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Asymptotic behavior for the Navier–Stokes equations with nonzero external forces☆

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ABSTRACT

We estimate the asymptotic behavior for the Stokes solutions, with external forces first. We found that if there are external forces, then the energy decays slowly even if the forces decay quickly. Then, we also obtain the asymptotic behavior in the temporal-spatial direction for weak solutions of the Navier–Stokes equations. We also provide a simple example of external forces which shows that the Stokes solution does not decay quickly.

1. Introduction

We study the asymptotic behavior in the weighted L^2 of solutions for the Navier–Stokes equations with external forces in the whole space $\mathbb{R}^n$:

$$\begin{aligned} &\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ &\nabla\cdot\mathbf{u} = 0, \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ &\mathbf{u}(x,0) = \mathbf{u}_0, \quad \text{for } x \in \mathbb{R}^n. \end{aligned} \tag{1.1}$$

Here, $\mathbf{u}_0$ is given initial data. The velocity $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and the pressure p are unknown.

The decay problem for weak solutions of the Navier–Stokes equations was first proposed by Leray [15] for the Cauchy problem in $\mathbb{R}^3$. Kato [14] obtained temporal decay rates for strong solutions, for the first time. Schonbek [19–22] worked on the temporal decay problem in $\mathbb{R}^n$. She obtained the lower and the upper bounds. In [20], she showed that if $\mathbf{u}_0 \in L^r \cap L^2$, $1 \le r < 2$, and the average of the initial data $\int \mathbf{u}_0 dx$ is nonzero, then

$$C_1(1+t)^{-\frac{3}{2}(1/r-1/2)} \le \|\mathbf{u}(t)\|_{L^2(\mathbb{R}^3)} \le C_2(1+t)^{-\frac{3}{2}(1/r-1/2)}.$$

In [21,22], it was shown that, if the average is zero, $\int_{\mathbb{R}^n} |\mathbf{u}_0|^2|x|dx < \infty$, and under some restrictions on $\mathbf{u}_0$, then

$$C_1(1+t)^{-n/4-1/2} \le \|\mathbf{u}(t)\|_{L^2(\mathbb{R}^n)} \le C_2(1+t)^{-n/4-1/2}$$

for $n = 2, 3$. See also Miyakawa and Schonbek [17] for the lower bound.

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Borchers and Miyakawa [7] studied the decay problem in half spaces $\mathbb{R}^n_+$. They obtained that if $\mathbf{u}_0 \in L^2 \cap L^r$, then

$$\|\mathbf{u}(t)\|_2 \le C_2(1+t)^{-\frac{n}{2}(1/r-1/2)}$$

provided $1 \le r < 2$. For example, if $r = 1$ then the decay rate is $t^{-n/4}$. For $r = 2$, they obtained that $\|\mathbf{u}(t)\|_2 \to 0$. We [3] showed that the decay rate of L^2-norm of the solutions for the Navier–Stokes equations in the half space is

$$t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{1}{2}}$$

if $\mathbf{u}_0 \in L^2 \cap L^r$ and $\int_{\mathbb{R}^n_+} |y_n \mathbf{u}_0(y)|^r \mathrm{d}y < \infty$.

For the spatial decay, Farwig and Sohr [9,10] showed the spatial decays for the exterior problems. He and Xin [12] showed that if $\mathbf{u}_0 \in L^1(\mathbb{R}^3)$ and $|x|\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, there exists a class of weak solutions satisfying

$$\|(1+|x|^2)^{1/2}\mathbf{u}\|_2^2 + \int_0^t \|(1+|x|^2)^{1/2}\nabla\mathbf{u}\|_2^2 \mathrm{d}\tau \le C$$

and also that if $|x|^{3/2}\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, then there is a class of weak solutions satisfying

$$\|(1+|x|^2)^{\alpha/2}\mathbf{u}(t)\|_2^2 + \int_0^t \|(1+|x|^2)^{\alpha/2}\nabla\mathbf{u}(\tau)\|_2^2 \mathrm{d}\tau \le C(1+\log(1+t))$$

for all $t > 0$, $0 \le \alpha \le 3/2$. If $\|e^{-tA}\mathbf{u}_0\|_1 \le C(1+t)^{-\gamma}$ for some $\gamma > 0$, then the right-hand side of the above inequality can be replaced by a constant independent of t. Schonbek and Schonbek [23] studied the decay properties of $\||x|^\alpha \mathbf{u}\|_2$ for $0 \le \alpha \le 3/4$, when u is smooth. We [4] showed the following. Let $1 \le \alpha < 5/2$. Assume that $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, $(1+|x|)\mathbf{u}_0 \in L^1(\mathbb{R}^3)$, $(1+|x|)^\alpha \mathbf{u}_0 \in L^2(\mathbb{R}^3)$ and $\operatorname{div}\mathbf{u}_0 = 0$. Then there is a weak solution of (1.1) satisfying the following inequality for all $t > 0$;

$$\|(1+|x|)^\alpha \mathbf{u}(t)\|_{L^2}^2 + \int_0^t \|(1+|x|)^\alpha \nabla\mathbf{u}(s)\|_{L^2}^2 \mathrm{d}s < C.$$

Interpolating with the temporal decays, we may obtain the temporal-spatial decay rates. Miyakawa [16] obtained pointwise upper bounds of the Navier–Stokes flows in $\mathbb{R}^n$. In [5], we obtained the lower bounds as in [21,22,17]. However, we include weights for the temporal-spatial decays

$$C_0(1+t)^{-\frac{5}{4}+\frac{\alpha}{2}} \le \|(1+|x|^2)^{\alpha/2}\mathbf{u}(\cdot,t)\|_{L^2} \le C_1(1+t)^{-\frac{5}{4}+\frac{\alpha}{2}}$$

for $0 \le \alpha \le 2$. The upper bound parts are estimated in several papers, for example [4,12]. For exterior domains, refer to [4,1]. Modifying methods in [12], we improved the rates for exterior domains in [4].

