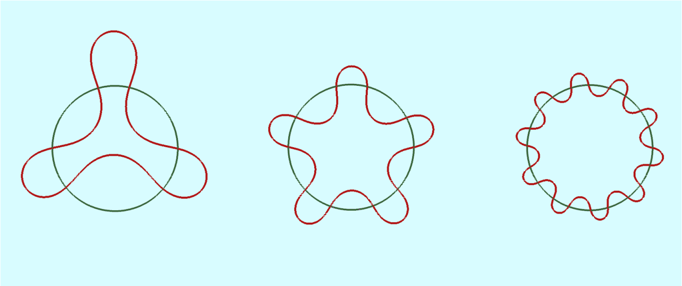
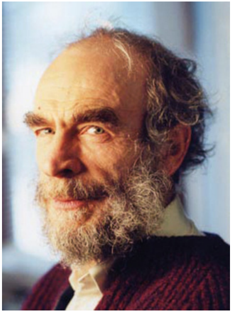
One-Dimensional Convex Integration

Vincent Borrelli

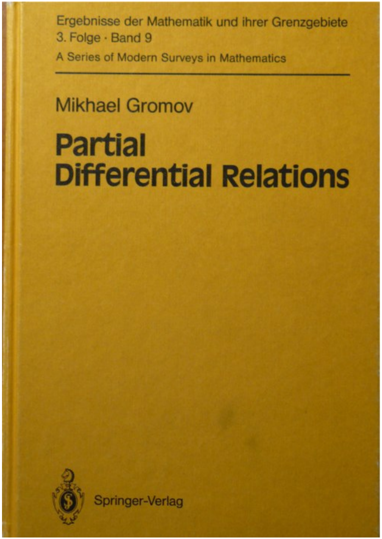
Université Lyon 1 - France



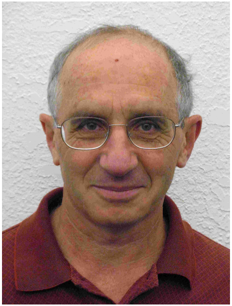
Bibliography



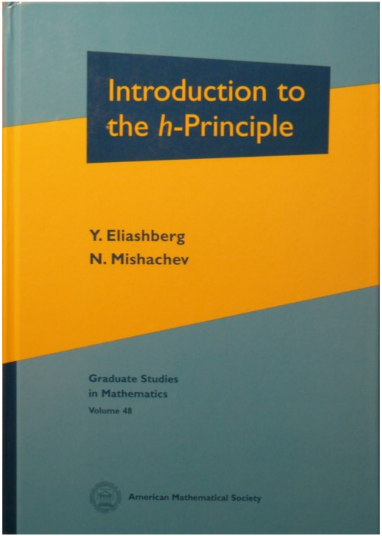
Mikhail Gromov



Bibliography

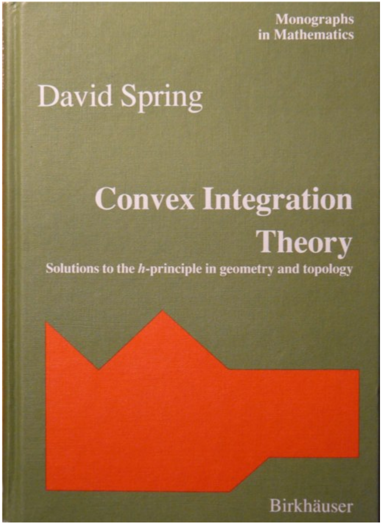


Yakov Eliashberg



Bibliography

David Spring



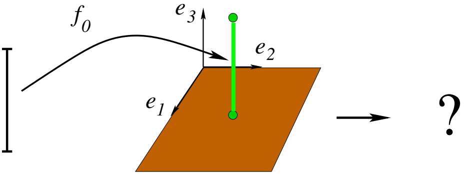
An Elementary Example

Exercice.– Let $f_0 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R}^3 \\ t & \longmapsto & (0,0,t) \end{array}$

Find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

$$\begin{array}{ll} i) & \forall\, t \in [0,1], \quad |\cos(f'(t), e_3)| < \epsilon \\ ii) & \|f - f_0\|_{C^0} < \delta \end{array}$$

where $\epsilon > 0$ and $\delta > 0$ are given.



