.22.

g=3. – Here $m = 1, 2, 4, 8$. Let G be the group of 3×3 $\mathbb{F}$-unitary matrices, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (quaternions), $\mathbb{O}$ (octonions), according to whether $m = 1, 2, 4, 8$. Note that $G = \mathbf{O}(3), \mathbf{U}(3), \mathbf{Sp}(3), \mathbf{F}_4$, respectively. Let V be the real vector space of 3×3 traceless $\mathbb{F}$-Hermitean matrices. Note that $\dim V = 3m+2$. Now $\rho : G \to \mathbf{O}(\mathbb{R}^{3m+2})$ is conjugation. The focal manifolds are the so called standard embeddings of the projective planes, namely $\mathbb{R}P^2$, $\mathbb{C}P^2$, $\mathbb{H}P^2$ and $\mathbb{O}P^2$. The construction gives minimal Legendrian embeddings

$$S^4 \to S^9, \quad S^7 \to S^{15}, \quad S^{13} \to S^{27}, \quad S^{25} \to S^{51}.$$

Since these embeddings are not totally geodesic (in fact, a simple computation shows that Ψ can be totally geodesic in S^{2n+1} only if M is totally geodesic in S^n), they are not congruent to the standard embedding $S^n \subset \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$ of the unit sphere. The case $S^4 \to S^9$ first appears in [52], Example 3.21.

g=4. – Here $m = 1, 2$. If $m = 1$ then the representation $\rho : \mathbf{U}(2) \to \mathbf{O}(\mathbb{R}^6)$ is on the space of 2×2 complex symmetric matrices and is given by $\rho(A)X = AXA^t$. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{U}(2)/\mathbb{Z}_2^2$ and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(\mathbf{U}(2)/\mathbb{Z}_2^2) \times_{\mathbb{Z}_2} S^1$. Since $\mathbf{U}(2)$ is diffeomorphic to $S^1 \times S^3$, in particular we get a minimal Legendrian immersion

$$S^1 \times S^1 \times S^3 \to S^{11}.$$

If $m = 2$ then $\rho : \mathbf{Sp}(2) \to \mathbf{O}(\mathbb{R}^{10})$ is the adjoint representation. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{Sp}(2)/T^2$ where $T^2 \subset \mathbf{Sp}(2)$ is a maximal torus and $\widetilde{M}/\mathbb{Z}_2$

is diffeomorphic to $\mathbf{Sp}(2)/T^2 \times_{\mathbb{Z}_2} S^1$.

g=6. – Here $m = 1, 2$. If $m = 1$ then ρ is the only almost faithful real irreducible 8-dimensional representation of $\mathbf{SO}(4)$. The isoparametric hypersurfaces are diffeomorphic to $\mathbf{SO}(4)/\mathbb{Z}_2^2$ and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $(\mathbf{SO}(4)/\mathbb{Z}_2^2) \times_{\mathbb{Z}_2} S^1$. Since $\mathbf{SO}(4)$ is finitely covered by $S^3 \times S^3$, in particular we get a minimal Legendrian immersion[12]

$$S^1 \times S^3 \times S^3 \to S^{15}.$$

If $m = 2$ then $\rho : \mathbf{G}_2 \to \mathbf{O}(\mathbb{R}^{14})$ is the adjoint representation. The isoparametric hypersurface are diffeomorphic to $\mathbf{G}_2/T^2$ where $T^2 \subset \mathbf{G}_2$ is a maximal torus and $\widetilde{M}/\mathbb{Z}_2$ is diffeomorphic to $\mathbf{G}_2/T^2 \times_{\mathbb{Z}_2} S^1$.

4.2.3 Orbits of isotropy representations of Hermitian symmetric spaces

We shall now show that certain orbits of the isotropy representations of certain Hermitian symmetric spaces give rise to examples of Legendrian submanifolds in spheres. The results quoted without proof can all be found in [54] and [63]. Let X be an Hermitian symmetric space of compact type, G its connected group of isometries and K the isotropy subgroup. So $X = G/K$ is a product of compact irreducible Hermitian symmetric spaces, and G is the product of the corresponding simple groups of isometries. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the decomposition of the Lie algebra $\mathfrak{g}$ of G into the ± 1-eigenspaces of the involution, which will be denoted by σ. Here $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is naturally identified with the tangent space of X at the base point. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. We choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{k}$ (this is possible, since $\mathfrak{k}$ is of maximal rank in $\mathfrak{g}$). Then $\mathfrak{h}^c$ is a Cartan subalgebra in $\mathfrak{g}^c$. The roots of $\mathfrak{g}^c$ which are also roots of $\mathfrak{k}^c$ are called compact roots. Introduce an ordering in the system of roots. To each root α we define the coroot $H_\alpha \in \mathfrak{h}^c$ by the rule $\alpha(H) = (H_\alpha, H)$ for all $H \in \mathfrak{h}^c$ (here $(\cdot,\cdot)$ denotes the Cartan-Killing form of $\mathfrak{g}^c$) and we associate a root vector X_α so that

$$(X_\alpha - X_{-\alpha}), \quad i(X_\alpha + X_{-\alpha}) \in \mathfrak{g} \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = \frac{2H_\alpha}{|\alpha|^2}.$$

There exists an element Z in the center of $\mathfrak{k}$ such that, for every noncompact positive root α, $[Z, X_\alpha] = -iX_\alpha$ and $[Z, X_{-\alpha}] = iX_{-\alpha}$. The restriction of $\mathrm{ad}(Z)$ to $\mathfrak{p}$ is a complex structure, which will be denoted by J. By means of this J, we view $\mathfrak{p}$ as $\mathbb{C}^{\dim X}$. The roots α, β are called strongly orthogonal if $\alpha \pm \beta$ are not roots. There exists a set Δ of strongly orthogonal noncompact positive roots such that the real subspace $\mathfrak{a}$ spanned by the $i(X_\alpha + X_{-\alpha})$, $\alpha \in \Delta$, is a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Define the Cayley transform $c = \exp \frac{\pi}{4} \sum_{\alpha\in\Delta} i(X_\alpha + X_{-\alpha}) \in G$ and let $\tau = \mathrm{ad}(c)^2$, viewed as an automorphism of $\mathfrak{g}$. Then $\tau^4 = 1$ and the following are equivalent (see [63], Remark 1, p. 286 and Proposition 4.4, p. 273) :

– the noncompact dual X^* of X is of tube type ;
– $\tau(Z) = -Z$;

[12] This immersion is of course different from the immersion of $S^1 \times S^3 \times S^3$ in S^{15} given above in the case $g = 2$, $n = 7$, because their images are not diffeomorphic.

– $\tau(\mathfrak{k}) = \mathfrak{k}$;
– $\tau^2 = 1$.

Hereafter we shall assume that the Hermitian symmetric space X^* is of tube type, that is, holomorphically equivalent to the tube domain over a self-dual cone. Since τ is an involution of $\mathfrak{g}$ which preserves $\mathfrak{k}$, and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, we have that τ also preserves $\mathfrak{p}$, and this yields the splittings $\mathfrak{k} = \mathfrak{k}^+ + \mathfrak{k}^-$, $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$ into the ± 1-eigenspaces of τ. Now $\mathrm{ad}(c)$ preserves $\mathfrak{k}^+$ and $\mathfrak{p}^+$, and interchanges $\mathfrak{k}^-$ with $\mathfrak{p}^-$, and J interchanges $\mathfrak{p}^+$ with $\mathfrak{p}^-$. Note that $Z \in \mathfrak{k}^-$ and define $p = \mathrm{ad}(c)Z \in \mathfrak{p}^-$. Let $\rho : K \to \mathbf{U}(\mathfrak{p})$ be the linear isotropy representation of X and consider the K-orbit through p. The tangent space $T_pK(p)$ is

$$\begin{aligned}
[\mathfrak{k}, p] &= \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}, Z] \\
&= \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}^+, Z] + \mathrm{ad}(c)[\mathrm{ad}(c)^{-1}\mathfrak{k}^-, Z] \\
&= \mathrm{ad}(c)[\mathfrak{k}^+, Z] + \mathrm{ad}(c)\, J\, \mathrm{ad}(c)^{-1}\mathfrak{k}^- \\
&= 0 + \mathrm{ad}(c)\, J\, \mathfrak{p}^- \\
&= \mathrm{ad}(c)\, \mathfrak{p}^+ \\
&= \mathfrak{p}^+.
\end{aligned}$$

Now $[\mathfrak{k}, p]$ is a Lagrangian subspace of $\mathfrak{p}$. Let $\mathfrak{k}'$ be the Cartan-Killing orthogonal complement of Z in $\mathfrak{k}$ and let K' be the analytic subgroup of K corresponding to $\mathfrak{k}'$. Since $[\mathfrak{k}, p] = [\mathfrak{k}', p] \oplus \mathbb{R}[Z, p] = [\mathfrak{k}', p] \oplus \mathbb{R}Jp$, we have that $[\mathfrak{k}', p] \oplus \mathbb{R}p$ is also a Lagrangian subspace of $\mathfrak{p}$. So the tangent space $T_pK'(p)$ is a Legendrian subspace of $\mathfrak{p}$. At this point, we assume that X is irreducible. Then $\rho(K') \subset \mathbf{SU}(\mathfrak{p})$, so that $K'(p)$ is a Legendrian submanifold and its Maslov form vanishes. Hence $K'(p)$ is a compact minimal Legendrian submanifold of the unit sphere in $\mathfrak{p} \cong \mathbb{C}^{\dim X}$. The remark after Proposition 4.4 in [63] can be interpreted as saying that X^* being of tube type is equivalent to the system of restricted roots of X^* being reduced (compare p. 456 in [54]). According to the classification of Hermitian symmetric spaces (see, for example, section 8.11 in [94]), one sees that there are five types of irreducible compact Hermitian symmetric spaces X whose noncompact duals are of tube type. These are listed in the table below, which also describes in each case the diffeomorphism type M of $K'(p)$.

X	$M \subset S^{\dim X - 1}$
$\mathbf{SO}(n+2)/\mathbf{SO}(2) \times \mathbf{SO}(n)$ $(n \geq 3)$	$S^{n-1} \subset S^{2n-1}$
$\mathbf{SU}(2n)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(n))$ $(n \geq 2)$	$\mathbf{SU}(n) \subset S^{2n^2-1}$
$\mathbf{SO}(4n)/\mathbf{U}(2n)$ $(n \geq 2)$	$\mathbf{SU}(2n)/\mathbf{Sp}(n) \subset S^{4n^2-2n-1}$
$\mathbf{Sp}(n)/\mathbf{U}(n)$ $(n \geq 3)$	$\mathbf{SU}(n)/\mathbf{SO}(n) \subset S^{n^2+n-1}$
$\mathbf{E}_7/\mathbf{SO}(2)\mathbf{E}_6$	$\mathbf{E}_6/\mathbf{F}_4 \subset S^{53}$

The first four minimal Legendrian embeddings in the table are given as follows : $S^{n-1} \subset S^{2n-1}$ is totally real ; $\mathbf{SU}(n) \subset S^{2n^2-1}$ comes from the standard inclusion of $\mathbf{SU}(n)$ into the $n \times n$ complex matrices ; $\mathbf{SU}(n)/\mathbf{SO}(n) \subset S^{n^2+n-1}$ is the $\mathbf{SU}(n)$-orbit $\{AA^t : A \in \mathbf{SU}(n)\}$

in the space of $n \times n$ complex symmetric matrices; $\mathbf{SU}(2n)/\mathbf{Sp}(n) \subset S^{4n^2-2n-1}$ is the $\mathbf{SU}(2n)$-orbit $\{A \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} A^t : A \in \mathbf{SU}(2n)\}$ in the space of $2n \times 2n$ complex skew-symmetric matrices. In [20, 21], these examples are discussed, with the curious exception of $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ for $n \neq 3$, on a case by case basis (note however the misprint in the title of [21]).

4.2.4 Yet another example

Consider the 4-dimensional complex irreducible representation of $\mathbf{SU}(2)$, namely $\rho : \mathbf{SU}(2) \to \mathbf{SU}(4)$ (ρ can be realized as the natural action of $\mathbf{SU}(2)$ on the space of homogeneous complex polynomials of degree 3 in two variables z_1, z_2). Upon suitable choices of basis elements, we can write

$$\rho \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} = \begin{pmatrix} \alpha^3 & \sqrt{3}\alpha\beta^2 & \beta^3 & \sqrt{3}\alpha^2\beta \\ \sqrt{3}\alpha\bar\beta^2 & \alpha\bar\alpha^2 - 2\bar\alpha\beta\bar\beta & \sqrt{3}\bar\alpha^2\beta & \beta\bar\beta^2 - 2\alpha\bar\alpha\bar\beta \\ -\bar\beta^3 & -\sqrt{3}\bar\alpha^2\bar\beta & \bar\alpha^3 & \sqrt{3}\bar\alpha\bar\beta^2 \\ -\sqrt{3}\alpha^2\bar\beta & 2\alpha\bar\alpha\beta - \beta^2\bar\beta & \sqrt{3}\bar\alpha\beta^2 & \alpha^2\bar\alpha - 2\alpha\beta\bar\beta \end{pmatrix},$$

where α, $\beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. Therefore the induced representation on the Lie algebra level maps the basis elements of the Lie algebra $\mathfrak{su}(2)$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

respectively into

$$\begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -3i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -2 \\ 0 & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & 2 & 0 & 0 \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} 0 & 0 & 0 & i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 2i \\ 0 & i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 2i & 0 & 0 \end{pmatrix}.$$

Let M be the orbit through the point $p = (1,0,1,0) \in \mathbb{C}^4$, which is the quotient of S^3 by a cyclic subgroup of order 3. The easily seen identities

$$< Xp, iYp >=< Xp, ip >= 0$$

for $X, Y \in \mathfrak{su}(2)$ show that the tangent space of M at p is a Legendrian subspace. Since $\rho[\mathbf{SU}(2)] \subset \mathbf{SU}(4)$, it follows that M is a Legendrian submanifold of S^7 with vanishing Maslov form. Therefore it is minimal and the cone over M is absolutely area-minimizing. Observe that this example is not the twisted normal bundle over a compact minimal surface in S^3 because of its topology.

4.3 Legendrian 2-tori in S^5

In this section we prove the theorem stated in the introduction.