All of the above are decay estimates without external forces. With external forces, Wiegner [24] and Ogawa [18] estimated temporal decays. In this paper, we estimate the decay rates of Stokes solutions and of the Navier–Stokes solutions with external forces.

In Section 2, we provide an example of external forces which indicates slow decays. Then, we obtain the temporal-spatial decays for the Stokes flow, and in Section 3 we obtain the decays with weight $(1+|x|^2)^{1/2}$ for the weak solutions of the Navier–Stokes equations.

2. Decay rate of solutions for the Stokes equations

In this section we obtain the decay rate for the Stokes equations in the whole space $\mathbb{R}^3$. If the initial data $\mathbf{u}_0$ and the external force $\mathbf{f}$ are divergence free in $\mathbb{R}^3$, then solution $\mathbf{u}$ for the Stokes equations is reduced to that of the heat equations with initial data $\mathbf{u}_0$ and inhomogeneous term $\mathbf{f}$. The main focus in this section is the estimation on $\mathbf{f}$ since the term concerning initial data is estimated in [5].

If the external force $\mathbf{f}$ is not divergence free, it can be decomposed in the form $\mathbf{f} = \mathbf{f}_{\mathrm{div}} + \nabla\Phi$, where $\mathbf{f}_{\mathrm{div}}$ is divergence free and $\Delta\phi = \nabla\cdot\mathbf{f}$ by the Helmholtz decomposition. Then we consider the Stokes equations

$$\mathbf{v}_t - \Delta\mathbf{v} + \nabla p = \mathbf{f}, \qquad \nabla\cdot\mathbf{v} = 0, \qquad \mathbf{v}(x,0) = \mathbf{a}$$

and the heat equations with the same initial data $\mathbf{a}$

$$V_t - \Delta V = \nabla\Phi, \qquad V(x,0) = \mathbf{a}.$$

Then, defining $\mathbf{w} = \mathbf{v} - V$, we have

$$\mathbf{w}_t - \Delta\mathbf{w} + \nabla p = \mathbf{f}_{\mathrm{div}}, \qquad \nabla\cdot\mathbf{w} = 0, \qquad \mathbf{w}(x,0) = 0,$$

of which the solution is also a solution of heat equations, since the external force is divergence free. The term $\nabla\Phi$ can be absorbed in the pressure term. So if we estimate solutions of heat equations of the form $\mathbf{v}_t - \Delta\mathbf{v} = \mathbf{f}$, then we obtain our goal.

Our motivation for this problem comes from Wiegner [24], in which it is stated that if the Stokes solution $\mathbf{v}$ and the inhomogeneous function f satisfy the decay property that $\|\mathbf{v}\|_{L^2} + (1+t)\|\mathbf{f}(t)\|_{L^2} \le C(1+t)^{-\alpha_0}$ with $\alpha_0 \ge 0$ then a weak solution $\mathbf{u}$ of the Navier–Stokes equations (1.1) satisfies that $\|\mathbf{u}(t)\|_{L^2}^2 \le C(1+t)^{-\alpha_1}$ with $\alpha_1 = \min\{\alpha_0, n/2+1\}$. We first take an example of $\mathbf{f}$ such that the solution of the heat equations with inhomogeneous term $\mathbf{f}$ has slower decay rates. Then, for general $\mathbf{f}$, we obtain upper and lower bounds of the asymptotic behavior of the heat solutions with weights.

By the usual notations, $\|f\|_s$ means $\left(\int_{\mathbb{R}^3} |f(x)|^s dx\right)^{1/s}$ for each s, and denotes $\|f\| = \|f\|_2$. We denote by $K(x,t)$ the heat kernel in the whole space $\mathbb{R}^n$,

$$K_t(x) = K(x,t) \equiv (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}.$$

Then the solution $\mathbf{V}(x,t)$ of heat equation $\mathbf{V}_t - \Delta \mathbf{V} = \mathbf{f}$ with the initial condition $\mathbf{V}(x,0) = \mathbf{a}(x)$ in $\mathbb{R}^3$ has a potential expression

$$\mathbf{V}(x,t) = \int_{\mathbb{R}^n} K(x-y,t)\mathbf{a}(y)\,dy + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)\mathbf{f}(y,s)\,dy\,ds. \tag{2.1}$$

Related to the results in [24], we first consider $\mathbf{f}$ of the form $\mathbf{f}(x,t) = (1+t)^{-\frac{1}{4}-\frac{\alpha}{2}}[K_t * \mathbf{b}](x)$, where $\mathbf{b}$ is independent of t. Schonbek [20] showed that if the average of $\mathbf{b}$ is nonzero and $\mathbf{b} \in L^r \cap L^2$ for $1 \le r < 2$, then

$$C_0(1+t)^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \le \|K * \mathbf{b}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le C_1(1+t)^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