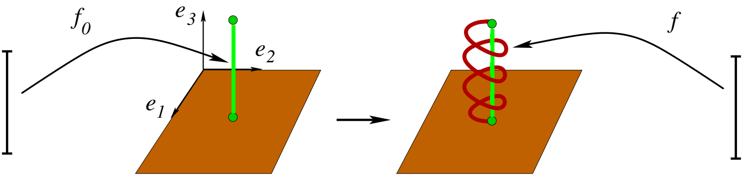
An Elementary Example

Exercice.– Let $\begin{array}{rccc} f_0 : & [0,1] & \longrightarrow & \mathbb{R}^3 \\ & t & \longmapsto & (0,0,t) \end{array}$

Find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

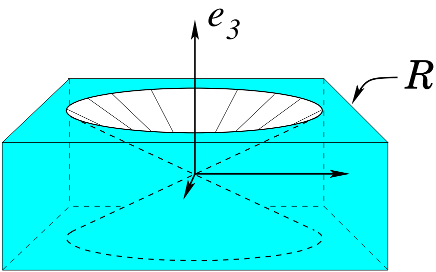
$$\begin{array}{ll} i) & \forall\, t \in [0,1], \quad |\cos(f'(t), e_3)| < \epsilon \\ ii) & \|f - f_0\|_{C^0} < \delta \end{array}$$

where $\epsilon > 0$ and $\delta > 0$ are given.



Rephrasing...

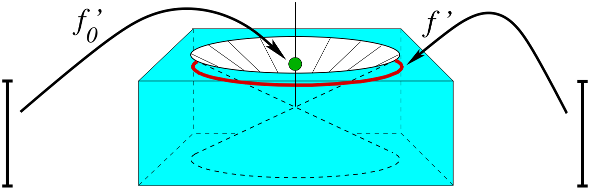
Condition (i) means that the image of f' lies inside the blue cone :



$\mathcal{R}$ is the *differential relation* of our problem.

... and new understanding

The map f must be C^0 near of f_0 :



The average of f' for each ≪ loop ≫ of f is $f_0'(t)$:

$$\frac{1}{Long(I)}\int_I f'(u)du = f_0'(t)$$

(I = the preimage of one loop by f).

A more general problem

Problem.– Let $\mathcal{R} \subset \mathbb{R}^3$ be a path-connected subset (=our differential relation) and $f_0 : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ be a map such

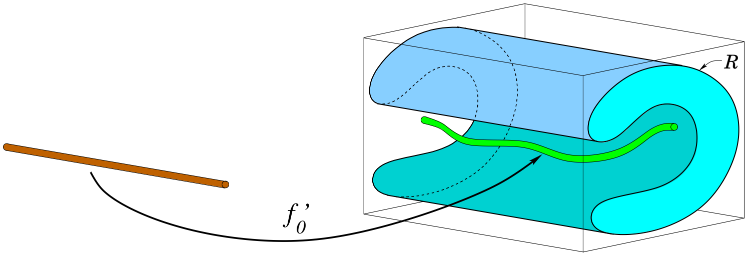
$$\forall t \in [0,1], \quad f_0'(t) \in \ Conv(\mathcal{R}).$$

Find $f : [0,1] \xrightarrow{C^1} \mathbb{R}^3$ such that :

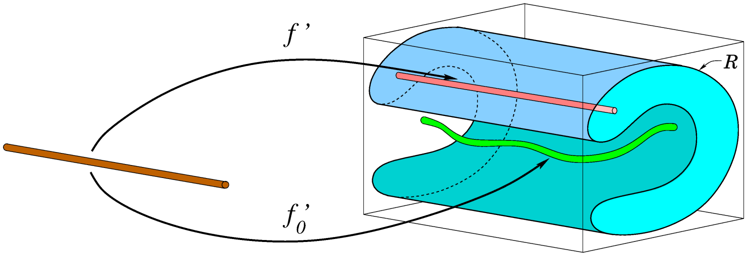
$$\begin{array}{ll} i) & \forall t \in [0,1], \quad f'(t) \in \mathcal{R} \\ ii) & \|f - f_0\|_{C^0} < \delta \end{array}$$

with $\delta > 0$ given.

