# The *H*-principle for Ample Relations

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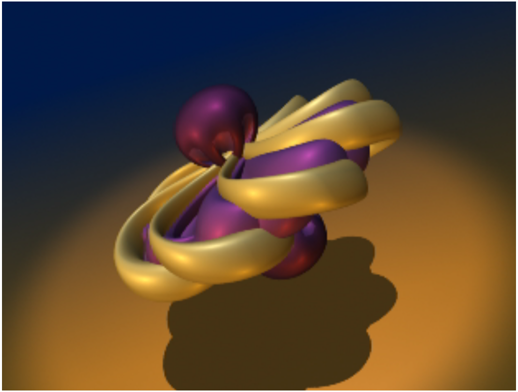
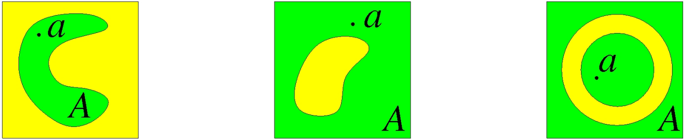


# Ample Relations

**Definition.–** A subset $A \subset \mathbb{R}^n$ is *ample* if for every $a \in A$ the interior of the convex hull of the connected component to which $a$ belongs is $\mathbb{R}^n$ *i. e.* : $IntConv(A, a) = \mathbb{R}^n$ (in particular $A = \emptyset$ is ample).

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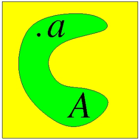

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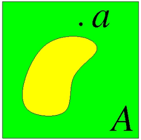

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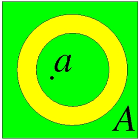

$A$ is not ample.

**Example.–** The complement of a linear subspace $F \subset \mathbb{R}^n$ is ample if and only if Codim $F \geq 2$.

# Ample Relations

**Definition.–** Let $E = P \times \mathbb{R}^n \xrightarrow{\pi} P$ be a fiber bundle, a subset $\mathcal{R} \subset E$ is said to be *ample* if, for every $p \in P$, $\mathcal{R}_p := \pi^{-1}(p) \cap \mathcal{R}$ is ample in $\mathbb{R}^n$.

**Remark.–** If $\mathcal{R} \subset E$ is ample and $z : P \longrightarrow E$ is a section, then, for every $p \in P$, the condition $z(p) \in Conv(\mathcal{R}_p, \sigma(p))$ necessarily holds.

# One Jet Space

- The **1-jet space of maps**

$$J^1(M,N) = \{(x,y,L) \mid x \in M, y \in N, L \in \mathcal{L}(T_xM, T_yN)\}$$

is a natural fiber bundle over $M \times N$

$$\mathcal{L}(T_xM, T_yN) \longrightarrow J^1(M,N) \xrightarrow{p} M \times N$$

and over $M$

$$\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \times N \longrightarrow J^1(M,N) \xrightarrow{\pi} M$$

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- A **differential relation** of order 1 is a subset $\mathcal{R} \subset J^1(M,N)$.

# Ample Relations in $J^1(M,N)$

- Locally, we identify $J^1(M,N)$ with

$$\begin{aligned} J^1(\mathcal{U},\mathcal{V}) &= \mathcal{U}\times\mathcal{V}\times\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n) = \mathcal{U}\times\mathcal{V}\times\textstyle\prod_{i=1}^m \mathbb{R}^n. \\ &= \{(x,y,v_1,...,v_m)\} \end{aligned}$$

where $\mathcal{U}$ and $\mathcal{V}$ are charts of $M$ and $N$.

- We set :

$$J^1(\mathcal{U},\mathcal{V})^\perp := \{(x,y,v_1,...,v_{m-1})\}.$$

- We have

$$\begin{array}{ccc} \mathcal{R}_{\mathcal{U},\mathcal{V}} & \longrightarrow & J^1(\mathcal{U},\mathcal{V}) \\ & & \downarrow p^\perp \\ & & J^1(\mathcal{U},\mathcal{V})^\perp. \end{array}$$

# Ample Relations in $J^1(M,N)$

• Let $z \in J^1(\mathcal{U},\mathcal{V})^\perp$, we set

$$\mathcal{R}_z^\perp = (p^\perp)^{-1}(z) \cap \mathcal{R}_{\mathcal{U},\mathcal{V}}.$$

• $\mathcal{R}^\perp$ is a differential relation of the bundle

$$J^1(\mathcal{U},\mathcal{V}) \xrightarrow{p^\perp} J^1(\mathcal{U},\mathcal{V})^\perp.$$

**Definition. –** A differential relation $\mathcal{R} \subset J^1(M,N)$ is *ample* if for every local identification $J^1(\mathcal{U},\mathcal{V})$ and for every $z \in J^1(\mathcal{U},\mathcal{V})^\perp$, the space $\mathcal{R}_z^\perp$ is ample in $(p^\perp)^{-1}(z) \simeq \mathbb{R}^n$.