Therefore, for $r = 1$, we obtain that

$$C_0(1+t)^{-1-\frac{\alpha}{2}} \le \|\mathbf{f}(\cdot,t)\| \le C_1(1+t)^{-1-\frac{\alpha}{2}}.$$

The second term in (2.1) is estimated as follows;

$$\begin{aligned}
\left\|\int_0^t [K_{t-s} * \mathbf{f}(\cdot,s)](x)ds\right\| &= \left\|\int_0^t (1+s)^{-\frac{1}{4}-\frac{\alpha}{2}} \left[K_{t-s} * [K_s * \mathbf{b}]\right](x)ds\right\| \\
&= \left\|[K_t * \mathbf{b}](x) \int_0^t (1+s)^{-\frac{1}{4}-\frac{\alpha}{2}} ds\right\| = C\,\|[K_t * \mathbf{b}](x)\| \\
&= C(1+t)^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \|\mathbf{b}\|_r
\end{aligned}$$

for $1 \le r < 2$, for some $C > 0$ if $\alpha > \frac{3}{2}$. This implies that even if $\mathbf{f}$ has fast decay rates, its heat solution $\mathbf{V}$ has slower decay rates, for example $(1+t)^{-\frac{3}{4}}$ when $r = 1$ and $n = 3$. Compare this with the rate $(1+t)^{-\frac{5}{4}}$ in Wiegner [24]. In other words, $\alpha_0 \le \frac{3}{4}$.

We now consider the more general function $\mathbf{f}$ to obtain upper and lower bounds of the temporal-spatial decays. We assume that $\mathbf{f}$ satisfies the following conditions for the upper bounds of the decays such that for some $C > 0$,

$$\int_0^\infty \|(1+|x|^2)^{\gamma/2}\mathbf{f}(s)\|_{L^r} ds \le C, \tag{2.2}$$

$$\text{for each } t > 0 \quad \|\mathbf{f}(t)\|_{L^2} \le C(1+t)^{-\beta_1}, \tag{2.3}$$

and the following for the lower bounds such that for each $t > 0$,

$$\||\cdot|\,\mathbf{f}(t)\|_{L^2}\, ds \le C(1+t)^{-\beta_2}, \tag{2.4}$$

$$\int_{\mathbb{R}^n} f_j(x,t)\,dx \ge C(1+t)^{-\beta_3} \quad j = 1,\ldots,n, \tag{2.5}$$

$$\int_{\mathbb{R}^n} \left(|x| + |x|^{\gamma+1}\right) |\mathbf{f}|(x,t)\,dx \le C(1+t)^{-\beta_4}, \quad \beta_4 > \frac{1}{2}, \tag{2.6}$$

where $\beta_3 \ge \beta_1 \ge 1 + \frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)$, and $\beta_2 \ge 1 + \frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)$.

In [5] for $n = 3$ and in [2] for $n = 2$, we have obtained that, with assumptions on $\mathbf{a}$ stated in Theorem 2.2

$$C_0(1+t)^{-\frac{n+2}{4}+\frac{\beta}{2}} \le \|(1+|x|^2)^{\frac{\beta}{2}}[K * \mathbf{a}](x,t)\| \le C_1(1+t)^{-\frac{n+2}{4}+\frac{\beta}{2}} \tag{2.7}$$

for $0 \le \beta \le (n+2)/2$ if the average of $\mathbf{a}$ is zero. In a similar way in [5] and in [20] if the average of $\mathbf{a}$ is nonzero, we can obtain similar estimates to (2.7):

$$C_0(1+t)^{-\frac{n}{4}+\frac{\beta}{2}} \le \|(1+|x|^2)^{\frac{\beta}{2}}[K * \mathbf{a}](x,t)\| \le C_1(1+t)^{-\frac{n}{4}+\frac{\beta}{2}}. \tag{2.8}$$

For our estimation, we need the following lemma, of which proof is provided in Lemma 3.3, [4] and in Lemma 2.3, [1].

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Lemma 2.1. *Let $a < 1,\ b > 0,\ d < 1$. If $b + d < 1$, then*

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}\mathrm{d}s \le ct^{1-a-d}(1+t)^{-b};$$

if $b + d = 1$, then

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}\mathrm{d}s \le ct^{-a}\ln(t+1);$$

if $b + d > 1$, then

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}\mathrm{d}s \le ct^{-a}.$$

With nonzero inhomogeneous function **f** satisfying (2.2)–(2.6) we obtain the following theorem:

Theorem 2.2. *Let $0 \le \gamma \le n/2$ be a number. Let $\mathbf{V}$ be the heat solution with initial data $\mathbf{a}$, where $\nabla\cdot\mathbf{a} = 0$ and $\int(1+|x|^2)^{(1+\gamma)/2}|\mathbf{a}(x)|\mathrm{d}x < \infty$ is integrable, and $\mathbf{f}$ satisfies (2.2)–(2.6).*

Then, there are positive constants C_0, C_1 such that if $\beta_3 > 1$ and $\beta_4 > 1/2$, then

$$C_0 t^{\frac{\gamma}{2}-\frac{n}{4}} \le \left(\int(1+|x|^2)^\gamma|\mathbf{V}(x,t)|^2\mathrm{d}x\right)^{1/2} \le C_1 t^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}$$

for sufficiently large $t > 0$ if $(b_{jk}) \ne 0$, where

$$b_{jk} \equiv \int y_k a_j(y)\mathrm{d}y.$$

If $0 \le \beta_3 = 1 - n + \frac{n}{r} < 1$ with $1 < r < 2$, and if $\beta_4 > \frac{1}{2} + \frac{n}{2r} - \frac{n}{2}$, then we have

$$C_0 t^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} \le \left(\int(1+|x|^2)^\gamma|\mathbf{V}(x,t)|^2\mathrm{d}x\right)^{1/2} \le C_1 t^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