Let λ be the Liouville form. We denote by $I_{Leg}(T^2, S^5)$ the space of Legendrian immersions of T^2 into S^5 and by $\mathrm{Mono}_\lambda(TT^2, TS^5)$ the space of bundle monomorphisms $\sigma : TT^2 \to TS^5$ such that $\lambda_{|\sigma(F)} = 0$ and $d\lambda_{|\sigma(F)} = 0$ for each fiber F of TT^2. (These two spaces are provided with the compact-open topology). The h-principle for Legendrian immersions implies that the map :

$$\begin{array}{ccc} I_{Leg}(T^2, S^5) & \longrightarrow & \mathrm{Mono}_\lambda(TT^2, TS^5) \\ g & \longmapsto & dg, \end{array}$$

is a weak homotopy equivalence (see [27], p. 144).

Lemma. – $\pi_0(I_{Leg}(T^2, S^5)) \simeq H^1(T^2, \mathbb{Z})$ *and the identification is given by half the Maslov class.*

Proof of the lemma. – A 2-frame of T_xT^2 is said to be Legendrian if the plane it spans is Legendrian. The space of all Legendrian 2-frames can be identified with $\mathbf{GL}(3, \mathbb{C})$. Let $R = (e_1, e_2)$ be a global section of the frame bundle $\pi : FT^2 \to T^2$. There is a bijection :

$$\Psi : \ \mathrm{Mono}_\lambda(TT^2, TS^5) \longrightarrow C^0(T^2, \mathbf{GL}(3, \mathbb{C})), \quad \sigma \longmapsto \Psi(\sigma) = f_\sigma,$$

where f_σ is the map $x \mapsto (\sigma(\pi(R_x)), \sigma(e_1(x)), \sigma(e_2(x)))$. Thus

$$\pi_0(I_{Leg}(T^2, S^5)) \simeq [T^2, \mathbf{GL}(3, \mathbb{C})] \simeq [T^2, \mathbf{U}(3)],$$

since $\mathbf{U}(3)$ is a deformation retract of $\mathbf{GL}(3, \mathbb{C})$. Consider the projection $\det : \mathbf{U}(3) \to \mathbf{U}(1)$ and let u be the pull back of the (normalized) volume form w of $\mathbf{U}(1)$. The class u is a generator of $H^1(\mathbf{U}(3), \mathbb{Z})$ and, since $\pi_1(\mathbf{U}(3)) = \mathbb{Z}$, $\pi_2(\mathbf{U}(3)) = 0$, the degree map :

$$\begin{array}{cccc} \deg : & [T^2, \mathbf{U}(3)] & \longrightarrow & H^1(T^2, \mathbb{Z}) \\ & f & \longmapsto & f^*u \end{array}$$

is a bijection (see [82], p. 451 or [11], p. 510 for example). Let $g \in I_{Leg}(T^2, S^5)$ and let $\hat{g}$ be its associated cone map $\hat{g} : T^2 \times (0, \infty) \to \mathbb{C}^3$ given by $\hat{g}(t, x) = tg(x)$. Let $p : \mathbf{U}(3) \to \mathbf{U}(3)/\mathbf{O}(3)$ be the projection. The Gauss map of $\hat{g}$ at $(x, 0)$ is given by

$$G(\hat{g})(x, 0) = p \circ r \circ (g(x), dg(e_1(x)), dg(e_2)(x)) = p \circ r \circ f_{dg}(x, 0),$$

where $r : \mathbf{GL}(3, \mathbb{C}) \to \mathbf{U}(3)$ is a retraction. Hence

$$[\mu(\hat{g})] = (\det{}^2 \circ G(\hat{g}))^* w = f_{dg}^* \circ r^* \circ p^* \circ (\det{}^2)^* w.$$

It is readily checked that $p^* \circ (\det^2)^* w = 2u$ and, of course, $r^* = id$ so

$$[\mu(\hat{g})] = 2 f_{dg}^* u.$$

Thus $\pi_0(I_{Leg}(T^2, S^5))$ is classified by half the Maslov class. $\square$

The proof of the theorem is a consequence of this lemma combined with the fact that there actually exists a minimal Legendrian embedding of the 2-torus into S^5.

The infimum of the Energy of Unit Vector Fields on Odd-Dimensional Spheres

Ann. of Glo. Anal. and Geom. 23 (2003), 129-140.

Abstract. – We construct a one-parameter family of unit smooth vector fields globally defined on the sphere $\mathbb{S}^{2k+1}$ for $k \geq 2$, with energy converging to the energy of the unit radial vector field, which is defined on the complementary of two antipodal points. So we prove that the infimum of the energy of globally defined unit smooth vector fields is :

$$(\frac{2k+1}{2} + \frac{k}{2k-1}) \text{ vol } (\mathbb{S}^{2k+1}).$$

2000 AMS subject classification : Primary 53C20 ; Secondary 58E15.

5.1 Introduction

Let M be a closed, n-dimensional Riemannian manifold and T^1M be the unit tangent bundle of M considered as a closed Riemannian manifold with the Sasaki metric. The energy of a unit smooth vector field $v \in \chi^1(M)$ is :

$$E(v) = \frac{1}{2} \int_M (n + \|\nabla v\|^2) \, dM,$$

where dM is the density defined by the metric and ∇ is its Levi-Civita connection [93].

It is clear that a lower bound for the functional E is given by $\frac{n}{2}$ vol (M). Moreover, it is realized if and only if v is parallel on M. The problem of minimizing the energy E among unit vector fields becomes interesting when M does not admit a global parallel unit vector field, as it has been pointed out in [93]. This is the case for an odd-dimensional sphere.

In [12], the second author established that Hopf vector fields are the unique absolute minima in the case of $\mathbb{S}^3$. The energy of Hopf vector fields on $\mathbb{S}^{2k+1}$ is

$$(\frac{2k+1}{2} + k) \text{ vol } (\mathbb{S}^{2k+1}).$$

Let us consider a unit vector field T, defined on the complementary of two antipodal points $\{p, -p\}$, such that it is tangent to radial geodesics issuing from p. It is easy to see that $E(T) = (\frac{2k+1}{2} + \frac{k}{2k-1})$ vol $(\mathbb{S}^{2k+1}) := c_k$ and then for $k \geq 2$ they have less energy that Hopf vector fields. In fact, in [16] *it has been shown that for every unit smooth vector field v (possibly with isolated singularities) $E(v) \geq c_k$*. Moreover the equality holds if and only if v is geodesic and $v^\perp$ is integrable and with umbilical leaves. Since the only vector fields with these properties are radial vector fields, *this lower bound is not attained on $\chi^1(\mathbb{S}^{2k+1})$*. In this paper, we prove the following theorem.

Theorem. – *The infimum of E among all globally defined unit smooth vector fields of the sphere $\mathbb{S}^{2k+1}$ ($k \geq 2$) is*
$$(\frac{2k+1}{2} + \frac{k}{2k-1})\ vol\ (\mathbb{S}^{2k+1}).$$
This value is not attained by any globally defined unit smooth vector field.

To obtain the result, we construct a family T_R of unit vector fields such that their energy goes to c_k when R goes to zero. The vector field T_1 ($R = 1$) is taken equal to T outside a neighbourghood G_1 of one of the geodesic segments joining the two antipodal singularities. The desingularization of T is done inside G_1 by a Riemannian analogue of the "insertion of a dipole", a method used by several authors to study the density problem in functional spaces [80]. The whole family $(T_R)_{R\leq 1}$ is obtained by performing a contraction of G_1 in the directions orthogonal to the geodesic segment. Inside the resulting neighbourghood G_R, the unit vector field T_R is the composition of this contraction with T_1 and outside, T_R is T. At this step, the problem is to carry out this construction while controling the energy. Since $Vol(G_R)$ decreases like R^{2k} when R goes to zero, for every odd dimension greater or equal to 5, $\|\nabla T_R\|^2$ should increase only like $R^{-\alpha}, \alpha < 4$. This is part of the difficulty and explains the technicalities of the construction.

For the construction of these vector fields it has been very convenient to use stereographic projection from a point situated on the equator determined by the singularities so that the choosen geodesic segment becames a segment in the coordinate axe. In this chart, T has rotational symmetry with respect to this axe and it is obtained by rotating a vector field defined on the plane (except on the two singular points). This symmetry will be preserved by T_R. But, since it is impossible to obtain a unit vector field without singularities on $\mathbb{S}^2$, the construction should be then more delicate and of course it can be only done when the dimension is odd.

This result has been announced in [10]. Variational aspects of E were focused in [57], [93], [34],[37] and [51].

Acknowledgements. – The first and the third authors are grateful to the second for his invitation at the University of São Paulo. During the preparation of this paper, the first author was supported by FAPESP, proj. tem. no. 1999/02684-5. The second author was supported by CNPq Brazil. The third was partially supported by FAPESP, proc. no. 2000/07143-1 and partially by DGES Project No. PB97-1425.

5.2 Proof of the Theorem

Last assertion in the statement has been shown in [16], therefore to prove the theorem we have to show :

There exists a family $\{T_R \mid R \in]0,1]\}$ of smooth globally defined unit vector fields on $\mathbb{S}^{2k+1}$ such that : $\lim_{R\to 0} E(T_R) = c_k$.

Let p be a point of $\mathbb{S}^{2k+1} \subset \mathbb{R}^{2k+2}$ and let T be the unit vector field tangent to radial geodesics from p. The field T is defined on $\mathbb{S}^{2k+1} - \{-p, p\}$. Our purpose is to construct a globally defined field T_R by modifying T in a neighbourhood of one geodesic joining p and $-p$.

Construction of T_R. – Let us assume for simplicity that $p = (1, 0, ..., 0)$. We represent by $(x, y) \in \mathbb{R} \times \mathbb{R}^{2k}$ the coordinates corresponding to the stereographic projection from the north pole $(0, ..., 0, 1)$ to the plane tangent to the sphere at the south pole. In this chart p has coordinates $(2, 0)$ and the geodesic segment joining p and $-p$ is just $I = [-2, 2] \times \{0\}$. Let E_0, E_y be two orthogonal vector fields that are unit with respect to the metric of the sphere and such that E_0 is on the direction of the x-axis and E_y is a radial field on the plane $\{x = 0\}$. Of course, since T is invariant by rotation around the x-axis it can be written as :

$$T(x, y) = a(x, \rho)E_0(x, y) + b(x, \rho)E_y(x, y),$$

where $\rho(y) = \sum_{i=1}^{2k} y_i^2$. (We have plotted in fig. 1 some trajectories of T).

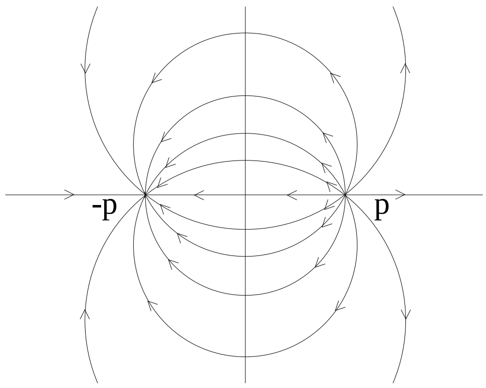


Fig.1

For each $R > 0$ we consider the open subset G_R bounded by the ellipsoide $\frac{x^2}{(2+R)^2} + \frac{\rho(y)}{R^2} = 1$. That is :

$$G_R = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{2k} | d_R(x, y) < 1\}$$

where :

$$d_R(x, y) = \frac{\sqrt{R^2 x^2 + \rho(y)(2 + R)^2}}{R(2 + R)}.$$

The field T_R will be defined such that :

- it is invariant by rotation,
- it is equal to T on the complementary of G_R.

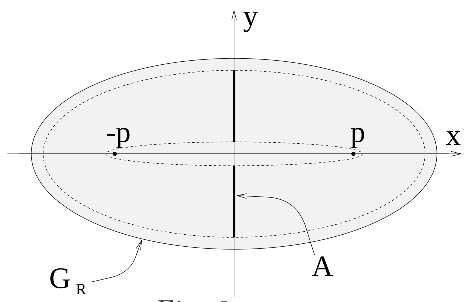


Fig. 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $0 \leq f \leq 1$, $f_{|]-\infty,\frac{1}{10}]} \equiv 0$ and $f_{|[\frac{9}{10},\infty[} \equiv 1$. The function $h_R = f \circ d_R$ vanishes on $G_{\frac{R}{10}}$ and is equal to 1 on the complementary of $G_{\frac{9R}{10}}$. We are going to use this function to define a smooth vector field U on the complementary of $A = \{0\} \times (B_{\frac{9R}{10}} - B_{\frac{R}{10}})$ (see fig. 2). Here $B_r = \{(0,y)| \quad |y| < r\}$ where $|y|$ is the Euclidean norm (we reserve the notation $\|.\|$ for the norm of the usual metric of the sphere). Let φ be the angle of T with the horizontal axis so that $a = \cos\varphi$ and $b = \sin\varphi$. Let $G_R^+ = \{(x,y) \in G_R | x > 0\}$. On $G_R^+ - \{p\}$ the angle φ is between 0 and π. We define $U_{|G_R^+}$ to be :

$$U = \cos(h_R\varphi)E_0 + \sin(h_R\varphi)E_y.$$

According to the definition of h_R, the angle of U with the horizontal axis is 0 near the axis and is φ near the boundary of G_R. On $G_R^- = \{(x,y) \in G_R | x < 0\}$ we define U with the same formula but now the angle φ is taken to be between $-\pi$ and 0. Notice that on the set $\{x = 0\}$ our choices do not match together except for $\{0\} \times (B_{\frac{R}{10}} \cup (B_R - B_{\frac{9R}{10}}))$. Outside G_R the vector field U is taken to be equal to T. So, U is a smooth unit vector field, invariant by rotation, defined on the connected set A^c. In order to overcome this difficulty, we introduce the disc D_R (resp. $\overline{D}_R$) on the plane $\{y_2 = ... = y_{2k} = 0\}$ with center $(0, R/2)$ and radius $R/2$ (resp. $\frac{2}{5}R$) (see fig. 3).

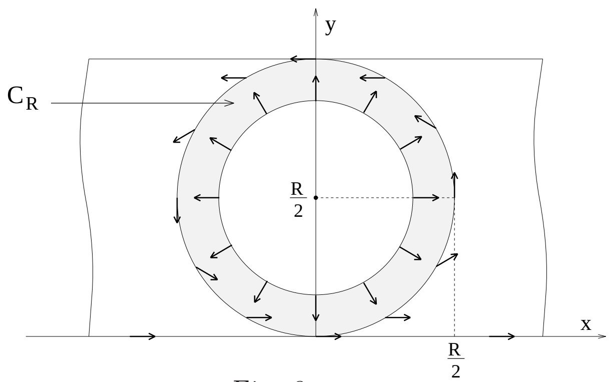


Fig. 3

We are going to modify U on $C_R = (D_R - \overline{D}_R) \times \mathbb{S}^{2k-1}$ such that the resulting vector field V has the following properties :

1) V restricted to $\partial D_R \times \mathbb{S}^{2k-1}$ is U,
2) V restricted to $\partial \overline{D}_R \times \mathbb{S}^{2k-1}$ is the outward normal field.