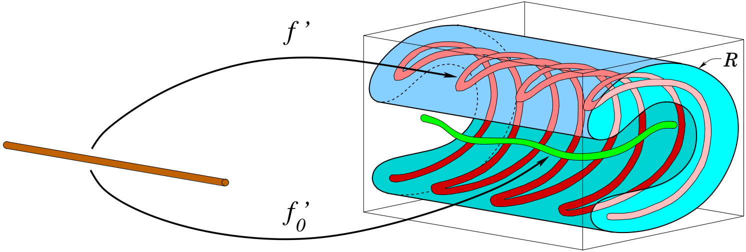
How to build a solution ?



How to build a solution ?



How to build a solution ?



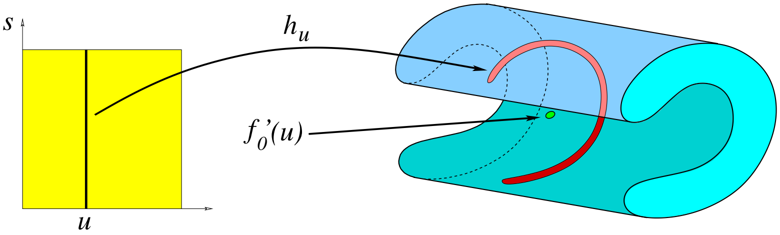
Construction of the solution : step 1

Choose a continuous family of loops

$$\begin{array}{rccc} h: & [0,1] & \longrightarrow & C^0(\mathbb{R}/\mathbb{Z}, \mathcal{R}) \\ & u & \longmapsto & h_u \end{array}$$

such that

$$\forall u \in [0,1], \quad \int_{[0,1]} h_u(s)ds = f_0'(u).$$

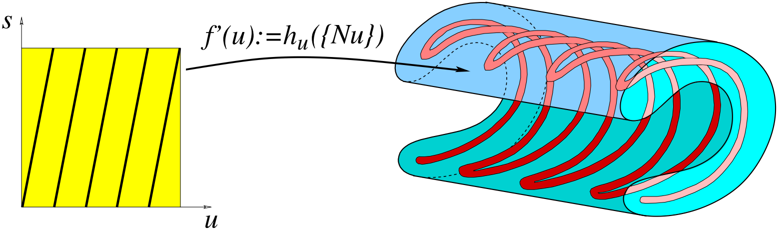


Construction of the solution : step 2

Then set

$$f(t) := f_0(0) + \int_0^t h_u(\{Nu\})du$$

where $N \in \mathbb{N}^*$ and $\{Nu\}$ is the fractional part of Nu.



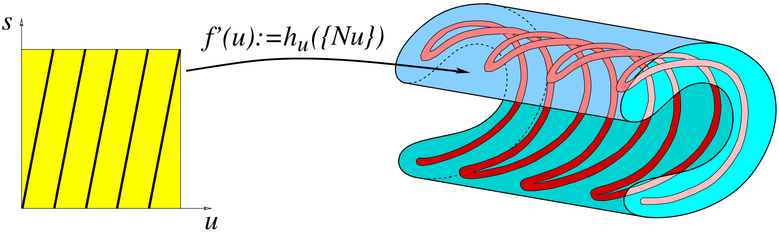
Construction of the solution : step 2

Then set

$$f(t) := f_0(0) + \int_0^t h_u(\{Nu\})du$$

where $N \in \mathbb{N}^*$ and $\{Nu\}$ is the fractional part of Nu.

We say that f is obtained from f_0 by a **convex integration.**



Fundamental Lemma

Notation.– Let $\mathcal{R} \subset \mathbb{R}^n$ and $\sigma \in \mathcal{R}$. We denote by $IntConv(\mathcal{R}, \sigma)$ the interior of the convex hull of the connected component of $\mathcal{R}$ to which σ belongs.