If $\beta_3 = 1$ with $1 < r < 2$, and if $\beta_4 > 1/2$, then we have

$$C_0 t^{\frac{\gamma}{2}-\frac{n}{4}}\ln(1+t) \le \left(\int(1+|x|^2)^\gamma|\mathbf{V}(x,t)|^2\mathrm{d}x\right)^{1/2} \le C_1 t^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

Proof. Since $C_0(1+|x|^\gamma) \le (1+|x|^2)^{\gamma/2} \le C_1(1+|x|^\gamma)$ for some positive numbers C_0, C_1, it is enough to estimate $\|\mathbf{V}(t)\|_{L^2(\mathbb{R}^2)}$ and $\|\mathbf{V}(t)|x|^\gamma\|_{L^2(\mathbb{R}^3)}$.

Owing to (2.7) and (2.8) it is enough to consider the term containing **f**. We first estimate upper bounds of the decay, then lower bounds. Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^n}|x|^{2\gamma}\left(\int_0^t[K_{t-s}*\mathbf{f}(\cdot,s)](x)\mathrm{d}s\right)^2\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^n}\left(\int_0^t\int_{\mathbb{R}^n}|x|^\gamma K(x-y,t-s)\mathbf{f}(y,s)\,\mathrm{d}y\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\quad\le C\int_{\mathbb{R}^n}\left(\int_0^t\int_{\mathbb{R}^n}(|x-y|^\gamma K(x-y,t-s)+|y|^\gamma K(x-y,t-s))\,|\mathbf{f}(y,s)|\,\mathrm{d}y\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\quad=I_1+I_2,
\end{aligned}$$

which can be estimated as follows; by the generalized Minkowski's and Young's convolution inequalities,

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n}\left(\int_0^t[|\cdot|^\gamma K_{t-s}*|\mathbf{f}|]\,(x,s)\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\le\left(\int_0^t\left(\int_{\mathbb{R}^n}|[|\cdot|^\gamma K_{t-s}*|\mathbf{f}|](x,s)|^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}s\right)^2\\
&\le 2\left(\int_0^{t/2}\|[|\cdot|^\gamma K_{t-s}*|\mathbf{f}|](s)\|\,\mathrm{d}s\right)^2+2\left(\int_{t/2}^t\|[|\cdot|^\gamma K_{t-s}*|\mathbf{f}|](s)\|\,\mathrm{d}s\right)^2
\end{aligned}$$

$$\le C\left(\int_0^{t/2}\|[|\cdot|^\gamma K_{t-s}\|_{r'}\,\|\mathbf{f}(s)\|_r\,\mathrm{d}s\right)^2+C\left(\int_{t/2}^{t}\|[|\cdot|^\gamma K_{t-s}\|_1\|\mathbf{f}(s)\|\,\mathrm{d}s\right)^2$$
$$\le C\left(\int_0^{t/2}(t-s)^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|\mathbf{f}(s)\|_r\,\mathrm{d}s\right)^2+C\left(\int_{t/2}^{t}(t-s)^{\frac{\gamma}{2}}\|\mathbf{f}(s)\|\,\mathrm{d}s\right)^2,$$

by (2.2) and (2.3),

$$\le C(t/2)^{\gamma-n\left(\frac{1}{r}-\frac{1}{2}\right)}\left(\int_0^{t/2}\|\mathbf{f}(s)\|_r\,\mathrm{d}s\right)^2+C\left(\int_{t/2}^{t}(t-s)^{\gamma/2}(1+s)^{-\beta_1}\,\mathrm{d}s\right)^2$$
$$\le Ct^{\gamma-n\left(\frac{1}{r}-\frac{1}{2}\right)}+C(t/2)^{\gamma+2-2\beta_1}=Ct^{\gamma-n\left(\frac{1}{r}-\frac{1}{2}\right)},$$

where $1+\frac{1}{2}=\frac{1}{r'}+\frac{1}{r}$. In a similar way, we obtain

$$I_2=\int_{\mathbb{R}^n}\left(\int_0^t[K_{t-s}*(|\cdot|^\gamma|\mathbf{f}|)](x,s)\,\mathrm{d}s\right)^2\mathrm{d}x$$
$$\le C\left(\int_0^t\left(\int_{\mathbb{R}^n}[K_{t-s}*(|\cdot|^\gamma|\mathbf{f}|)]^2(x,s)\,\mathrm{d}x\right)^{1/2}\mathrm{d}s\right)^2$$
$$\le C\left(\int_0^{t/2}\|K_{t-s}\|_{r'}\;\||\cdot|^\gamma|\mathbf{f}|(s)\|_r\;\mathrm{d}s+\int_{t/2}^{t}\|K_{t-s}\|_1\;\||\cdot|^\gamma|\mathbf{f}|(s)\|\,\mathrm{d}s\right)^2$$
$$\le C\left(\int_0^{t/2}(t-s)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\||\cdot|^\gamma|\mathbf{f}|(s)\,\|_r\;\mathrm{d}s+\int_{t/2}^{t}\||\cdot|^\gamma|\mathbf{f}|(s)\|_{L^2}\mathrm{d}s\right)^2,$$