Since V is going to be also invariant by rotation, it is enough to define it on $D_R - \overline{D}_R$. To do this, let us use polar coordinates on D_R. On these coordinates, U has the form :

$$U(r,\theta) = \cos\psi(r,\theta)E_r + \sin\psi(r,\theta)E_\theta.$$

Note that $\psi(r,\theta) \in]-\pi,\pi[$. We take V to be :

$$V = \cos(g_R\psi)E_r + \sin(g_R\psi)E_\theta$$

where $g_R(r,\theta) = f(\frac{10r}{R} - 4)$. By construction, V satisfies conditions 1) and 2).

It remains to define the vector field on $\overline{D}_R \times \mathbb{S}^{2k-1}$. Let $X \in \chi^1(\mathbb{S}^{2k-1}(1))$ and $\alpha(r) = \frac{\pi}{2}f(\frac{10r}{9R})$. On $\{(r,\theta,s) \in \mathbb{R}^{2k+1} | r < \frac{9R}{10} \text{ and } |s| = \frac{R}{2}\}$ we put :

$$W_R(r,\theta,s) = \cos\alpha(r)X_R(s) + \sin\alpha(r)E_r(r,\theta,s)$$

with $X_R(s) = \frac{1}{\lambda}X(\frac{2s}{R})$ where λ is the conformal factor of the metric on the sphere, that is $\lambda(x,y) = 4/(4+x^2+\rho)$. In particular $X_R(s) = \frac{16+R^2}{16}X(\frac{2s}{R})$.

Now T_R is defined to be W_R on $\overline{D}_R \times \mathbb{S}^{2k-1}$, V on C_R, U on $G_R - (D_R \times \mathbb{S}^{2k-1})$ and T otherwise. The figure 4 shows some of the trajectories of the field T_R outside $D_R \times \mathbb{S}^{2k-1}$.

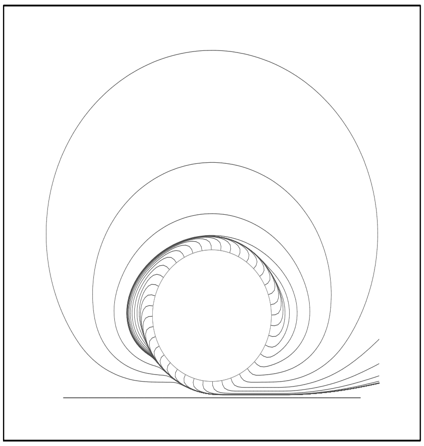


Fig. 4

Remark. − It is easy to see that the distribution orthogonal to T_R is integrable if $X \in \chi^1(\mathbb{S}^{2k-1})$ is such that $X^\perp$ is integrable. The figure below shows the leaves of the orthogonal distribution out of $D_R \times \mathbb{S}^{2k-1}$.

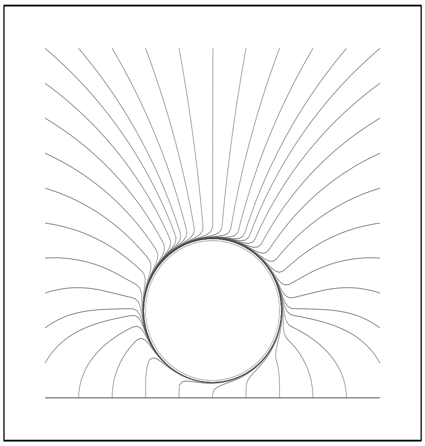


Fig. 5

ENERGY OF T_R. − Let $T = aE_0 + bE_y$ with $a = \cos\varphi$ and $b = \sin\varphi$, then (see Appendix) :

$$\begin{aligned} \|\nabla T\|^2 &= \left(\frac{1}{\lambda}\frac{\partial\varphi}{\partial x} + \frac{\sqrt{\rho}}{2}\right)^2 + \left(\frac{2\sqrt{\rho}}{\lambda}\frac{\partial\varphi}{\partial\rho} - \frac{x}{2}\right)^2 \\ &+(n-2)\left((4+x^2-\rho)\frac{\sin\varphi}{4\sqrt{\rho}} - \frac{x}{2}\sqrt{\rho}\cos\varphi\right)^2 . \end{aligned}$$

We use this formula to prove the following lemma :

Lemma 1. −*One has :*

$$\forall R \leq 1, \ \ \forall m \in G_1, \quad \|\nabla T_R(m)\|^2 \leq F(R) + 4h_R^2(m)\|\nabla T(m)\|^2$$

where T is the unit radial vector field and $F(R)$ is of the form $\dfrac{Const_3}{R^3} + \dfrac{Const_2}{R^2} + \dfrac{Const_1}{R} + Const_0$.

Note that the theorem is a consequence of this lemma. Indeed :

$$\lim_{R\to 0}\int_{G_R} h_R^2\|\nabla T\|^2\, dM = 0$$

since $\int \|\nabla T\|^2 < \infty$, and :

$$\int_{G_R} F(R)\, dM \sim \frac{1}{R^3} vol(G_R) \sim R^{2k-3}.$$

The hypothesis $2k+1 \geq 5$ implies that :

$$\lim_{R\to 0}\int_{G_R} F(R)\, dM = 0$$

and thus : $\lim_{R\to 0} E(T_R) = E(T) = c_k$.

Proof of lemma 1.– Let us compute first $\|\nabla T_R\|^2$ on the domain $G_R - (D_R \times \mathbb{S}^{2k-1})$. Here $T_R = \cos(\varphi_R)E_0 + \sin(\varphi_R)E_y$ with $\varphi_R = h_R\varphi$ and $h_R = f \circ d_R$. It is easy to see that on G_1,

$$|\frac{\partial d_R}{\partial x}| \leq \frac{1}{2+R} \leq \frac{1}{2} \text{ and } |\frac{\partial d_R}{\partial \rho}| \leq \frac{1}{2R\sqrt{\rho}}.$$

Thus :

$$\begin{aligned}\left(\tfrac{1}{\lambda}\tfrac{\partial \varphi_R}{\partial x} + \tfrac{\sqrt{\rho}}{2}\right)^2 &= \left(\tfrac{1}{\lambda}\tfrac{\partial d_R}{\partial x} f'\varphi + \tfrac{1}{\lambda}h_R\tfrac{\partial \varphi}{\partial x} + \tfrac{\sqrt{\rho}}{2}\right)^2 \\ &\leq 2\left(|\tfrac{1}{\lambda}\tfrac{\partial d_R}{\partial x} f'\varphi + \tfrac{(1-h_R)\sqrt{\rho}}{2}|^2 + h_R^2|\tfrac{1}{\lambda}\tfrac{\partial \varphi}{\partial x} + \tfrac{\sqrt{\rho}}{2}|^2\right) \\ &\leq 2\left((\tfrac{\pi}{\lambda}\tfrac{1}{2}\sup|f'| + \tfrac{\sqrt{\rho}}{2})^2 + h_R^2\|\nabla T\|^2\right).\end{aligned}$$

On the other hand :

$$\begin{aligned}\left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial \varphi_R}{\partial \rho} - \tfrac{x}{2}\right)^2 &= \left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial d_R}{\partial \rho} f'\varphi + \tfrac{2\sqrt{\rho}}{\lambda}h_R\tfrac{\partial \varphi}{\partial \rho} - \tfrac{x}{2}\right)^2 \\ &\leq 2\left(|\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial d_R}{\partial \rho} f'\varphi + \tfrac{(h_R-1)x}{2}|^2 + h_R^2|\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial \varphi}{\partial \rho} - \tfrac{x}{2}|^2\right) \\ &\leq 2\left((\tfrac{2\pi}{\lambda R}\sup|f'| + \tfrac{3}{2})^2 + h_R^2\|\nabla T\|^2\right).\end{aligned}$$

It remains to bound : $(n-2)\left((4+x^2-\rho)\frac{\sin h_R\varphi}{4\sqrt{\rho}} - \frac{x}{2}\sqrt{\rho}\cos\varphi\right)^2$. The only non-trivial term is the quotient $\frac{\sin h_R\varphi}{\sqrt{\rho}}$. Let $B_R = \{\rho \leq \left(\frac{R}{15}\right)^3\} \cap G_1$. Of course on $B_R^c \cap G_1$, $\frac{|\sin h_R\varphi|^2}{\rho} \leq \left(\frac{15}{R}\right)^3$ and on $B_R \cap G_{\frac{R}{10}}$, $\frac{\sin h_R\varphi}{\sqrt{\rho}} = 0$. Thus we have to find a bound of $\frac{\sin h_R\varphi}{\sqrt{\rho}}$ on $D = G^c_{\frac{R}{10}} \cap B_R \cap G_1 = D_- \cup D_+$ (see the figure below).

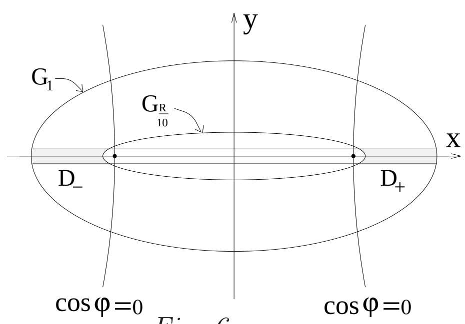


Fig. 6

Note that the hyperboloid of two sheets $H = \{\cos\varphi = 0\} = \{x^2 = \rho + 4\}$ (see Appendix) intersects the boundary of $G_{\frac{R}{10}}$ at

$$\rho = \left(\frac{R}{10}\right)^3 \frac{4 + \frac{R}{10}}{\left(\frac{R}{10}\right)^2 + (2 + \frac{R}{10})^2} \geq \frac{4\frac{R^3}{1000}}{\left(\frac{1}{10}\right)^2 + \left(2 + \frac{1}{10}\right)^2} > \left(\frac{R}{15}\right)^3.$$

Thus on D_+, $0 \leq \varphi < \frac{\pi}{2}$ and $\frac{|\sin h_R\varphi|}{\sqrt{\rho}} \leq \frac{|\sin\varphi|}{\sqrt{\rho}}$. By the lemma 3 of the appendix :

$$\frac{|\sin\varphi|^2}{\rho} = \frac{4x^2}{(4-x^2)^2 + \rho^2 + 2x^2\rho + 8\rho}.$$

It is easily checked that :

$$\min_{\partial G_R} (4-x^2)^2+\rho^2+2x^2\rho+8\rho = \frac{4R^3(2+R)^2}{1+R},$$

and therefore :

$$\min_{G_1-G_{\frac{R}{10}}} (4-x^2)^2+\rho^2+2x^2\rho+8\rho = 4\frac{R^3}{10^3}\frac{(2+\frac{R}{10})^2}{1+\frac{R}{10}} \geq \frac{4R^3}{275}.$$

Thus on D_+ and by symmetry on D_-,

$$\frac{|\sin\varphi|^2}{\rho} \leq 4(2+1)^2\frac{275}{4R^3}.$$

Finally on $G_R - (D_R \times \mathbb{S}^{2k-1})$, we have obtained :

$$\|\nabla T_R(m)\|^2 \leq \frac{Const_3}{R^3}+\frac{Const_2}{R^2}+\frac{Const_1}{R}+Const_0+4h_R^2\|\nabla T(m)\|^2$$

where the $Const_i$ depend on $\sup|f'|$.

The same method obviously works for the remaining part of G_R. Note that there is a subtility concerning W_R because, in this case, the unit vector field X_R depends on R. So, instead of the function $\cos\alpha$, one must consider the function $\frac{16+R^2}{16}\cos\alpha$. Anyway, the behaviour of this function near 0 is similar to the other cases. □

5.3 Appendix

Lemma 2. – *Let $T(x,y)=a(x,\rho)E_0(x,y)+b(x,\rho)E_y(x,y)$ as in §2, then :*

$$\begin{aligned}\|\nabla T\|^2 \quad &= \quad \left(\tfrac{1}{\lambda}\tfrac{\partial a}{\partial x}-\tfrac{\sqrt{\rho}}{2}b\right)^2+\left(\tfrac{1}{\lambda}\tfrac{\partial b}{\partial x}+\tfrac{\sqrt{\rho}}{2}a\right)^2+\left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial a}{\partial \rho}+\tfrac{x}{2}b\right)^2\\ &\quad +\left(\tfrac{2\sqrt{\rho}}{\lambda}\tfrac{\partial b}{\partial \rho}-\tfrac{x}{2}a\right)^2+(n-2)\left(\tfrac{4+x^2-\rho}{4\sqrt{\rho}}b-\tfrac{xa}{2}\right)^2.\end{aligned}$$

Proof. – We have that $E_0=\frac{1}{\lambda}\frac{\partial}{\partial x}$, $E_i=\frac{1}{\lambda}\frac{\partial}{\partial y^i}$, for $i=1,\ldots,2k$ and then

$$E_y=\frac{1}{\sqrt{\rho}}\sum_{i=1}^{n} y^i E_i.$$

Since the Christoffel symbols in this chart are :

$$\Gamma_{ij}^l = -\frac{\lambda}{2}\left(\delta_{il}y^j+\delta_{jl}y^i-\delta_{ij}y^l\right),$$

then :

$$g(\nabla_{E_i}E_j,E_l)=\frac{1}{2}\left(\delta_{ij}y^l-\delta_{il}y^j\right).$$

Here we take $i = 0, \dots\ 2k$ and $y^0 = x$. Now, a straightforward computation shows :

$$\begin{array}{rcl}
g(\nabla_{E_0}T, E_0) & = & E_0(a) - \frac{\sqrt{\rho}}{2}b = \frac{1}{\lambda}\frac{\partial a}{\partial x} - \frac{\sqrt{\rho}}{2}b \\
g(\nabla_{E_0}T, E_l) & = & y^l\left(\frac{E_0(b)}{\sqrt{\rho}} + \frac{a}{2}\right) \\
 & = & y^l\left(\frac{1}{\lambda\sqrt{\rho}}\frac{\partial b}{\partial x} + \frac{a}{2}\right) \\
g(\nabla_{E_l}T, E_0) & = & E_l(a) + y^l\frac{bx}{2\sqrt{\rho}} \\
 & = & y^l\left(\frac{2}{\lambda}\frac{\partial a}{\partial \rho} + \frac{bx}{2\sqrt{\rho}}\right) \\
g(\nabla_{E_l}T, E_j) & = & \delta_{lj}\left(-\frac{ax}{2} - \frac{b\sqrt{\rho}}{2}\right) + E_l(b)\frac{y^j}{\sqrt{\rho}} \\
 & & +bE_l\left(\frac{y^j}{\sqrt{\rho}}\right) + \frac{y^l y^j b}{2\sqrt{\rho}} \\
 & = & \delta_{lj}\left(\frac{b}{\lambda\sqrt{\rho}} - \frac{ax}{2} - \frac{b\sqrt{\rho}}{2}\right) \\
 & & +\frac{y^l y^j}{\sqrt{\rho}}\left(\frac{2}{\lambda}\frac{\partial b}{\partial \rho} - b\frac{4+x^2-\rho}{4\rho}\right).
\end{array}$$