Definition.– A (continuous) loop $g : [0, 1] \to \mathbb{R}^n$, $g(0) = g(1)$, *strictly surrounds* $z \in \mathbb{R}^n$ if

$$IntConv(g([0, 1])) \supset \{z\}.$$

Fundamental Lemma

Fundamental Lemma.– *Let* $\mathcal{R} \subset \mathbb{R}^n$ *be an open set,* $\sigma \in \mathcal{R}$ *and* $z \in IntConv(\mathcal{R}, \sigma)$ *There exists a loop* $h : [0,1] \xrightarrow{C^0} \mathcal{R}$ *with base point* σ *that strictly surrounds* z *and such that :*

$$z = \int_0^1 h(s)ds.$$

Fundamental Lemma

Remark.– *A priori* $h \in \Omega_\sigma(\mathcal{R})$, but it is obvious that we can choose h among "round-trips" *i.e* the space :

$$\Omega_\sigma^{AR}(\mathcal{R}) = \{h \in \Omega_\sigma(\mathcal{R}) \mid \forall s \in [0,1] \;\; h(s) = h(1-s)\}.$$

The point is that the above space is contractible. For every $u \in [0,1]$ we then denote by $h_u : [0,1] \longrightarrow \mathcal{R}$ the map defined by

$$h_u(s) = \begin{cases} h(s) & \text{if} \quad s \in [0, \frac{u}{2}] \cup [1-\frac{u}{2}] \\ h(u) & \text{if} \quad s \in [\frac{u}{2}, 1-\frac{u}{2}]. \end{cases}$$

This homotopy induces a deformation retract of $\Omega_\sigma^{AR}(\mathcal{R})$ to the constant map

$$\begin{array}{rccc} \widetilde{\sigma} : & [0,1] & \longrightarrow & \mathcal{R} \\ & s & \longmapsto & \sigma. \end{array}$$

Parametric Fundamental Lemma

Parametric version of the Fundamental Lemma. – *Let* $E = [a,b] \times \mathbb{R}^n \xrightarrow{\pi} [a,b]$ *be a trivial bundle and* $\mathcal{R} \subset E$ *be an open set. Let* $\sigma \in \Gamma(\mathcal{R})$ *and* $z \in \Gamma(E)$ *such that :*

$$\forall p \in [a,b],\ z(p) \in IntConv(\mathcal{R}_p, \sigma(p))$$

where $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$. *Then, there exists* $h : [a,b] \times [0,1] \xrightarrow{C^\infty} \mathcal{R}$ *such that :*

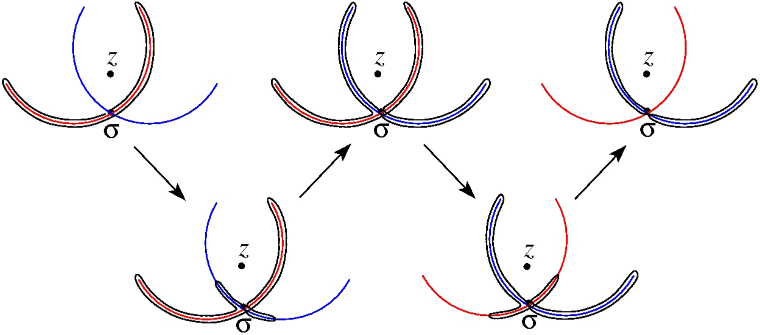
$$h(.,0) = h(.,1) = \sigma \in \Gamma^\infty(\mathcal{R}),$$

$$\forall p \in [a,b],\ h(p,.) \in \Omega^{AR}_{\sigma(p)}(\mathcal{R}_p)$$

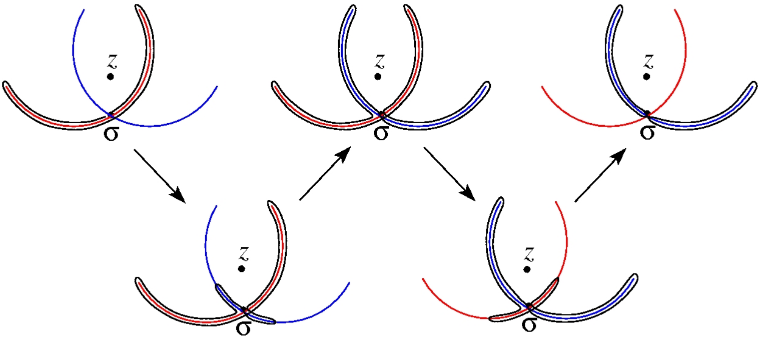
and

$$\forall p \in [a,b],\ z(p) = \int_0^1 h(p,s)ds.$$

Proof



Proof



Observation.– The parametric lemma still holds if the parameter space $[a, b]$ is replaced by a compact manifold P.