Q1

by (2.2) and (2.4),

$$\le Ct^{-n\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

Therefore, we have

$$\int_{\mathbb{R}^n}|x|^{2\gamma}\left(\int_0^t[K_{t-s}*\mathbf{f}(\cdot,s)](x)\mathrm{d}s\right)^2\mathrm{d}x\le C\left(t^{\gamma-n\left(\frac{1}{r}-\frac{1}{2}\right)}+t^{-n\left(\frac{1}{r}-\frac{1}{2}\right)}\right)$$

for large t. Since $\|(1+|x|^2)^{\gamma/2}K_t*\mathbf{a}\|_{L^2}=Ct^{-\frac{n+2}{4}+\frac{\gamma}{2}}$, we conclude that

$$\left(\int|x|^{2\gamma}|\mathbf{V}(x,t)|^2\mathrm{d}x\right)^{1/2}\le\||\cdot|^\gamma(K_t*\mathbf{a})\|+\left\||\cdot|^\gamma\int_0^t K_{t-s}*\mathbf{f}(\cdot,s)\,\mathrm{d}s\right\|$$
$$\le Ct^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}$$

for large t.

We now estimate the lower bounds. Observe that, for each $j=1,2,\ldots,n$,

$$\int_0^t[K_{t-s}*f_j](x,s)f\,\mathrm{d}s=\int_0^t\int_{\mathbb{R}^n}K(x-y,t-s)f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s$$
$$=\int_0^t\int_{\mathbb{R}^n}(K_{t-s}(x-y)-K_{t-s}(x))f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}K_{t-s}(x)f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s$$
$$=J_{1,j}+J_{2,j}.$$

Consider $J_{2,j}=\int_0^t\int_{\mathbb{R}^n}K(x,t-s)f_j(y,s)\,\mathrm{d}y\,\mathrm{d}s$; for each $j=1,\ldots,n$, by (2.5),

$$\int_{\mathbb{R}^n}|x|^{2\gamma}\left(J_{2,j}\right)^2\mathrm{d}x\ge C\int_{\mathbb{R}^n}|x|^{2\gamma}\left(\int_0^t(t-s)^{-\frac{n}{2}}\mathrm{e}^{-\frac{|x|^2}{4(t-s)}}(1+s)^{-\beta_3}\,\mathrm{d}s\right)^2\mathrm{d}x$$
$$\ge C\int_{|x|\le\sqrt{t}}|x|^{2\gamma}\left(\int_0^{\frac{t}{2}}(t-s)^{-\delta}\mathrm{e}^{-\frac{|x|^2}{4(t-s)}}(t-s)^{-\frac{n}{2}+\delta}(1+s)^{-\beta_3}\,\mathrm{d}s\right)^2\mathrm{d}x$$
$$\ge C\left(\int_{|x|\le\sqrt{t}}|x|^{2\gamma}t^{-2\delta}\mathrm{e}^{-\frac{|x|^2}{2t}}\mathrm{d}x\right)\left(\int_0^{t/2}(t-s)^{-\frac{n}{2}+\delta}(1+s)^{-\beta_3}\,\mathrm{d}s\right)^2$$

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$$\geq Ct^{-n}\left(\int_{|x|\leq\sqrt{t}}|x|^{2\gamma}e^{-\frac{|x|^2}{2t}}\,dx\right)\left(\int_0^{t/2}(1+s)^{-\beta_3}\,ds\right)^2$$

for small $\delta > 0$. If $\beta_3 > 1$, then

$$\int_{\mathbb{R}^n}|x|^{2\gamma}\left(J_{2,j}\right)^2 dx \geq t^{-n}\int_{|x|\leq\sqrt{t}}|x|^{2\gamma}e^{-|x|^2/(2t)}dx = Ct^{-n}(2t)^{\frac{n}{2}+\gamma} = Ct^{-\frac{n}{2}+\gamma},$$

If $0 \leq \beta_3 < 1$ then

$$\int_{\mathbb{R}^n}|x|^{2\gamma}\left(J_{2,j}\right)^2 dx \geq Ct^{\gamma-\frac{n}{2}+1-\beta_3},$$

in particular, if $0 \leq \beta_3 = 1-n+\frac{n}{r} < 1$ with $1 < r < 2$, then

$$\int_{\mathbb{R}^n}|x|^{2\gamma}\left(J_{2,j}\right)^2 dx \geq Ct^{\gamma-n\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

If $\beta_3 = 1$, then

$$\int_{\mathbb{R}^n}|x|^{2\gamma}(J_2)^2\,dx \geq Ct^{-\frac{n}{2}+\gamma}\left(\ln(1+t)\right)^2.$$

We now estimate J_1. We use the same method used in [3];

$$\begin{aligned}
J_{1,j} &= \int_0^t\int\left(K_{t-s}(x-y)-K_{t-s}(x)\right)f_j(y,s)\,dy\,ds\\
&= \int_0^t(4\pi(t-s))^{-n/2}\int\left(\int_0^1\frac{d}{d\tau}e^{-\frac{|x-\tau y|^2}{4(t-s)}}d\tau\right)f_j(y,s)dy\,ds\\
&= -\int_0^t(4\pi(t-s))^{-n/2}\int\int_0^1\partial_{x_k}e^{-\frac{|x-\tau y|^2}{4(t-s)}}y_kf_j(y,s)\,d\tau\,dy\,ds.
\end{aligned}$$