By taking the squares of these expressions, adding up and simplifying we get the result. □

Lemma 3. – *The expression of the radial vector field is* $T(x,y) = a(x,\rho)E_0(x,y) + b(x,\rho)E_y(x,y)$ *where :*

$$\begin{array}{rcl}
a(x,\rho) & = & \frac{\lambda(x^2-\rho-4)}{4\sqrt{1-\lambda^2x^2}} \\
b(x,\rho) & = & \frac{\lambda\sqrt{\rho}x}{2\sqrt{1-\lambda^2x^2}}.
\end{array}$$

Proof.- At a point of coordinates (x,y), the differential of the stereographic projection maps T into the unit vector that is tangent to the circle passing through $(2,0)$, $(-2,0)$ and (x,y), oriented from $(2,0)$ to $(-2,0)$ and contained in the plane determined by the x-axis and the vector $(0,y)$, if $y \neq 0$. If $y = 0$, this circle degenerates in a line. The proof follows now from elementary arguments. □

A critical radius for unit Hopf vector fields on spheres

Accepté à Math. Ann. sous réserve de modifications mineures

Abstract. – The volume of a unit vector field V of the sphere $\mathbb{S}^n$ (n odd) is the volume of its image $V(\mathbb{S}^n)$ in the unit tangent bundle. Unit Hopf vector fields, that is, unit vector fields that are tangent to the fibre of a Hopf fibration $\mathbb{S}^n \to \mathbb{C}P^{\frac{n-1}{2}}$, are well known to be critical for the volume functional. Moreover, Gluck and Ziller proved that these fields achieve the minimum of the volume if $n = 3$ and they opened the question of whether this result would be true for all odd dimensional spheres. It was shown to be inaccurate on spheres of radius one. Indeed, Pedersen exhibited smooth vector fields on the *unit* sphere with less volume than Hopf vector fields for a dimension greater than five. In this article, we consider the situation for any odd dimensional spheres, but not necessarily of radius one. We show that the stability of the Hopf field *actually depends on radius,* instability occurs precisely if and only if $r > \frac{1}{\sqrt{n-4}}$. In particular, the Hopf field cannot be minimum in this range. On the contrary, for r small, a computation shows that the volume of vector fields built by Pedersen is greater than the volume of the Hopf one thus, in this case, the Hopf vector field remains a candidate to be a minimizer. We then study the asymptotic behaviour of the volume; for small r it is ruled by the first term of the Taylor expansion of the volume. We call this term the *twisting* of the vector field. The lower this term is, the lower the volume of the vector field is for small r. It turns out that unit Hopf vector fields are absolute minima of the twisting. This fact, together with the stability result, gives two positive arguments in favour of the Gluck and Ziller conjecture *for small r*.

2000 Mathematics Subject Classification : 53C20
Keywords and phrases : Volume, Hopf vector field, Stability of minimal vector fields, Twisting

6.1 Introduction and main results

The volume of a unit vector field V on a compact oriented Riemannian manifold M can be defined (see [40]) as the volume of the submanifold $V(M)$ of the unit tangent bundle equipped with the restriction of the Sasaki metric. It is given by

$$Vol(V) = \int_M \sqrt{det(Id + {}^T\nabla V \circ \nabla V)}dvol$$

where $dvol$ is the volume element determined by the metric and ∇ is the Levi-Civita connection. There is a trivial absolute minimum of the volume functional when unit parallel vector fields exist, but this is not always the case, since such a vector field will determine two mutually orthogonal, complementary and totally geodesic foliations.

An odd dimensional sphere admits unit vector fields but not parallel ones. A natural unit vector field is then given by the tangents to the fibers of the Hopf fibration and it is shown in [61] that this field is critical for the volume functional. In fact, we can go further, for [40] Gluck and Ziller proved that Hopf vector fields achieved the minimum of the volume among all unit vector fields of the unit sphere of dimension three. The method they used could not be extended to higher dimensions and they opened the question of whether that result was still true for all odd-dimensional spheres. This was shown to be inaccurate on spheres of radius one. Indeed, Pedersen [77] exhibited smooth vector fields on the unit sphere with less volume than Hopf vector fields for a dimension greater than five.

One remarkable fact with the volume functional is that it is not homogeneous with a dilatation of the metric. The influence of radius on the index and on the nullity of Hopf vector fields of the sphere is studied in [37], [38]. It is shown that Hopf vector fields remain critical for any radius and that, for $n \geq 5$, instability occurs if the radius is strictly greater than $\frac{1}{\sqrt{n-4}}$ (or equivalently, if the curvature k is less than $n-4$). Whether Hopf vector fields are unstable for $k \geq n-4$ was an open question. In this article, we completely solve the stability problem, which *actually depends on curvature.*

Stability Theorem. – *Let $n \geq 5$. The Hopf vector field is stable if and only if $k \geq n-4$.*

Note that the situation for $n = 3$ is independent of curvature. Hopf vector fields are not only stable but absolute minimizers of the volume [40], [12].

We then study the asymptotic behaviour of the volume functional with the curvature. If V is a unit vector field on $\mathbb{S}^n(1)$, we consider the function $k \mapsto Vol(V^k)$ where $Vol(V^k)$ is the volume of the corresponding unit vector field V^k on the sphere $\mathbb{S}^n(r)$ of radius $r = k^{-\frac{1}{2}}$. If k is small, a Taylor expansion shows that this behaviour is ruled by the *bending* $B(V)$ of V, a notion previously introduced by Wiegmink [93] (see the end of section 3). If k is large, this behaviour is ruled by another quantity which we call the *twisting* of V :

$$Tw(V) = \int_{\mathbb{S}^n(1)} \sqrt{\sigma_{n-1}(\,{}^T\nabla V \circ \nabla V)}\, dvol$$

(in this expression, σ_{n-1} denotes the $(n-1)$-th elementary symmetric polynomial function).

Twisting Theorem. –

1) For every unit vector field V of $\mathbb{S}^n(1)$, $Tw(V) \geq Tw(H)$.

2) If $Tw(V) > Tw(H)$ then there is $k_0 > 0$ such that for all $k > k_0$, we have : $Vol(V^k) > Vol(H^k)$.

As a consequence, vector fields constructed by Pedersen cannot achieve the minimum of the volume functional for large values of curvature since their twisting is strictly greater than that of the Hopf vector field (see Lemma 4). In fact, we do not know of any unit vector field with less volume than the Hopf one for these large values. So Gluck and Ziller's conjecture is still open for spheres of small radii.

Besides the Hopf field, there is another field which plays an important role in that problem, namely the radial field R (see the end of section 3 for a description of it). Indeed, it is shown in [13] that the volume of any smooth unit vector field is greater than or equal to the volume of R and that equality uniquely holds for this vector field. Nevertheless, since it has two singularities this vector field is not the minimum of the volume functional among globally defined smooth unit vector fields. We conclude this article with an appendix in which we consider a family of globally defined unit vector fields (R_ϵ) which is built from the radial field. We explicitly compute its volume and compare it with the volume of the Hopf field H and the volume of the field P obtained by Pedersen in [77]. Lower values of the volume are reached by R_ϵ for small k, P for medium k and H for large k.

Acknowledgments. – This work was done in part during the first author's visit to the University of Valencia and the second author's visit to the University of Lyon. Both authors are grateful to each other and their institutions for promoting these visits. The second author was partially supported by DGI Grant No. BFM2001-3548.

6.2 Hessian of the volume functional

In this section, we state three different expressions of the Hessian, each of them giving interesting insight into the stability problem. Nevertheless, only the two last are needed to prove the Stability Theorem.

Let $\mathbb{S}^{2m+1}(r) \subset \mathbb{C}^{m+1}$ be the sphere of curvature $k = r^{-2}$, $N(p) = \sqrt{k}p$ the outwards unit normal at p and J the usual complex structure on $\mathbb{C}^{m+1}$, so the Hopf vector field can then be written in the form $H = JN$.

We denote by $\bar{\nabla}$ the Levi-Civita connection on $\mathbb{R}^{2m+2}$, the Levi-Civita connection ∇ on $\mathbb{S}^{2m+1}(r)$ is $\nabla_X Y = \bar{\nabla}_X Y - < \bar{\nabla}_X Y, N > N$ and $\bar{\nabla}_X H = J\bar{\nabla}_X N = \frac{1}{r}JX$. Therefore $\nabla_H H = 0$ and if $< X, H >= 0$ then $\nabla_X H = \frac{1}{r}JX$.

n° Let $W : \mathcal{U} \subset \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be a vector field, we put $D^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W - J\bar{\nabla}_X W$ and $\bar{D}^{\mathbb{C}}_X W = \bar{\nabla}_{JX} W + J\bar{\nabla}_X W$. Recall that W is *holomorphic* (resp.*anti-holomorphic*) if for all X, $D^{\mathbb{C}}_X W = 0$ (resp. $\bar{D}^{\mathbb{C}}_X W = 0$).

Let $H^\perp$ be the distribution $Span(x, Jx)^\perp$ on $\mathbb{C}^{m+1} \setminus \{0\}$ and $\pi : T(\mathbb{C}^{m+1} \setminus \{0\}) \to H^\perp$ be the natural projections $\{x\} \times \mathbb{C}^{m+1} \longrightarrow H^\perp_x$. We denote by $\|\pi \circ D^{\mathbb{C}} W\|_{H^\perp}$ the norm of $\pi \circ D^{\mathbb{C}} W_{|H^\perp} : H^\perp \to H^\perp$ that is :

$$\|\pi \circ D^{\mathbb{C}} W\|^2_{H^\perp} = \sum_{i=1}^{2m} \|\pi \circ D^{\mathbb{C}}_{E_i} W\|^2$$

where $E_1, ..., E_{2m}$ is a local orthonormal frame of $H^\perp$. Similarly :

$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{H^\perp} = \sum_{i=1}^{2m} \|\pi \circ \bar{D}^{\mathbb{C}}_{E_i} W\|^2,$$

but in that case

$$\pi \circ \bar{D}^{\mathbb{C}} W_{|H^\perp} = \bar{D}^{\mathbb{C}} W_{|H^\perp} : H^\perp \to H^\perp$$

so that :

$$\|\pi \circ \bar{D}^{\mathbb{C}} W\|^2_{H^\perp} = \|\bar{D}^{\mathbb{C}} W\|^2_{H^\perp}.$$

Recall that, for every r, the unit Hopf vector field H is critical for the volume functional defined on the space $\Gamma^\infty(T^1\mathbb{S}^{2m+1}(r))$ of smooth unit vector fields of $\mathbb{S}^{2m+1}(r)$ (see [38]). An element of the tangent space to the Fréchet manifold $\Gamma^\infty(T^1\mathbb{S}^{2m+1}(r))$ at the point H is a smooth vector field A everywhere orthogonal to H. In [37] Lemma 10, the Hessian of the volume at H is computed and the following result gives a new expression of it in which the role of radius (equivalently the role of curvature) becomes clearer.

Main proposition. – *Let A be a smooth vector field on $\mathbb{S}^{2m+1}(r)$ such that $< A, H >= 0$, then :*

$$\begin{aligned} i)\ (Hess\,Vol\,)_H(A) = \ & (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)} \Big(-2mk\|A\|^2 + \|\nabla A\|^2 \\ & +k\|\nabla_H A + \sqrt{k}JA\|^2\Big)\, dvol. \end{aligned}$$

$$\begin{aligned} ii)\ (Hess\,Vol\,)_H(A) = \ & (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)} \left(\frac{k(1-m)}{k+1}(3+4k+m)\|A\|^2 \right. \\ & \left. +(k+1)\|\nabla_H A + \sqrt{k}\frac{k+m}{k+1}JA\|^2 + \frac{1}{2}\|\pi\circ D^{\mathbb{C}}A\|^2_{H^\perp}\right) dvol. \end{aligned}$$

$$\begin{aligned} iii)\ (Hess\,Vol\,)_H(A) = \ & (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)} \left(-k\frac{(m+1)^2}{k+1}\|A\|^2 \right. \\ & \left. +(k+1)\|\nabla_H A + \sqrt{k}\frac{k-m}{k+1}JA\|^2 + \frac{1}{2}\|\bar{D}^{\mathbb{C}}A\|^2_{H^\perp}\right) dvol. \end{aligned}$$

Proof. – Let $(M, < .,. >)$ be a Riemannian manifold and let W be a unit Killing vector field which is critical for the volume functional V. Let A be a vector field orthogonal to W, then the Hessian of the volume at W can be expressed using Theorem 7 and Lemma 9 of [37] as :

$$\begin{aligned} (Hess\,Vol\,)_W(A) = & \int_M \Big(\|A\|^2\omega_W(W) + f(W)tr(L_W^{-1}\circ {}^T\nabla A\circ L_W^{-1}\circ\nabla A) \\ & + \frac{2}{f(W)}\sigma_2(K_W\circ\nabla A)\Big) dvol. \end{aligned}$$

where $L_W = Id + {}^T\nabla W\circ\nabla W$, $f(W) = \sqrt{det\ L_W}$, $K_W = f(W)L_W^{-1}\circ{}^T\nabla W$, $\omega_W = C_1^1\nabla K_W$ is the tensor contraction of ∇K_W, and $2\sigma_2(C) = tr(C)^2 - tr(C^2)$. This expression is useful for our purpose since Hopf vector fields can also be characterized as the unit Killing vector fields of the sphere $\mathbb{S}^{2m+1}(r)$. In that case $L_H|_{H^\perp} = (k+1)Id$ and $L_H(H) = H$ (see [37]). So :

$$f(H) = (1+k)^m,\ K_H = (1+k)^{m-1\ T}(\nabla H),\ \omega_H(H) = -2mk(1+k)^{m-1}.$$

Moreover, $K_H H = 0$ and if $< X, H >= 0$ then :

$$< K_H(X), Y >= (1+k)^{m-1} < X, \nabla_Y H >= -\sqrt{k}(1+k)^{m-1} < JX, Y > .$$