# Ample Relations in $J^1(M, N)$

**Example. –** *The differential relation $\mathcal{I}$ of immersions from $M^m$ to $N^n$ is ample if $n > m$.*

# Formal solutions

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• The natural inclusion

$$\begin{array}{rccc} J : & C^1(M, N) & \longrightarrow & J^1(M, N) \\ & f & \longmapsto & j^1 f. \end{array}$$

induces a map

$$J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R}).$$

# H-principle

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This is asking the map $J$ to induce an onto mapping between the components of $\mathcal{S}ol(\mathcal{R})$ and $\Gamma(\mathcal{R})$

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**Definition.–** A differential relation $\mathcal{R}$ satisfies **the parametric *h*-principle** if the map

$$J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$$

is a weak homotopy equivalence.

# $H$-principle for Ample Relations

**Theorem (Gromov 69-73). –** *Let $\mathcal{R} \subset J^1(M, N)$ be an open and ample differential relation. Then $\mathcal{R}$ satisfies the parametric h-principle i. e.*

$$J : \mathcal{S}ol(\mathcal{R}) \longrightarrow \Gamma(\mathcal{R})$$

*is a weak homotopy equivalence.*

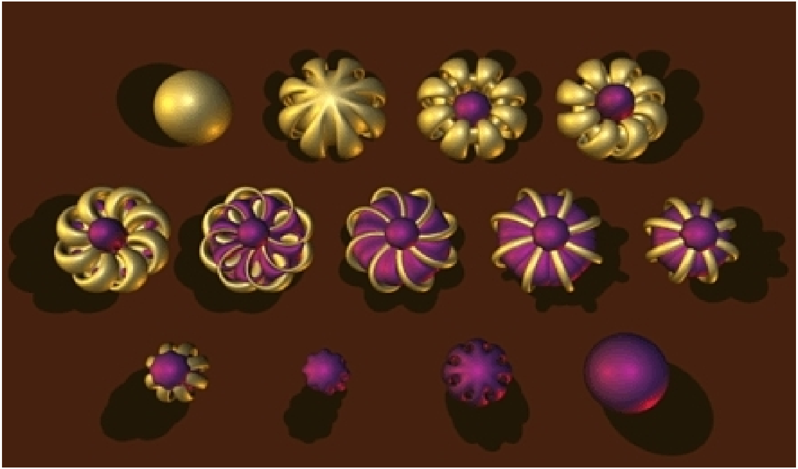
# Immersions

**Smale Paradox (1958).–** The parametric $h$-principle holds for the differential relation of immersions of $M^m$ into $N^n$ with $n > m$. A homotopic computation shows that if $M^m = \mathbb{S}^2$ and $N^n = \mathbb{R}^3$ then

$$\pi_0(I(\mathbb{S}^2, \mathbb{R}^3)) = \pi_2(Gl_+(3, \mathbb{R})) = 0.$$

Thus there is only one class of immersions of the sphere inside the three dimensional space and in particular, the sphere can be everted among immersions.

# Eversion of the Sphere



*Thurston eversion, 1994*

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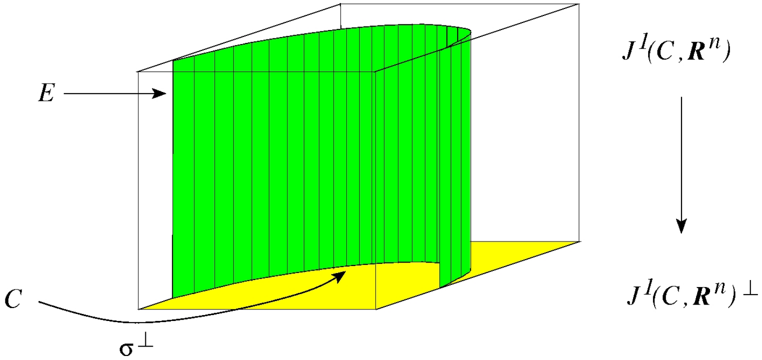
• It is likely that, at that time, Thurston was unaware of the Convex Integration Theory.

# Proof of Gromov Theorem




*The pull-back bundle E*

# $C^0$-density

A $C^0$**-dense** $h$**-principle** holds for $\mathcal{R} \subset J^1(M, N)$ if the (usual) $h$-principle holds and if for every formal solution $\sigma : M \longrightarrow \mathcal{R}$ and every arbitrarily small neighborhood $U \subset N$ of the image of the underlying map $f_0 = bs\, \sigma : M \longrightarrow N$, the homotopy $\sigma_t : M \longrightarrow \mathcal{R}$ joining $f_0$ to a solution $f := bs\, \sigma_1$ can be chosen such that $bs\, \sigma_t(M) \subset U$, for all $t \in [0, 1]$.

Similar definition for the $C^0$**-dense parametric** $h$**-principle.**

**Theorem (Gromov). –** *Let* $\mathcal{R} \subset J^1(M, N)$ *be open and ample, then* $\mathcal{R}$ *satisfies to the* $C^0$*-dense parametric h-principle .*

# William Thurston