Taking $z = (x-\tau y)/\sqrt{4(t-s)}$, and by the Minkowski's inequality, and the fundamental theorem of calculus, we have

$$\begin{aligned}
&\left(\int\left|\int_0^t(t-s)^{-n/2}\int\int_0^1|x|^\gamma\partial_{x_k}e^{-\frac{|x-\tau y|^2}{4(t-s)}}y_kf_j(y,s)\,d\tau\,dy\,ds\right|^2dx\right)^{1/2}\\
&\leq \int_0^t(t-s)^{-n/2}\int\int_0^1\left(\int|x|^{2\gamma}\left|\partial_{x_k}e^{-\frac{|x-\tau y|^2}{4(t-s)}}\right|^2dx\right)^{1/2}d\tau\,|y||\mathbf{f}|(y,s)\,dy\,ds\\
&= C\int_0^t(t-s)^{-\frac{n+2}{4}+\frac{\gamma}{2}}\int\int_0^1\left(\int\left(|z|^{2\gamma}+\tau^{2\gamma}|y|^{2\gamma}\right)\left|\partial_{z_k}e^{-|z|^2}\right|^2dz\right)^{1/2}|y||\mathbf{f}|(y,s)\,d\tau\,dy\,ds\\
&\leq C\int_0^t(t-s)^{-\frac{n+2}{4}+\frac{\gamma}{2}}\int\left(|y|+|y|^{\gamma+1}\right)|\mathbf{f}|(y,s)\,dy\,ds\\
&\leq C\int_0^t(t-s)^{-\frac{n+2}{4}+\frac{\gamma}{2}}(1+s)^{-\beta_4}\,ds\\
&\leq C(1+t)^{-\frac{n-2}{4}+\frac{\gamma}{2}-\beta_4}
\end{aligned}$$

by (2.6) and by Lemma 2.1.

We finally obtain that

$$\begin{aligned}
\left\||\cdot|^\gamma\int_0^t[K_{t-s}*\mathbf{f}(\cdot,s)]ds\right\| &\geq \left\||\cdot|^\gamma\int_0^t\int_{\mathbb{R}^n}K(x,t-s)\mathbf{f}(y,s)\,dy\,ds\right\|_{L^2}\\
&\quad-\left\||\cdot|^\gamma\int_0^t\int_{\mathbb{R}^n}\left(K(x-y,t-s)-K(x,t-s)\right)\mathbf{f}(y,s)\,dy\,ds\right\|_{L^2}\\
&\geq Ct^{\frac{\gamma}{2}-\frac{n}{4}}-Ct^{\frac{\gamma}{2}-\frac{n-2}{4}-\beta_4}\geq Ct^{\frac{\gamma}{2}-\frac{n}{4}}
\end{aligned}$$

when $\beta_3 > 1$ and $\beta_4 > 1/2$, where in the last inequality we use the upper bounds of the second term. If $0 \leq \beta_3 = 1-n+\frac{n}{r} < 1$ with $1 < r < 2$, and if $\beta_4 > \frac{1}{2}+\frac{n}{2r}-\frac{n}{2}$, then we have

$$\left\||\cdot|^\gamma\int_0^t[K_{t-s}*\mathbf{f}(\cdot,s)]ds\right\| \geq Ct^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}-Ct^{\frac{\gamma}{2}-\frac{n-2}{4}-\beta_4}\geq Ct^{\frac{\gamma}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

If $\beta_3 = 1$ and $\beta_4 > 1/2$, then we have

$$\left\| |\cdot|^{\gamma} \int_0^t [K_{t-s} * \mathbf{f}(\cdot, s)](x)\mathrm{d}s \right\| \geq Ct^{\frac{\gamma}{2}-\frac{n}{4}} \ln(1+t) - Ct^{\frac{\gamma}{2}-\frac{n-2}{4}-\beta_4} \geq Ct^{\frac{\gamma}{2}-\frac{n}{4}} \ln(1+t).$$

Hence, combining the results in Theorem 2.1, [5] for $n = 3$, and [2] for $n = 2$, we complete the proof. □

We remark, here, that if the average of $\mathbf{f}(x, t)$ is zero, then we may obtain the lower bounds as $t^{\frac{\gamma}{2}-\frac{n+2}{4}}$. If the average of $\mathbf{f}$ is zero, then we use the upper bounds of the terms concerning $\mathbf{f}$, and the lower bounds of the terms concerning $\mathbf{a}$ as in [5].

Let $\mathbf{u}$ be a weak solution of (1.1). Define

$$c_{kl}^0 = \int_0^\infty \int_{\mathbb{R}^3} (u_k u_l)(x, s)\mathrm{d}x\,\mathrm{d}s,$$

and $F_{l,k} = (F_{l,1k}, F_{l,2k}, F_{l,3k})$ by

$$F_{l,jk}(x, t) \equiv (\delta_{jk}\partial_l K_t)(x) + \int_t^\infty (\partial_l\partial_j\partial_k K_s)(x)\mathrm{d}s.$$

Notice that $|c_{kl}^0| < \infty$. Define $\mathbf{v} = (v_1, v_2, v_3)$ by

$$v_j(x, t) = [K_t * u_{0,j}](x) + F_{l,jk}(x, t)c_{kl}^0 + \int_0^t [K_{t-s} * f_j](x, s)\,\mathrm{d}s, \quad t > 0, \tag{2.9}$$

where $\mathbf{u}_0 = (u_{0,1}, u_{0,2}, u_{0,3})$ is the given initial data. Observe that $\mathbf{v}$ is a solution of a heat solution with inhomogeneous term $\mathbf{f}$.