So, $K_H(X) = -\sqrt{k}(1+k)^{m-1}J(X- < X, H > H)$ and therefore :

$$K_H(\nabla_X A) = -\sqrt{k}(1+k)^{m-1}J(\nabla_X A- < \nabla_X A, H > H).$$

Since $\nabla_X JA = J\nabla_X A - \sqrt{k} < X, A > H + \sqrt{k} < JA, X > N$, then :

$$(K_H \circ \nabla A)(X) = -\sqrt{k}(1+k)^{m-1}((\nabla JA)(X) + \sqrt{k} < X, A > H).$$

To compute $\sigma_2(K_H \circ \nabla A)$ in terms of $\sigma_2(\nabla JA)$, we choose a local orthonormal frame $(E_1, ..., E_{2m+1})$ such that $E_{2m+1} = H$ and $E_{m+j} = JE_j$, for $1 \leq j \leq m$. So, $(E_1, ..., E_m)$ is an orthonormal J-frame of $H^\perp$. It is easy to see that :

$$2\sigma_2(K_H \circ \nabla A) = k(1+k)^{2m-2}\Big(2\sigma_2(\nabla JA)$$

$$-2\sqrt{k}\sum_{i=1}^{2m} < E_i, A >< \nabla_H JA, E_i > \Big),$$

thus :

$$2\sigma_2(K_H \circ \nabla A) = k(1+k)^{2m-2}(2\sigma_2(\nabla JA) + 2\sqrt{k} < \nabla_H A, JA >).$$

On the other hand :

$$tr(L_H^{-1} \circ^T \nabla A \circ L_H^{-1} \circ \nabla A) = (1+k)^{-2}(\|\nabla A\|^2 + k^2\|JA\|^2 + k\|\nabla_H A\|^2).$$

Consequently, we have :

$$(Hess\ Vol\,)_H(A) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)}\Big(- 2mk(1+k)\|A\|^2 + 2k\sigma_2(\nabla JA)$$

$$+2k\sqrt{k} < \nabla_H A, JA > +\|\nabla A\|^2 + k^2\|JA\|^2 + k\|\nabla_H A\|^2\Big)dvol.$$

Now for any vector field X on a Riemannian manifold M, we have (see, for example, [78] p. 170) :

$$\int_M Ricci(X, X)dvol = 2\int_M \sigma_2(\nabla X)dvol,$$

from where part i follows immediately. To show ii and iii we compute $\|\pi \circ D^{\mathbb{C}}A\|^2_{H^\perp}$ and $\|\bar{D}^{\mathbb{C}}A\|^2_{H^\perp}$ in terms of the matrix B of ∇A in a local frame, i. e. $B_i^j =< \nabla_{E_i} A, E_j >$, obtaining :

$$\begin{aligned}
\frac{1}{2}\|\pi \circ D^{\mathbb{C}}A\|^2_{H^\perp} &= \sum_{i,j=1}^{m}(B_{i+m}^{j+m} - B_i^j)^2 + (B_{i+m}^j + B_i^{j+m})^2 \\
&= \sum_{i,j=1}^{2m}(B_i^j)^2 + 2\sum_{i,j=1}^{m}(B_{i+m}^j B_i^{j+m} - B_{i+m}^{j+m} B_i^j),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\|\bar{D}^{\mathbb{C}}A\|^2_{H^\perp} &= \sum_{i,j=1}^{m}(B_{i+m}^{j+m} + B_i^j)^2 + (B_{i+m}^j - B_i^{j+m})^2 \\
&= \sum_{i,j=1}^{2m}(B_i^j)^2 - 2\sum_{i,j=1}^{m}(B_{i+m}^j B_i^{j+m} - B_{i+m}^{j+m} B_i^j).
\end{aligned}$$

n° To go further, we need the following lemma which relates H with vectors on the orthogonal distribution.

Lemma 1. – *We have :*

$$a) \quad 2m\sqrt{k}H = -\sum_{i=1}^{m}[E_i, JE_i] + \sum_{i=1}^{m} div(JE_i)E_i - \sum_{i=1}^{m} div(E_i)JE_i.$$

$$b) \quad 2m\sqrt{k}\int_{\mathbb{S}^{2m+1}(r)} < \nabla_H A, JA > \;=\; -2k\int_{\mathbb{S}^{2m+1}(r)} \|A\|^2 + 2\int_{\mathbb{S}^{2m+1}(r)} \sum_{i=1}^{m} < J\nabla_{E_i}A, \nabla_{JE_i}A > .$$

Proof. – A simple computation shows that :

$$\sum_{i=1}^{m}[E_i, JE_i] = \sum_{i,j}^{2m+1} (< E_i, \nabla_{E_i}JE_j > E_j + < JE_i, \nabla_{JE_i}JE_j > E_j)$$

$$-\sum_{i,j}^{2m+1} (< E_i, \nabla_{E_i}E_j > JE_j + < JE_i, \nabla_{JE_i}E_j > JE_j) - 2m\sqrt{k}H$$

hence a) and then :

$$2m\sqrt{k} <\nabla_H A, JA>= \sum_{i=1}^{m}\Big(- <\nabla_{[E_i,JE_i]}A, JA> + div(JE_i) <\nabla_{E_i}A, JA>\Big)$$

$$-\sum_{i=1}^{m} div(E_i) < \nabla_{JE_i}A, JA > .$$

On the other hand :

$$k\|A\|^2 = \sum_{i=1}^{m} R(E_i, JE_i, A, JA) = \sum_{i=1}^{m}(JE_i)(< \nabla_{E_i}A, JA >)$$

$$-\sum_{i=1}^{m} E_i(< \nabla_{JE_i}A, JA >) + \sum_{i=1}^{m} < \nabla_{[E_i,JE_i]}A, JA >$$

$$-\sum_{i=1}^{m} < \nabla_{E_i}A, \nabla_{JE_i}JA > + \sum_{i=1}^{m} < \nabla_{JE_i}A, \nabla_{E_i}JA > .$$

By integrating and using Stokes Theorem and the identity $div(fX) = fdiv(X) + X(f)$, we get :

$$2m\sqrt{k}\int_{\mathbb{S}^{2m+1}(r)} < \nabla_H A, JA >=$$

$$\int_{\mathbb{S}^{2m+1}(r)} \Big(- k\|A\|^2 - \sum_{i=1}^{m} < \nabla_{E_i}A, \nabla_{JE_i}JA > + \sum_{i=1}^{m} < \nabla_{JE_i}A, \nabla_{E_i}JA > \Big)$$

hence the result. □

Using the lemma :

$$\int_{\mathbb{S}^{2m+1}(r)} \frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp} dvol =$$

$$\int_{\mathbb{S}^{2m+1}(r)} \Big(\|\nabla A\|^2 - 3k\|A\|^2 - \|\nabla_H A\|^2 - 2m\sqrt{k} < \nabla_H A, JA > \Big) dvol,$$

and

$$\int_{\mathbb{S}^{2m+1}(r)} \frac{1}{2}\|\bar{D}^{\mathbb{C}} A\|^2_{H^\perp} dvol =$$

$$\int_{\mathbb{S}^{2m+1}(r)} \Big(\|\nabla A\|^2 + k\|A\|^2 - \|\nabla_H A\|^2 + 2m\sqrt{k} < \nabla_H A, JA > \Big) dvol.$$

From here :

$$(Hess\ Vol)_H(A) = \int_{\mathbb{S}^{2m+1}(r)} \Big((1+k)^{m-2}\frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp}$$

$$+(-2m+3+k)k\|A\|^2 + (k+1)\|\nabla_H A\|^2 + 2(k+m)\sqrt{k} < \nabla_H A, JA > \Big) dvol,$$

and

$$(Hess\ Vol)_H(A) = \int_{\mathbb{S}^{2m+1}(r)} \Big((1+k)^{m-2}\frac{1}{2}\|D^{\mathbb{C}} A\|^2_{H^\perp}$$

$$+(-2m-1+k)k\|A\|^2 + (k+1)\|\nabla_H A\|^2 + 2(k-m)\sqrt{k} < \nabla_H A, JA > \Big) dvol.$$

That the Hessian admits expressions *ii* and *iii* is now an easy computation. □

Remarks. − We will use *ii* and *iii* to show the Stability Theorem. Expression *i)* relates the Hessian of the volume with the Hessian of the energy functional that is the functional

$$V \overset{E}{\longmapsto} \frac{1}{2}\int_{\mathbb{S}^{2m+1}(r)} tr(Id + {}^T\nabla V \circ \nabla V) dvol.$$

Just as for the volume, Hopf vector fields are critical for the energy and from Lemma 10 of [37] we have

$$(Hess\ E)_H(A) = \int_{\mathbb{S}^{2m+1}(r)} -2mk\|A\|^2 + \|\nabla A\|^2.$$

This gives us some idea why the curvature could have an influence on the signature of $(Hess\ Vol)_H$; the term $-2mk\|A\|^2 + \|\nabla A\|^2$ is of order k and $k\|\nabla_H A + \sqrt{k}JA\|^2$ is of order k^2.

Putting $m = 1$ in the expression *ii* gives :

$$(Hess\ Vol)_H(A) = \frac{1}{1+k}\int_{\mathbb{S}^3(r)} \Big((k+1)\|\nabla_H A + \sqrt{k}JA\|^2 + \frac{1}{2}\|\pi \circ D^{\mathbb{C}} A\|^2_{H^\perp}\Big) dvol$$

so we immediately get that the Hopf vector field on $\mathbb{S}^3(r)$ is stable for any radius.

6.3 Proof of the Stability Theorem

The Stability Theorem will follow from the main proposition of the preceding section and the two lemmas below.

Let $A : \mathbb{S}^{2m+1}(r) \to H^\perp \subset \mathbb{C}^{m+1}$ be a smooth vector field orthogonal to H, that is $A(p) \in Span(p, Jp)^\perp$ for every $p \in \mathbb{S}^{2m+1}(r)$. We set :

$$A_l(p) = \frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}p)e^{-il\theta}d\theta \in H_p^\perp$$

so that the Fourier serie of this smooth map A is :

$$A(p) = \sum_{l\in\mathbb{Z}} A_l(p).$$

Since $A_l(e^{i\theta}p) = e^{il\theta}A_l(p)$, we have :

$$\nabla_H A = \bar{\nabla}_H A = \sum_{l\in\mathbb{Z}} i\sqrt{k}lA_l$$

and, if $\mathcal{C}(p)$ denotes the fiber of the Hopf fibration $\mathbb{S}^{2m+1} \to \mathbb{C}P^m$ passing through p, :

$$\int_{\mathcal{C}(p)} < A_l, A_q >= 0,$$

if $l \neq q$. We denote the symmetric bilinear form associated to $(Hess\,Vol\,)_H$ by B_{Vol}, that is : $B_{Vol}(A, A) = (Hess\,Vol\,)_H(A)$.

Lemma 2. – *If $l \neq q$ then $B_{Vol}(A_l, A_q) = 0$, thus :*

$$(Hess\,Vol\,)_H(A) = \sum_{l\in\mathbb{Z}}(Hess\,Vol\,)_H(A_l).$$

Proof. – We write :

$$\begin{aligned} B_{Vol}(A_l, A_q) = \quad & (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)}\Big(-k\frac{(m+1)^2}{k+1} < A_l, A_q > \\ & +(k+1) < \nabla_H A_l + \sqrt{k}\frac{k-m}{k+1}JA_l, \nabla_H A_q + \sqrt{k}\frac{k-m}{k+1}JA_q > \\ & +\frac{1}{2}\sum_{j=1}^{2m} < \bar{D}^{\mathbb{C}}_{E_j}A_l, \bar{D}^{\mathbb{C}}_{E_j}A_q > \Big)dvol. \end{aligned}$$

Let $l \neq q$, we have :

$$\int_{\mathcal{C}(p)} -k\frac{(m+1)^2}{k+1} < A_l, A_q >= 0$$

and

$$\int_{\mathcal{C}(p)} (k+1) < \nabla_H A_l + \sqrt{k}\frac{k-m}{k+1}JA_l, \nabla_H A_q + \sqrt{k}\frac{k-m}{k+1}JA_q >= 0.$$

Let $(E_1, ..., E_m)$ be an orthonormal J-frame of $H^\perp$ along $\mathcal{C}(p)$ satisfying :

$$\forall\, 1 \leq j \leq m, \qquad E_j(e^{i\theta}p) = e^{i\theta}E_j(p),$$

then :

$$(\bar{D}^{\mathbb{C}}_{E_j}A)_l = \bar{D}^{\mathbb{C}}_{E_j}A_l.$$

Indeed, let $X \in H^\perp$ such that $X(e^{i\theta}p) = e^{i\theta}X(p)$, and γ_{p,X_p} be a path on $\mathbb{S}^{2m+1}(r)$ such that $\gamma(0) = p$ and $\gamma'(0) = X_p$, we have :

$$\begin{aligned}
(\bar{\nabla}_{X_p}A_l)(p) &= \frac{d}{dt}\left(A_l \circ \gamma_{p,X_p}(t)\right)_{|t=0} \\
&= \frac{d}{dt}\left(\frac{1}{2\pi}\int_0^{2\pi} A(e^{i\theta}\gamma_{p,X_p}(t))e^{-il\theta}d\theta\right)_{|t=0} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{dt}\left(A \circ \gamma_{e^{i\theta}p,e^{i\theta}X_p}(t)\right)_{|t=0} \\
&= \frac{1}{2\pi}\int_0^{2\pi}(\bar{\nabla}_{e^{i\theta}X_p}A)(e^{i\theta}p)e^{-il\theta}d\theta.
\end{aligned}$$

On the other hand :

$$(\bar{\nabla}_X A)_l(p) = \frac{1}{2\pi}\int_0^{2\pi}(\bar{\nabla}_{X_{e^{i\theta}p}}A)(e^{i\theta}p)e^{-il\theta}d\theta.$$

Thus, if $l \neq q$:

$$\int_{\mathcal{C}(p)} < D^{\mathbb{C}}_{E_j}A_l, D^{\mathbb{C}}_{E_j}A_q >= \int_{\mathcal{C}(p)} < (\bar{D}^{\mathbb{C}}_{E_j}A)_l, (\bar{D}^{\mathbb{C}}_{E_j}A)_q >= 0$$

□

Lemma 3. − *If $k \geq 2m-3$ then for all $l \in \mathbb{Z}$, $(Hess\,Vol\,)_H(A_l) \geq 0$.*

Proof. − Note that :

$$\int_{\mathcal{C}(p)} \|\nabla_H A_l + \alpha\sqrt{k}JA_l\|^2 = (l+\alpha)^2 k \int_{\mathcal{C}(p)} \|A_l\|^2.$$

Thus, from expression *ii* of the main proposition, we have :

$$(Hess\,Vol\,)_H(A_l) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)}\left(\frac{1}{2}\|\pi \circ D^{\mathbb{C}}A_l\|^2_{H^\perp}\right.$$

$$\left.(3 - 2m + k + l((k+1)l + 2(k+m)))k\|A_l\|^2\right)dvol$$

Since $k \geq 2m-3$, $(Hess\,Vol\,)_H(A_l) < 0$ implies $-3 \leq \frac{-2(k+m)}{k+1} < l < 0$, hence $l = -1$ or -2. But, from expression *iii* of the main proposition :

$$(Hess\,Vol\,)_H(A_{-1}) = (1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)}\frac{1}{2}\|\bar{D}^{\mathbb{C}}A_{-1}\|^2_{H^\perp}dvol \geq 0$$

and

$$(Hess\,Vol\,)_H(A_{-2}) =$$
$$(1+k)^{m-2}\int_{\mathbb{S}^{2m+1}(r)}\Big((k+2m+3)k\|A_{-2}\|^2+\frac{1}{2}\|\bar{D}^{\mathbb{C}}A_{-2}\|^2_{H^\perp}\Big)dvol \geq 0$$

□

Lemmas 2 and 3 prove the Stability Theorem.