The Convex Integration Process

Let $\mathcal{R} \subset \mathbb{R}^n$ be a arc-connected subset, $f_0 \in C^\infty(I, \mathbb{R}^n)$ be a map such that $f_0'(I) \subset \ IntConv(\mathcal{R})$. We set

$$\forall t \in I, \qquad F(t) := f_0(0) + \int_0^t h(s, Ns)ds$$

with $N \in \mathbb{N}^*$.

Definition.– We say that $F \in C^\infty(I, \mathbb{R}^n)$ is a solution of $\mathcal{R}$ obtained from f_0 by a *convex integration process*.

Obviously $F'(t) = h(t, Nt) \in \mathcal{R}$ and thus F is a solution of the differential relation $\mathcal{R}$.

C^0-Density

One crucial property of the convex integration process is that the solution F can be made arbitrarily close to the initial map f_0.

Proposition (C^0-density).– *We have*

$$\|F - f_0\|_{C^0} \leq \frac{1}{N}\left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0}\right)$$

where $\|g\|_{C^0} = \sup_{p\in D}\|g(p)\|_{\mathbb{E}^3}$ *denotes the C^0 norm of a function* $g : D \to \mathbb{E}^3$.

C^0-Density

Remark.– Even if $f_0(0) = f_0(1)$, the map F obtained by a convex integration from f_0 does not satisfy $F(0) = F(1)$ in general. This can be easily corrected by defining a new map f with the formula

$$\forall t \in [0,1] \ , \quad f(t) = F(t) - t\,(F(1) - F(0)) \,.$$

C^0-Density

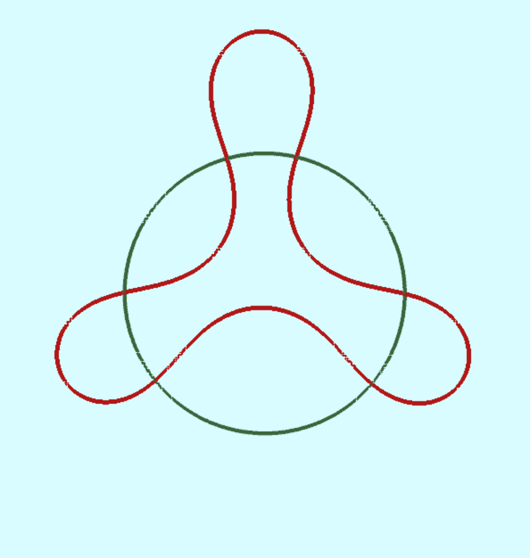
Remark.– Even if $f_0(0) = f_0(1)$, the map F obtained by a convex integration from f_0 does not satisfy $F(0) = F(1)$ in general. This can be easily corrected by defining a new map f with the formula

$$\forall t \in [0,1] \;\; , \quad f(t) = F(t) - t\,(F(1) - F(0))\,.$$

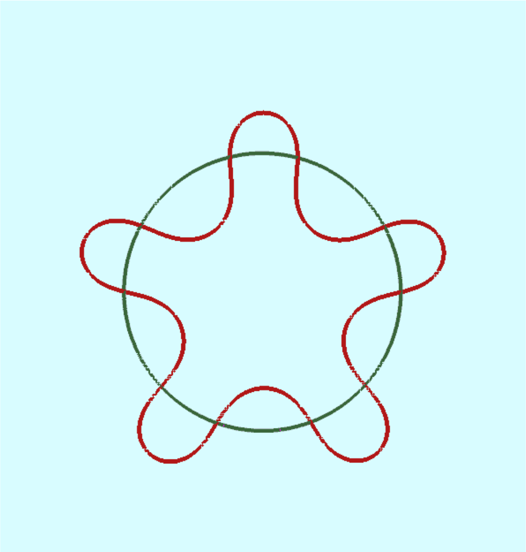
Proposition (C^0-density).– *We have*

$$\|f - f_0\|_{C^0} \leq \frac{2}{N}\left(2\|h\|_{C^0} + \|\frac{\partial h}{\partial t}\|_{C^0}\right)$$