The following theorem is shown in [5] for $n = 3$ and in [2] for $n = 2$ when $\mathbf{f} = 0$. But if we look at the proofs carefully, it is not important whether or not $\mathbf{f} = 0$, so the following theorem also works, even for $\mathbf{f} \neq 0$.

Theorem 2.3. *Let $(1 + |x|^2)^{(1+\gamma)/2}\mathbf{u}_0 \in L^1(\mathbb{R}^n)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Let $\mathbf{u}(\cdot, t)$ be a solution to the Navier–Stokes equations (1.1) with initial data $\mathbf{u}_0$. Suppose that $(b_{kl}) \equiv \left(\int y_l u_{0,k}\mathrm{d}y\right) \neq 0$ or $(c_{kl}^0) \neq (c\delta_{kl})$ for any $c \in \mathbb{R}$. Then, there are positive constants $C_0, C_1 > 0$ such that*

$$C_0(1+t)^{\frac{2\gamma-n-2}{4}} \leq \|(1 + |x|^2)^{\frac{\gamma}{2}} (K_t * \mathbf{u}_{0,j} + c_{kl}F_{l,jk})\| \leq C_1(1+t)^{\frac{2\gamma-n-2}{4}}$$

for $j = 1, 2, 3$, where $0 \leq \gamma < (n + 2)/2$.

With the help of Theorem 2.2 for the inhomogenous term $\mathbf{f}$, we also obtain the following theorem:

Theorem 2.4. *Let $(1 + |x|^2)^{(1+\gamma)/2}\mathbf{u}_0 \in L^1(\mathbb{R}^3)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Assume that $\mathbf{f}$ satisfies (2.2)–(2.6). Let $\mathbf{u}(\cdot, t)$ be a solution to the Navier–Stokes equations (1.1) with initial data $\mathbf{u}_0$, and let $\mathbf{v}$ be a solution of heat equation defined by (2.9). Suppose that $(b_{kl}) \equiv \left(\int y_l u_{0,k}\mathrm{d}y\right) \neq 0$ or $(c_{kl}^0) \neq (c\delta_{kl})$ for any $c \in \mathbb{R}$. Then, we have the same conclusion in Theorem 2.2, except that $\mathbf{V}$ is replaced with $\mathbf{v}$ and that $0 \leq \gamma < n/2$.*

The following propositions are also shown in [5] for $n = 3$, in [2] for $n = 2$ without $\mathbf{f}$:

Proposition 2.5. *Let $n = 3$. Let $\mathbf{v}$ be the solution of the heat equation defined by (2.9) with $\mathbf{f} = 0$, where $\nabla \cdot \mathbf{u}_0 = 0$ and $\int(1 + |x|)|\mathbf{u}_0(x)|\mathrm{d}x < \infty$ is integrable. Then, there are positive constants C such that*

$$\|\nabla\mathbf{v}\| \leq C(1+t)^{-7/4}.$$

Furthermore, if $\int |x|^2|\mathbf{u}_0(x)|\mathrm{d}x < \infty$, then

$$\|(1 + |\cdot|^2)^{1/2}\nabla\mathbf{v}\|_\infty \leq C(1+t)^{-2}$$

for sufficiently large $t > 0$. If $\int(1 + |x|^2)^{3/2}|\mathbf{u}_0(x)|\mathrm{d}x < \infty$, then

$$\|(1 + |\cdot|^2)\nabla\mathbf{v}\| \leq C(1+t)^{-3/4}, \qquad \|(1 + |\cdot|^2)\nabla\mathbf{v}\|_\infty \leq C(1+t)^{-3/2}.$$

Proposition 2.6. *Let $n = 2$. Let $\mathbf{v}$ be the solution of the heat equation defined by (2.9) with $\mathbf{f} = 0$, where $\nabla \cdot \mathbf{u}_0 = 0$ and $\int(1 + |x|)|\mathbf{u}_0(x)|\mathrm{d}x < \infty$ is integrable. Then, there are positive constants C such that*

$$\|\nabla\mathbf{v}\| \leq C(1+t)^{-3/2}.$$

Furthermore, if $\int |x|^2|\mathbf{u}_0(x)|\mathrm{d}x < \infty$, then

$$\|(1 + |\cdot|^2)^{1/2}\nabla\mathbf{v}\|_\infty \leq C(1+t)^{-3/2}$$

for sufficiently large $t > 0$.