6.4 Asymptotic behaviour of the volume

In this section we prove the Twisting Theorem and then compare the asymptotic behaviour of different vector fields, namely the Hopf vector field, the perturbed parallel field (Pedersen construction) and the perturbed radial field.

Let V be a unit vector field on $\mathbb{S}^{2m+1}(1)$, the volume of the corresponding unit vector field V^k on the sphere $\mathbb{S}^{2m+1}(r)$ with radius $r = k^{-\frac{1}{2}}$ is given by :

$$\begin{aligned} Vol(V^k) &= \int_{\mathbb{S}^{2m+1}(1)}\sqrt{det(\frac{1}{k}Id+M)dvol} \\ &= \int_{\mathbb{S}^{2m+1}(1)}\sqrt{\frac{1}{k^n}+\frac{1}{k^{n-1}}\sigma_1(M)+...+\frac{1}{k}\sigma_{2m}(M)+\sigma_{2m+1}(M)dvol} \end{aligned}$$

where M denotes ${}^T\nabla V\circ\nabla V$ and the σ_j's are the elementary symmetric polynomial functions. It is well-known that, since $\|V\|=1$, $\sigma_{2m+1}(M)$ is zero. Hence when k is large, the behaviour of the volume is ruled by :

$$\int_{\mathbb{S}^{2m+1}(1)}\sqrt{\sigma_{2m}({}^T\nabla V\circ\nabla V)}dvol.$$

that is the *twisting of* V. In particular, if W is another unit vector field such that $Tw(W) > Tw(V)$ then we have $k_0 > 0$ such that for every $k > k_0$ the volume of W^k is strictly greater than that of V^k. This states part 2) of the Twisting Theorem.

Proof of the Twisting Theorem, part 1.– Since

$${}^T\nabla H\circ\nabla H = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix},$$

then $Tw(H) = vol(\mathbb{S}^{2m+1}(1))$. It remains to show that if V is any unit vector field then $Tw(V) \geq vol(\mathbb{S}^{2m+1}(1))$. Note that :

$$\begin{aligned} \sigma_{2m}({}^T\nabla V\circ\nabla V) &= \sum_{i_1<...<i_{n-1}}\|\nabla_{e_{i_1}}V\wedge...\wedge\nabla_{e_{i_{n-1}}}V\|^2 \\ &\geq \|\nabla_{e_1}V\wedge...\wedge\nabla_{e_{n-1}}V\|^2. \end{aligned}$$

So, if $S = \nabla V_{|V^\perp}$, we obtain :

$$\sigma_{2m}(\,^T\nabla V \circ \nabla V) \geq \sigma_{2m}(\,^T S.S) = det^2(S).$$

Thus :

$$\sqrt{\sigma_{2m}(\,^T\nabla V \circ \nabla V)} \geq |det(S)| \geq det(S) = \sigma_{2m}(S).$$

According to [15],

$$\int_{\mathbb{S}^{2m+1}(1)} \sigma_{2m}(S)\, dvol = vol(\mathbb{S}^{2m+1}(1)),$$

henceforth :

$$Tw(V) \geq vol(\mathbb{S}^{2m+1}(1)).$$

Since $Tw(H) = vol(\mathbb{S}^{2m+1}(1))$, the Hopf field actually achieves the minimum. □

The Pedersen construction. – In [77], we can see the construction of smooth vector fields on $\mathbb{S}^{2m+1}(1)$ with less volume than the Hopf one. They are obtained from parallel vector fields, that is a parallel translation along the geodesics of a given vector at a point p. These parallel vector fields have a singularity of index 0 at $-p$ but a C^∞-pertubation on a small neighbourhood of $-p$ gives smooth vector fields on the whole sphere. The volume of these vector fields approaches the volume of the parallel vector field and similarly so for the twisting.

Lemma 4. – *Let P be a unit vector field obtained by the parallel transport of any given unit vector. Then*

$$Tw(P) = \frac{1}{2^{2m}} \frac{C_{4m}^{2m}}{C_{2m-1}^{m}}\, vol(\mathbb{S}^{2m+1}(1)).$$

In particular $Tw(P) > Tw(H)$, thus the Pedersen construction cannot yield to the minimum of the volume for large k.

Proof. – n° Let $\mathbb{S}^n(1) \subset \mathbb{R}^{n+1}$ with $n = 2m+1$, and $S = (0, ..., 0, -1)$ be the south pole. The parallel vector field P on $\mathbb{S}^n(1) \setminus \{S\}$ obtained by the parallel transport of $\partial_n = (0, ..., 0, 1, 0)$ along geodesics has the following expression :

$$P(x) = x_n(h(x)\overline{x} - \partial_{n+1}) + \partial_n$$

where $\overline{x} = \sum_{j=1}^n x_j \partial_j$ and $h(x) = -(1 + x_{n+1})^{-1}$. Let $E_i = \sum_{l=1}^n (\delta_{li} + h x_l x_i)\partial_l - x_i \partial_{n+1}$. According to [38], the $n \times n$ matrix of ∇P in the basis (E_i) is given by

$$\nabla P = h \begin{pmatrix} x_n Id & ^T a \\ 0 & 0 \end{pmatrix}$$

with $a = (-x_1, ..., -x_{n-1})$. Thus

$$\frac{1}{k} Id + \,^T\nabla P \circ \nabla P = \begin{pmatrix} (h^2 x_n^2 + \frac{1}{k})Id & x_n h^2\, ^T a \\ x_n h^2 a & h^2 a.\, ^T a + \frac{1}{k} \end{pmatrix},$$

and a simple computation shows that :

$$\begin{aligned} det(\tfrac{1}{k}Id + {}^T\nabla P \circ \nabla P) &= \tfrac{1}{k}(h^2x_n^2 + \tfrac{1}{k})^{n-2}(h^2x_n^2 + h^2\|a\|^2 + \tfrac{1}{k}) \\ &= \tfrac{1}{k}(h^2x_n^2 + \tfrac{1}{k})^{n-2}(-2h - 1 + \tfrac{1}{k}). \end{aligned}$$

Since $\sigma_{2m}({}^T\nabla P \circ \nabla P)$ is the $\frac{1}{k}$ term of the expansion of the above determinant, we obtain :

$$\sigma_{2m}({}^T\nabla P \circ \nabla P) = h^{2n-4}x_n^{2n-4}(-2h-1).$$

Note that $h^{2n-4}(-2h-1) = (1-x_{n+1})(1+x_{n+1})^{3-2n}$, so

$$\sqrt{\sigma_{2k}({}^T\nabla P \circ \nabla P)} = (1-x_{n+1})^{1/2}(1+x_{n+1})^{3/2-n}|x_n|^{n-2}.$$

Let $x_{n+1} = \cos t$, $x_n = \sin t \cos\alpha$, $0 < t < \pi$, $0 < \alpha < \pi$, we then have

$$Tw(P) = I \cdot vol(\mathbb{S}^{n-2}(1))$$

with

$$I = \int_0^\pi\int_0^\pi (1-\cos t)^{\frac{1}{2}}(1+\cos t)^{\frac{3}{2}-n}\sin^{n-2}t|\cos\alpha|^{n-2}\sin^{n-1}t\sin^{n-2}\alpha dt d\alpha.$$

Let

$$\begin{aligned} I_1 &= \int_0^\pi (1-\cos t)^{\frac{1}{2}}(1+\cos t)^{\frac{3}{2}-n}(1-\cos^2 t)^{n-3/2}dt \\ &= \int_0^\pi (1-\cos t)^{n-1}dt = \int_0^\pi 2^{n-1}\sin^{2n-2}\frac{t}{2}\,dt \\ &= 2^n\int_0^{\frac{\pi}{2}}\sin^{2n-2}t\,dt = C_{4m}^{2m}\frac{\pi}{2^{2m}}, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^\pi |\cos\alpha|^{n-2}\sin^{n-2}\alpha\,d\alpha = 2\int_0^{\frac{\pi}{2}}\cos^{n-2}\alpha\sin^{n-2}\alpha\,d\alpha \\ &= \frac{1}{2^{n-3}}\int_0^{\frac{\pi}{2}}\sin^{n-2}2\alpha\,d\alpha = \frac{1}{2^{n-2}}\int_0^\pi \sin^{n-2}\beta\,d\beta \\ &= \frac{1}{2^{n-3}}\int_0^{\frac{\pi}{2}}\sin^{n-2}\beta\,d\beta = \frac{1}{(2m-1)C_{2m-2}^{m-1}}. \end{aligned}$$

Since

$$\begin{aligned} Tw(P) &= I_1I_2vol(\mathbb{S}^{n-2}(1)) \\ &= C_{4m}^{2m}\frac{\pi}{2^{2m}}\frac{1}{(2m-1)C_{2m-2}^{m-1}}\frac{m}{\pi}vol(\mathbb{S}^{2m+1}(1)), \end{aligned}$$

we get the result.

We know from the proof of the Twisting Theorem part 1 that $Tw(H) = vol(\mathbb{S}^{2m+1}(1))$. From this, it is easily checked that $Tw(P) > Tw(H)$. $\square$

The perturbed radial field. – Analogously, the behaviour of the volume for k small is ruled by the bending, that is :

$$B(V) = \int_{\mathbb{S}^{2m+1}(1)}\sigma_1(M)dvol = \int_{\mathbb{S}^{2m+1}(1)}\|\nabla V\|^2 dvol.$$

In [13] it is shown that the volume of any smooth unit vector field is greater than or equal to the volume of the radial field R and equality uniquely holds for this vector field. This field is the one given by the unit tangent vectors to the radial geodesics issuing from a given point p. It has two singularities at points p and $-p$ of opposite index ± 1. Due to the non-vanishing of these indices, it is impossible to obtain a smooth unit vector field from the field by a C^∞-perturbation on small neighbourhoods of p and $-p$. This perturbation has to be realised on a small tubular neighbourhood of an arc joining p and $-p$. In [10] a construction following these lines is performed to obtain a family R_ϵ of smooth unit vector fields. Unfortunately, as ϵ tends to 0, the volume of R_ϵ does not converge to the volume of R (see appendix). Although the twisting of H is equal to the twisting of R, the limit $\lim_{\epsilon\to 0} Tw(R_\epsilon)$ is strictly bigger than the twisting of H and then by the Twisting Theorem, for large k :

$$\lim_{\epsilon\to 0} Vol(R_\epsilon^k) > Vol(H^k).$$

On the contrary, in [10] and [16] it is proven that for every smooth vector field V :

$$\lim_{\epsilon\to 0} B(R_\epsilon) = B(R) < B(V).$$

So for any given V, we have :

$$\lim_{\epsilon\to 0} Vol(R_\epsilon^k) < Vol(V^k)$$

for sufficiently small k. To sum up, we have :

Proposition 5 ("Bending Theorem"). – *Let V be a unit vector field of $\mathbb{S}^{2m+1}(1)$, then :*

1) $B(V) > \lim_{\epsilon\to 0} B(R_\epsilon) = B(R)$ ([10], [16]),

2) There are $\epsilon > 0$ and $k_0 > 0$ such that, for all $k < k_0$, we have : $Vol(V^k) > Vol(R_\epsilon^k)$.

6.5 Appendix

In this appendix we explicitly compute and then compare the volume of the parallel field and the volume of the perturbed radial field.

Proposition 6. – *Let $m \geq 1$ and $r = k^{-\frac{1}{2}}$.*

1) $Vol(H^k) = (1+k)^m vol(\mathbb{S}^{2m+1}(r)$,

2) $Vol(P^k) =$

$$\frac{k(2m+1)4^{2m}}{2\,C_{4m}^{2m}}\,{}_3F_2\left(\frac{1}{2}, \frac{2m+1}{2}, 2m+1, 1, 2m+\frac{1}{2}; 1-k\right) vol(\mathbb{S}^{2m+1}(r)),$$

3) $$\lim_{\epsilon\to 0} Vol(R_\epsilon^k) = \left(\sum_{j=0}^{m} \frac{(C_m^j)^2}{C_{2m}^{2j}} k^j + \frac{4^m}{C_{2m}^m} k^m\right) vol(\mathbb{S}^{2m+1}(r))$$

Remark. – In 2) ${}_3F_2$ denotes the generalized hypergeometric series. This series is convergent if $k < 2$. Since $k \mapsto Vol(P^k)$ is analytic, the formula for $k \geq 2$ is the analytic continuation of the above expression.

Let

$$\mathcal{V}(k) = \min(Vol(H^k), Vol(P^k), \lim_{\epsilon\to 0} Vol(R^k_\epsilon)).$$

The study of these three functions shows that :

Corollary 7. – *For every $m \geq 2$ there is $0 < k_1(m) < k_2(m)$ such that :*

$$\begin{array}{ll} \mathcal{V}(k) = \lim_{\epsilon\to 0} Vol(R_\epsilon) & \text{if } k \leq k_1(m), \\ \mathcal{V}(k) = Vol(P^k) & \text{if } k_1(m) \leq k \leq k_2(m), \\ \mathcal{V}(k) = Vol(H^k) & \text{if } k_2(m) \leq k. \end{array}$$

A numerical computation gives $k_1(2) = 0.269...$, $k_1(3) = 0.494...$, $k_1(4) = 0.618...$, $k_1(10) = 0.848...$, $k_1(100) = 0.985...$ and $k_2(2) = 1.815...$, $k_2(3) = 5.563...$, $k_2(4) = 8.729...$, $k_2(10) = 26.29...$, $k_2(100) = 286.1...$

Recall that

$$\begin{array}{lll} Tw(H) & = & vol(\mathbb{S}^{2m+1}(1)), \\ Tw(P) & = & \dfrac{1}{2^{2m}} \dfrac{C^{2m}_{4m}}{C^m_{2m-1}}\, vol(\mathbb{S}^{2m+1}(1)), \end{array}$$

(see the proof of the Twisting Theorem and Lemma 4). The mere power expansion of the expression 3) in Proposition 6 shows that :

Corollary 8. – $\lim_{\epsilon\to 0} Tw(R_\epsilon) = (1 + \frac{4^m}{C^m_{2m}})\, vol(\mathbb{S}^{2m+1}(1))$.