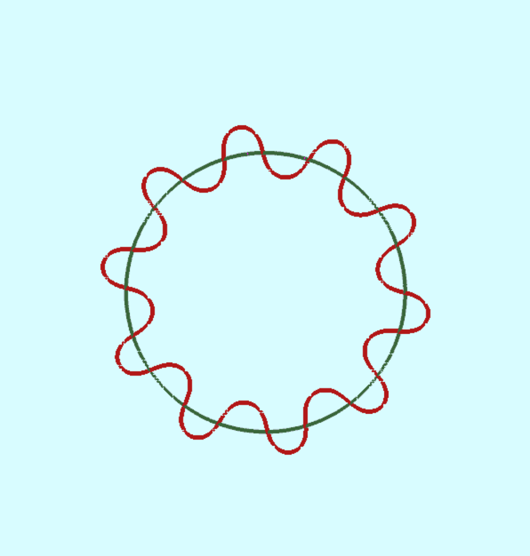
C^0 Density, $N = 3$



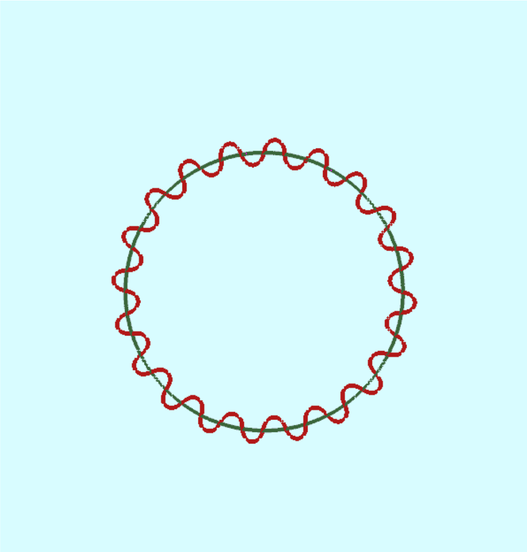
C^0 Density, $N = 5$



C^0 Density, $N = 10$



C^0 Density, $N = 20$

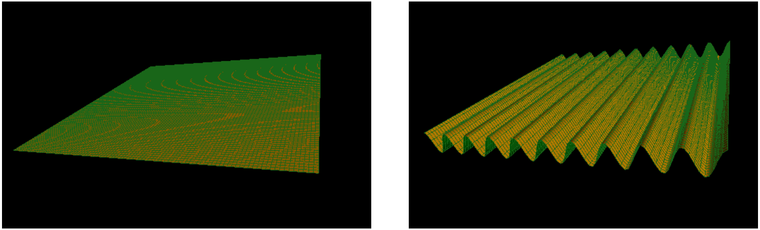


Multi Variables Setting

• In a multi-variable setting, the convex integration formula takes the following natural form :

$$f(c_1, \ldots, c_m) := f_0(c_1, \ldots, c_{m-1}, 0) + \int_0^{c_m} h(c_1, \ldots, c_{m-1}, s, Ns)\,ds$$

where $(c_1, \ldots, c_m) \in [0, 1]^m$.



A corrugated plane

Multi Variables Setting

• The C^0-density property can be enhanced to a $C^{1,\widehat{m}}$-density property where the notation $C^{1,\widehat{m}}$ means that the closeness is measured with the following norm

$$\|f\|_{C^{1,\widehat{m}}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, ..., \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0}).$$

Multi Variables Setting

• The C^0-density property can be enhanced to a $C^{1,\widehat{m}}$-density property where the notation $C^{1,\widehat{m}}$ means that the closeness is measured with the following norm

$$\|f\|_{C^{1,\widehat{m}}} = \max(\|f\|_{C^0}, \|\frac{\partial f}{\partial c_1}\|_{C^0}, \ldots, \|\frac{\partial f}{\partial c_{m-1}}\|_{C^0}).$$

Proposition ($C^{1,\widehat{m}}$-density).– *We have*

$$\|f - f_0\|_{C^{1,\widehat{m}}} = O\left(\frac{1}{N}\right).$$

Mikhaïl Gromov