The next two subsections are devoted to the proof of points 2) and 3) of Proposition 6. Point 1) is well-known and easy.

6.5.1 Volume of the field obtained by parallel transport

The volume of the parallel field P^k on the n-dimensional sphere $\mathbb{S}^n(r)$ of radius $r = k^{-\frac{1}{2}}$ is given by :

$$Vol(P^k) = \int_{\mathbb{S}^n(1)} \sqrt{det\, (\frac{1}{k} Id + {}^T\nabla P \circ \nabla P)}\, dvol.$$

Let $f = \sqrt{det\, (\frac{1}{k} Id + {}^T\nabla P \circ \nabla P)}$, from the proof of Lemma 4 we know that f only depends on x_n and x_{n+1}. Putting $x_{n+1} = \cos t$, $x_n = \sin t \cos\alpha$, $0 < t < \pi$, $0 < \alpha < \pi$, we get

$$g(t,\alpha) = f(\sin t\cos\alpha, \cos t) =$$
$$r(1+\cos t)^{\frac{1-n}{2}}((1-\cos t)(\cos^2\alpha - r^2) + 2r^2)^{\frac{n-2}{2}}((r^2-1)(1+\cos t)+2)^{\frac{1}{2}}.$$

Thus

$$Vol(P^k) = \int_{\mathbb{S}^n(1)} f\, dvol = vol(\mathbb{S}^{n-2}(1))\, I.$$

with

$$I = \int_0^\pi \int_0^\pi g(t,\alpha) \sin^{n-1} t \sin^{n-2}\alpha\, dt d\alpha.$$

Let $u = \frac{1}{2}(1-\cos t)$, we obtain

$$\begin{aligned} I &= \int_0^\pi \int_0^1 r^n 2^{n-1} u^{\frac{n-2}{2}} (1-u)^{-\frac{1}{2}} (1-(1-\frac{\cos^2\alpha}{r^2})u)^{\frac{n-2}{2}} \\ & \qquad (1-(1-\frac{1}{r^2})u)^{\frac{1}{2}} \sin^{n-2}\alpha du d\alpha \\ &= \int_0^\pi F_1(\frac{n}{2}, 1-\frac{n}{2}, -\frac{1}{2}, \frac{n+1}{2}; 1-\frac{\cos^2\alpha}{r^2}, 1-\frac{1}{r^2}) \\ & \qquad r^n 2^{n-1} B(\tfrac{n}{2}, \tfrac{1}{2}) \sin^{n-2}\alpha d\alpha, \end{aligned}$$

where B is the Beta function and F_1 is the first hypergeometric function of two variables. Since

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = (1-y)^{-\alpha} F_1(\alpha, \beta, \gamma-\beta-\beta', \gamma; \frac{y-x}{y-1}, \frac{y}{y-1})$$

we have

$$I = 2^n r^{2n} B(\frac{n}{2}, \frac{1}{2}) \int_0^{\pi/2} F_1(\frac{n}{2}, 1-\frac{n}{2}, n, \frac{n+1}{2}, \sin^2\alpha, 1-r^2) \sin^{n-2}\alpha d\alpha.$$

Let $x = \sin^2\alpha$, we finally obtain

$$I = 2^{n-1} r^{2n} B(\frac{n}{2}, \frac{1}{2})\ J$$

with

$$J = \int_0^1 x^{\frac{n-3}{2}} (1-x)^{-\frac{1}{2}} F_1(\frac{n}{2}, 1-\frac{n}{2}, n, \frac{n+1}{2}, x, 1-r^2) dx.$$

We put $\rho - 1 = \frac{n-3}{2}$, $\sigma - 1 = -\frac{1}{2}$, $\alpha = \frac{n}{2}$, $\beta = 1-\frac{n}{2}$, $\beta' = n$, $\gamma = \frac{n+1}{2}$ and $y = 1-r^2$. As usual $(\alpha)_k$ denotes $\Gamma(\alpha+k)/\Gamma(\alpha)$. Then

$$\begin{aligned} J &= \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} F_1(\alpha, \beta, \beta', \gamma, x, y) dx \\ &= \int_0^1 x^{\rho-1}(1-x)^{\sigma-1} \sum_{k,l} \frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_{k+l} k!\ l!} x^k y^l dx \\ &= \sum_{k,l} B(\rho+k, \sigma) \frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_{k+l} k!\ l!} y^l \\ &= \sum_{k,l} B(\rho, \sigma) \frac{(\rho)_k}{(\rho+\sigma)_k} \frac{(\alpha)_{k+l}(\beta)_k(\beta')_l}{(\gamma)_{k+l} k!\ l!} y^l \\ &= B(\rho, \sigma) \sum_l \frac{(\alpha)_l(\beta')_l}{(\gamma)_l l!} \left(\sum_k \frac{(\alpha+l)_k(\beta)_k(\rho)_k}{(\gamma+l)_k(\rho+\sigma)_k k!} \right) y^l \\ &= B(\rho, \sigma) \sum_l \frac{(\alpha)_l(\beta')_l}{(\gamma)_l l!}\ {}_3F_2(\alpha+l, \beta, \rho, \gamma+l, \rho+\sigma; 1) y^l. \end{aligned}$$

Since

$${}_3F_2(a, b, c, e, f; 1) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(b+s)\Gamma(c+s)}\ {}_3F_2(e-a, f-a, s, s+b, s+c; 1)$$

with $s = e + f - a - b - c$ we deduce

$$_3F_2(\alpha + l, \beta, \rho, \gamma + l, \rho + \sigma; 1) = \frac{(\gamma)_l}{(\alpha)_l} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2})}{\Gamma(n - \frac{1}{2})} \, _3F_2(\frac{1}{2}, -l, \frac{n}{2}, 1, n - \frac{1}{2}; 1)$$

and therefore

$$J = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n - \frac{1}{2})} K$$

where

$$K = \sum_l \frac{(n)_l}{l!} \, _3F_2(\frac{1}{2}, -l, \frac{n}{2}, 1, n - \frac{1}{2}; 1) y^l.$$

We have

$$K = \sum_{k,l} \frac{(n)_l (\frac{1}{2})_k (-l)_k (\frac{n}{2})_k}{(1)_k (n - \frac{1}{2})_k l! \, k!} y^l = \sum_k \frac{(\frac{1}{2})_k (\frac{n}{2})_k}{(1)_k (n - \frac{1}{2})_k k!} \mathcal{L}_k(y)$$

with $\mathcal{L}_k(y) = \sum_l \frac{(n)_l (-l)_k}{l!} y^l$. Since $(-l)_k = (-1)^k l(l-1)...(l-k+1)$ we get

$$\mathcal{L}_k(y) = \sum_{l \geq k} \frac{(n)_l (-l)_k}{l!} y^l = (-1)^k \sum_{l \geq k} \frac{(n)_l}{(l-k)!} y^l.$$

It is straightforward that

$$\mathcal{L}_0(y) = (1 - y)^{-n}$$

and that :

$$\mathcal{L}_k(y) = (-1)^k y^k \mathcal{L}_0^{(k)}(y) = (-1)^k (n)_k y^k (1 - y)^{-(n+k)}.$$

Thus

$$\begin{aligned} K &= \sum_k \frac{(\frac{1}{2})_k (\frac{n}{2})_k (n)_k}{(1)_k (n - \frac{1}{2})_k k!} \left(\frac{y}{y-1}\right)^k \frac{1}{(1-y)^n} \\ &= \frac{1}{(1-y)^n} \, _3F_2(\frac{1}{2}, \frac{n}{2}, n, 1, n - \frac{1}{2}; \frac{y}{y-1}) \\ &= \frac{1}{r^{2n}} \, _3F_2(\frac{1}{2}, \frac{n}{2}, n, 1, n - \frac{1}{2}; \frac{r^2 - 1}{r^2}). \end{aligned}$$

Putting all these successive expressions together, we obtain

$$Vol(P^k) = C_n \cdot \, _3F_2(\frac{1}{2}, \frac{n}{2}, n, 1, n - \frac{1}{2}; \frac{r^2 - 1}{r^2}),$$

with

$$C_n = 2^{n-1} B(\frac{n}{2}, \frac{1}{2}) \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n - \frac{1}{2})} \frac{n-1}{2\pi} vol(\mathbb{S}^n(1)).$$

It is easy to check that

$$C_n = \frac{4^{n-1}}{C_{2n-2}^{n-1}} vol(\mathbb{S}^n(1)),$$

thus

$$Vol(P^k) = k^{\frac{n}{2}} \frac{4^{n-1}}{C_{2n-2}^{n-1}} \, {}_3F_2(\frac{1}{2}, \frac{n}{2}, n, 1, n - \frac{1}{2}; 1-k) vol(\mathbb{S}^n(r)).$$

The preceding computations are valid only if $\frac{1}{\sqrt{2}} < r < \sqrt{2}$. Nevertheless, since $k \mapsto Vol(P^k)$ is analytic and ${}_3F_2(\frac{1}{2}, \frac{n}{2}, n, 1, n - \frac{1}{2}; 1-k)$ is converging if $k < 2$, the above expression is in fact valid as soon as $k < 2$. For $n = 2m+1$, we obtain the result stated in Proposition 6. □

6.5.2 Volume of the perturbed radial fields

The perturbed radial field R_ϵ built in [10] is such that $R_\epsilon = R$ out of an ϵ-neighbourhood G_ϵ of a geodesic segment joining the two singular points p and $-p$ of the radial field. When computing the volume of R_ϵ^k and then passing to the limit, we obtain a result which splits in two parts, one term coming from the underlying radial field R^k and another taking into account (the limit of) the pertubation on G_ϵ.

Lemma 9. – $\lim_{\epsilon \to 0} Vol(R_\epsilon^k) = Vol(R^k) + \pi \frac{1}{\sqrt{k}} vol(\mathbb{S}^{2m}(1))$.

Since

$$Vol(R^k) = (\sum_{j=0}^{m} \frac{(C_m^j)^2}{C_{2m}^{2j}} k^j) \; vol(\mathbb{S}^{2m+1}(r))$$

and

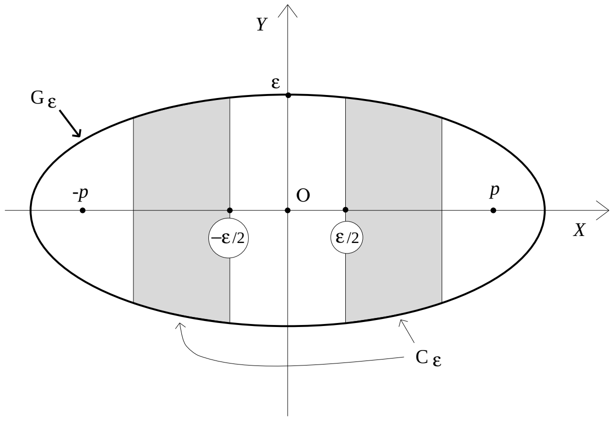
$$\frac{vol(\mathbb{S}^{2m}(1))}{vol(\mathbb{S}^{2m+1}(r))} = \frac{1}{\pi} \frac{4^m}{C_{2m}^m} k^{m+\frac{1}{2}},$$

we obtain the expression 3) of Proposition 6.

Proof. – See [10] for the construction of R_ϵ (in this article, R_ϵ is denoted T_ϵ). Let O be the middle point of the geodesic segment $[-p, p]$. In the image of the stereographic projection from $-O$, let G_ϵ be the full $(2m+1)$-dimensional ellipsoid with focal points $\{-p, p\}$ defined by

$$G_\epsilon = \{q \in \mathbb{R}^{2m+1} \mid d(q,p) + d(q,-p) \leq 2\sqrt{\epsilon^2 + d^2(O,p)}\}.$$

We denote by x the coordinate on the line $D =]-p, p[$ and by $y = (y_1, ..., y_{2m})$ those of $D^\perp$.



Let $A_\epsilon = G_\epsilon \cap \{|x| \leq \frac{\epsilon}{2}\}$, $B_\epsilon^+ = G_\epsilon \cap \{x \geq p - \epsilon/2\}$, $B_\epsilon^- = G_\epsilon \cap \{x \geq -p + \epsilon/2\}$ and $C_\epsilon = G_\epsilon \setminus (A_\epsilon \cup B_\epsilon^+ \cup B_\epsilon^-)$. By dimensional arguments :

$$\lim_{\epsilon \longrightarrow 0} Vol(R_\epsilon^k|_{A_\epsilon}) = 0 \quad \text{and} \quad \lim_{\epsilon \longrightarrow 0} Vol(R_\epsilon^k|_{B_\epsilon^\pm}) = 0.$$

Let $S_\epsilon = R_\epsilon^k(C_\epsilon)$, S_ϵ is a submanifold of $T^1\mathbb{S}^{2m+1}(r)|_{C_\epsilon}$ and obviously

$$\lim_{\epsilon \longrightarrow 0} Vol(R_\epsilon^k) = Vol(R^k) + \lim_{\epsilon \longrightarrow 0} vol(S_\epsilon).$$

Now let $\Sigma = \pi^{-1}(]-p,p[) \subset T^1\mathbb{S}^{2m+1}(r)$ where $\pi : T^1\mathbb{S}^{2m+1}(r) \to \mathbb{S}^{2m+1}(r)$ is the projection. Since π is a Riemannian submersion, when we consider in the first manifold the Sasaki metric, the volume of Σ is πr times the common volume of each fiber, which is equal to $vol(\mathbb{S}^{2m}(1))$.

To conclude, we only need to show that $\lim_{\epsilon \longrightarrow 0} vol(S_\epsilon) = vol(\Sigma)$. Let

$$\begin{array}{rccc} f: & C_\epsilon & \longrightarrow & S_\epsilon \\ & (x,y) & \longmapsto & (x,y,R_\epsilon^r(x,y)), \end{array} \qquad \begin{array}{rccc} h: & C_\epsilon & \longrightarrow & \Sigma \\ & (x,y) & \longmapsto & (x,R_\epsilon^r(x,y)) \end{array}$$

and g_S be the Sasaki metric on $T^1\mathbb{S}^{2m+1}(r)$. Of course $f^*g_S = h^*g_S + d^2y$. Thus if F, H and Y are the corresponding matrices of f^*g_S, h^*g_S and d^2y, we have

$$dS_\epsilon = det(F)^{\frac{1}{2}} dxdy = det(H+Y)^{\frac{1}{2}} dxdy.$$

Since $\|\frac{\partial h}{\partial y_i}\| = O(\frac{1}{\epsilon})$, $\|\frac{\partial h}{\partial x}\| = O(1)$ (see [10]) we deduce $det(H)^{\frac{1}{2}} = O(\frac{1}{\epsilon^{2m}})$ and $det(H+Y)^{\frac{1}{2}} = det(H)^{\frac{1}{2}} + o(\frac{1}{\epsilon^{2m}})$, thus

$$vol(S_\epsilon) = vol(h(C_\epsilon)) + o(1).$$

When ϵ goes to zero, $h(C_\epsilon)$ covers Σ except on a set of zero measure. Therefore, $\lim_{\epsilon \longrightarrow 0} vol(h(C_\epsilon)) = vol(\Sigma)$. $\square$

Bibliographie

[1] P. Ahern et W. Rudin, *Totally real embeddings of* $\mathbb{S}^3$ *in* $\mathbb{C}^3$, Proc. of the Am. Math. Soc. 94 (1985), 460-462.

[2] M. Atiyah, *K-Theory*, W. A. Benjamin Inc., New-York (1967).

[3] M. Audin, *Fibrés normaux d'immersions en dimension double, points doubles d'immersions lagrangiennes et plongements totalement réels*, Comment. Math. Helvetici 63 (1988), 593-623.

[4] H. Baues, *Obstruction theory*, Lecture Notes in Mathematics 628, Springer-Verlag, New-York (1977).

[5] D. Blair et A. Carriazo, *The contact Whitney sphere*, Note Mat. 20 (2000/01), 125-133.

[6] V. Borrelli, B.-Y. Chen et J.-M. Morvan, *Une caractérisation géométrique de la sphère de Whitney*, C. R. Acad. Sc. Paris, Série I, 321 (1995), 1485-1490.

[7] V. Borrelli, *Classe de Maslov des sous-variétés totalement réelles*, C. R. Acad. Sc. Paris, Série I, 323 (1996), 1035-1038.

[8] V. Borrelli, *Maslov form and J-volume of totally real immersions*, J. of Geo. and Phy. 25 (1998), 271-290.

[9] V. Borrelli, *Linking classes of Lagrangian or totally real embeddings*, Ann. Global. Anal. Geom. 17 (1999), 371-384.

[10] V. Borrelli, F. Brito and O. Gil-Medrano, *The infimum of the Energy of unit vector fields on odd-dimensional spheres*, Ann. Global Anal. Geom. 23 (2003) 129-140.

[11] G. E. Bredon, *Topology and Geometry*, GTM 139, Springer, 1993.

[12] F. Brito, *Total Bending of flows with mean curvature correction*, Diff. Geom. and its Appl. 12 (2000), 157-163.

[13] F. Brito, P. Chacón and A. M. Naveira, *On the volume of unit vector fields on spaces of constant sectional curvature*, A paraître dans Comment. Math. Helv.

[14] E. Boeckx, J. C. González-Dávila and L. Vanhecke *Energy of Radial Vector Fields on Compact Rank One Symmetric Spaces* Ann. of Glo. Anal. and Geom. 23 (2003), 29-52.

[15] F. Brito, R. Langevin and H. Rosenberg, *Intégrale de courbure sur des variétés feuilletées*, J. Diff. Geom. 16 (1981), 19-50.

[16] F. Brito and P. Walczak, *On the Energy of Unit vector Fields with Isolated Singularities*, Ann. Math. Polon. 73 (2000), 269-274.

[17] I. Castro et F. Urbano, *Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form*, Tohoku Math. 45 (1993), 565-582.

[18] B.-Y Chen, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, Proc. Third. Pacific Reim. Geom. Conf., Intern. Press, Cambridge, MA, 1996.

[19] B.-Y Chen, *Some new obstructions to minimal and Lagrangian isometric immersions*, Japan. J. Math. 26 (2000), 105-127.

[20] B. Cheng, *Area-minimizing cones and coflat fibrations*, Indiana J. of Math. 3 (1988), 196–213.

[21] B. Cheng, *The special Lagrangian cones over $E(6)/F_4$ and $SU(6)/Sp(3)$*, Yokohama Math. J. 42 (1994), 103–105.

[22] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Mathematics 107, Pitman, Boston, 1985.

[23] I. Castro and F. Urbano, *New examples of minimal Lagrangian tori in the complex projective plane*, Manuscripta Math. 85 (1994), 265–281.

[24] T. E. Cecil and P. J. Ryan, *Tight and Taut immersions of Manifolds*, Research Notes in Mathematics 107, Pitman, Boston, 1985.

[25] J.-P. Dax, *Etude homotopique des espaces de plongements*, Ann. Scien. Ec. Norm. Sup. 5 (1972), 303-377.

[26] Y. Eliashberg *On symplectic manifolds with some contact properties*, J. Diff. Geom. 33 (1991), 233-238.

[27] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, GSM vol 48, AMS 2000.

[28] Y. Eliashberg and L. Polterovich, *New applications of Luttinger's surgery*, Comment. Math. Helv. 69 (1994),512-522.

[29] D. Ferus and H. Karcher and H. F. Muenzner, *Cliffordalgebren and neue isoparametrische Hyperflaechen*, Math. Z, 177 (1981), 479–502.

[30] T. Fiedler, *Totally real embeddings of the torus into $\mathbb{C}^2$*, Ann. Global Anal. Geom. 5 (1987), 117-121.

[31] F. Forstnerič, *On totally real embeddings into $\mathbb{C}^n$*, Expo. Math. 4 (1986), 243-255.

[32] F. Forstnerič, *Some totally real embeddings of 3-manifolds*, Manuscripta Math. 55 (1986), 1-7.

[33] T. Goodwillie, J. Klein and M. Weiss, *A Haefliger style description of the embedding calculus tower*, Preprint.

[34] O. Gil-Medrano, *Relationship between volume and energy of vector fields*, Diff. Geom. and its Appl. 12 (2001), 137-152.

[35] O. Gil-Medrano *Volume and Energy of vector fields on spheres. A survey* Differential Geometry, Valencia 2001, 167-178, World Sci. Publishing, River Edge, NJ 2002.

[36] O. Gil-Medrano *Unit vector fields that are critical points of the volume and the energy : characterization and examples* à paraître dans Complex, contact and symmetric spaces : papers in honour of Lieven Vanheche. Progress in Math. Birkhauser.

[37] O. Gil-Medrano and E. Llinares-Fuster, *Second variation of Volume and Energy of vector fields. Stability of Hopf vector fields*, Math. Ann. 320 (2001), 531-545.

[38] O. Gil-Medrano and E. Llinares-Fuster, *Minimal unit vector fields*, Tohoku Math. J. 54 (2002), 71-84.

[39] J. C. Gonzalez-Davila and L. Vanhecke, *Vanhecke Examples of minimal unit vector fields. Special issue in memory of Alfred Gray (1939–1998)*, Ann. Global Anal. Geom. 18 (2000), no. 3-4, 385–404.

[40] H. Gluck and W. Ziller, *On the volume of a unit vector field on the three-sphere*, Comment. Math. Helv. 61 (1986), 177-192.

[41] J. W. Gray, *Some global properties of contact structures,* Ann. of Math. 69 (1959), 421-450.

[42] M. Gromov, *Convex integration of differential relation*, Math. USSR Izvestija 7 (1973), 329-343.

[43] M. Gromov, *Pseudo holomorphic curves in symplectic manifold*, Invent. Math. 82 (1985), 307-347.

[44] M. Gromov, *Partial Differential Relation*, Springer-Verlag, New-York (1986).

[45] M. Gromov, *Convex integration of differential relation*, Math. USSR Izvestija 7 (1973), 329-343.

[46] A. Haefliger, *Proceedings of Liverpool-Singularities Symposium II*, LNM 209, Springer-Verlag (1971), 128-141.

[47] A. Haefliger, *Plongements differentiable dans le domaine stable*, Comm. Math. Helv. 37 (1962), 155-176.

[48] A. Haefliger, *Differentiable embeddings of $\mathbb{S}^n$ in $\mathbb{S}^{n+q}$ for $q > 2$*, Ann. Math. 83 (1966), 402-436.

[49] A. Haefliger and M. Hirsch, *Immersions in the stable range*, Ann. Math. 75 (1962), 231-241.

[50] A. Haefliger and M. Hirsch, *On the existence and classification of differentiable embeddings*, Topology 2 (1963), 129-135.

[51] D.-S. Han and J.-W. Yim, *Unit vector fields on spheres which are harmonic maps*, Math Z. 227 (1998), 83-92.

[52] R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta Math. 148 (1982), 47-157.

[53] M. Haskins, *Special Lagrangian cones*, arXiv :math.DG/0005164, 2000, to appear in Amer. J. Math.

[54] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics 80, Academic Press, New York-London, 1978.

[55] M. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. 93 (1959) 242-276.

[56] W. Y. Hsiang and H. B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. (1971) 5, 1–38.

[57] T. Ishihara, *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. 13 (1979), 23-27.

[58] D. Joyce, *Special Lagrangian 3-folds and integrable systems*, arXiv :math.DG/0101249, 2001.

[59] D. Joyce, *Special Lagrangian submanifolds with conical singularities. V. Survey and applications.*, arXiv :math.DG/0303272, 2003.

[60] D. Joyce, *Lectures on special Lagrangian geometry*, arXiv :math.DG/0111111, 2003.

[61] D. L. Johnson, *Volume of flows*, Proc. of the Amer. Math. Soc. 104 (1988), 923-932.

[62] T. Kawashima, *Some remarks on lagrangian imbeddings*, J. Math. Soc. Japan 33 (1981), 281-294.

[63] A. Korányi and J. A. Wolf, *Realization of Hermitian symmetric spaces as generalized half-planes*, Ann. Math. 81 (1965), 265–288.

[64] L. L. Larmore, *The cohomology of* $(\Lambda^2 X, \Delta X)$, Can. J. Math., 25 (1973), 908-921.

[65] L. L. Larmore and R. D. Rigdon, *Enumerating immersions and embeddings of projective spaces*, Pacific J. Math. 64 (1976), 471-492.

[66] H. B. Lawson, *Minimal varieties in real and complex geometry*, Séminaire de Mathématiques Supérieures, Les Presses de l'Université de Montréal, Montreal, Que., 1974.

[67] J. A. Lees, *On the classification of Lagrange immersions*, Duke Math. J. 43 (1976), 217-224.

[68] D. McDuff, *Symplectic manifolds with contact type boundaries*, Invent. math. 103 (1991), 651-671.

[69] I. McIntosh, *Special Lagrangian cones in* $\mathbb{C}^3$ *and primitive harmonic maps*, arXiv :math.DG/0201157, 2002, to appear in J. London Math. Soc.

[70] J.-M. Morvan, *Classe de Maslov d'une immersion lagrangienne et minimalité*, C. R. A. S. série I, 292 (1981), 633–636.

[71] H. F. Münzner, *Isoparametrische Hyperflächen in Sphären*, I : Math. Ann. 251 (1980), no. 1, 57-71 ; II : Math. Ann. 256 (1981), no. 2, 215–232.

[72] K. Nomizu, *Some results in É. Cartan's theory of isoparametric families of hypersurfaces*, Bull. Amer. Math. Soc. 79 (1973), 1184–1188.

[73] L. Niglio, *Classes caractéristiques lagrangiennes*, Thèse, Université des sciences et techniques du Languedoc, 1987.

[74] Y.-G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. Math. 101 (1990), 501-519.

[75] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres, I*, Tôhoku Math. J. 27 (1975), 515–559.

[76] H. Ozeki and M. Takeuchi, *On some types of isoparametric hypersurfaces in spheres, II*, Tôhoku Math. J. 28 (1976), 7–55.

[77] S. L. Pedersen, *Volumes of vector fields on spheres*, Trans. Amer. Math. Soc. 336 (1993), 69-78.

[78] W. A. Poor, *Differential geometric structures*, Mc Graw-Hill, New-York, 1981.

[79] L. Polterovich, *New invariants of totally real embeddings of the torus and a problem from Hamiltonnian mechanic*, Methods of Qualitative Theory and of Bifucations Theory, Gorki (1988) in Russian.

[80] T. Riviere, *Dense subset of* $H^{1/2}(S^2, S^1)$, Annals of Global Analysis and Geometry 18 (2000), 517-528.

[81] W. Rudin, *Totally real Klein bottles in* $\mathbb{C}^2$, Proc. of the Am. Math. Soc. 82 (1981), 653-654.

[82] E. H. Spanier, *Algebraic Topology*, McGraw Hill, New-York, 1966.

[83] E. L. Stout et W. R. Zame, *Totally real imbeddings and the universal covering spaces of domains of holomorphy : some examples*, Manuscripta Math. 50 (1985), 29-48.

[84] D. Spring, *Convex Integration Theory*, Monographs in Mathematics, Vol. 92, Birkhäuser (1998).

[85] N. Steenrod, *The topology of fibre bundles*, Princeton University Press (1951).

[86] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror Symmetry is T-Duality*, Nuclear Physics B479 (1996), 243-259.

[87] G. Thorbergsson, *A survey on isoparametric hypersurfaces and its generalizations*, Handbook of Differential Geometry, vol. 1, Eds. F. J. E. Dillen and L. C. A. Verstraelen, Elsevier Science, 2000.

[88] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, Differential Geometry in honor of K. Yano, Kinokuniya, Tokyo, 1972, 469–481.

[89] C. Viterbo, *A new obstruction to embedding Lagrangian tori*, Invent. Math. 100 (1990), 301-320.

[90] T. Yasui, *On the map defined by regarding embeddings as immersions*, Hiroshima Math. J. 13 (1983), 457-476.

[91] A. Weinstein, *Lectures on symplectic manifolds*, C. B. M. S. Reg. Conf. Series in Math. n. 29, Amer. Math. Soc., Providence, R. I.,1979.

[92] G. W. Whitehead, *Elements of Homotopy Theory*, Springer, New-York, 1978.

[93] G. Wiegmink, *Total bending of vector fields on Riemannian manifolds*, Math. Ann. 303 (1995), 325-344.

[94] J. A. Wolf, *Spaces of constant curvature*, McGraw-Hill Book Company, New-York, 1967.

[95] C. M. Wood, *On the energy of a unit vector field*, Geometriae Dedicata 64 (1997), 319-330.