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With nonzero $\mathbf{f}$, we may obtain the similar results to the above for $n \geq 2$:

Theorem 2.7. *Let $\mathbf{v}$ be the solution of the heat equation defined by (2.9) with $\mathbf{f}$ satisfying (2.2)–(2.6), where $\nabla \cdot \mathbf{u}_0 = 0$ and $\int (1+|x|)|\mathbf{u}_0(x)|\mathrm{d}x < \infty$ is integrable. Then, there are positive constants C such that*

$$\|\nabla \mathbf{v}\| \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}. \tag{2.10}$$

Furthermore, if $\int |x|^\gamma |\mathbf{u}_0(x)|\mathrm{d}x < \infty$, then

$$\|(1+|\cdot|^2)^{\gamma/2}\nabla \mathbf{v}\|_\infty \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2r}+\frac{\gamma}{2}} \tag{2.11}$$

for sufficiently large $t > 0$. If $\int (1+|x|^2)^{\gamma/2}|\mathbf{u}_0(x)|\mathrm{d}x < \infty$, then

$$\|(1+|\cdot|^2)^{\gamma/2}\nabla \mathbf{v}\| \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{1}{2}+\frac{\gamma}{2}}. \tag{2.12}$$

Proof. It is enough to consider the term concerning $\mathbf{f}$. Observe that

$$\begin{aligned}
\left\|\partial_{x_k}\int_0^t [K_{t-s}*f_j](x,s)\,\mathrm{d}s\right\| &\leq \int_0^{t/2} \|\partial_k K_{t-s}\|_{r'}\,\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t \|\partial_k K_{t-s}\|_1\,\|f_j(\cdot,s)\|\,\mathrm{d}s \\
&\leq \int_0^{t/2} (t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-\beta_1}\,\mathrm{d}s \\
&\leq Ct^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)},
\end{aligned}$$

where $1/r' + 1/r = 3/2$.

Observe that for $1/r' + 1/r = 1$,

$$\begin{aligned}
\left\|\partial_{x_k}\int_0^t [K_{t-s}*f_j](x,s)\,\mathrm{d}s\right\|_\infty &\leq \int_0^{t/2} \|\partial_k K_{t-s}\|_{r'}\,\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t \|\partial_k K_{t-s}\|\,\|f_j(\cdot,s)\|\,\mathrm{d}s \\
&\leq \int_0^{t/2} (t-s)^{-\frac{1}{2}-\frac{n}{2r}}\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t (t-s)^{-\frac{n+2}{4}}(t+s)^{-\beta_1}\,\mathrm{d}s \\
&\leq Ct^{-\frac{1}{2}-\frac{n}{2r}}.
\end{aligned}$$

Since $|x| \leq |x-y| + |y|$, we have that $1/r' + 1/r = 1$,

$$\begin{aligned}
&\left\||x|^\gamma \partial_{x_k}\int_0^t [K_{t-s}*f_j](x,s)\,\mathrm{d}s\right\|_\infty \\
&\leq C\left\|\int_0^t \left[(|\cdot|^\gamma \partial_k K_{t-s})*f_j\right](x,s)\,\mathrm{d}s\right\|_\infty + \left\|\int_0^t \left[\partial_k K_{t-s}*(|\cdot|^\gamma f_j)\right](x,s)\,\mathrm{d}s\right\|_\infty \\
&\leq \int_0^{t/2} \||\cdot|^\gamma \partial_k K_{t-s}\|_{r'}\,\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t \|\partial_k K_{t-s}\|\,\left\||\cdot|^\gamma f_j(\cdot,s)\right\|\,\mathrm{d}s \\
&\leq \int_0^{t/2} (t-s)^{-\frac{1}{2}-\frac{n}{2r}+\frac{\gamma}{2}}\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t (t-s)^{-\frac{n+2}{4}}(t+s)^{-\beta_2}\,\mathrm{d}s \\
&\leq Ct^{-\frac{1}{2}-\frac{n}{2r}+\frac{\gamma}{2}}.
\end{aligned}$$

We also notice that $1/r' + 1/r = 3/2$,

$$\begin{aligned}
&\left\||x|^\gamma \partial_{x_k}\int_0^t [K_{t-s}*f_j](x,s)\,\mathrm{d}s\right\| \\
&\leq C\left\|\int_0^t \left[(|\cdot|^\gamma \partial_k K_{t-s})*f_j\right](x,s)\,\mathrm{d}s\right\| + \left\|\int_0^t \left[\partial_k K_{t-s}*(|\cdot|^\gamma f_j)\right](x,s)\,\mathrm{d}s\right\| \\
&\leq \int_0^{t/2} \||\cdot|^\gamma \partial_k K_{t-s}\|_{r'}\,\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t \|\partial_k K_{t-s}\|_1\,\left\||\cdot|^\gamma f_j(\cdot,s)\right\|\,\mathrm{d}s \\
&\leq \int_0^{t/2} (t-s)^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{1}{2}+\frac{\gamma}{2}}\|f_j(\cdot,s)\|_r\,\mathrm{d}s + \int_{t/2}^t (t-s)^{-\frac{1}{2}}(t+s)^{-\beta_2}\,\mathrm{d}s \\
&\leq Ct^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)-\frac{1}{2}+\frac{\gamma}{2}}. \quad \square
\end{aligned}$$

3. Decay rates with weight $(1+|x|^2)^{1/2}$ for the Navier–Stokes equations

In this section we consider the decay rates for weak solutions with weight $(1+|x|^2)^{1/2}$ for the Navier–Stokes equations. For $n=3$, since the strong solution is still unknown, we should consider the approximate solutions $\mathbf{u}^N$, $N=1,2,\ldots,$ of (1.1) with initial data $\mathbf{u}_0 \in L^1 \cap L^2$, $\operatorname{div} \mathbf{u}_0 = 0$, of the following equations:

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{u}^N - \Delta \mathbf{u}^N + (U^N \cdot \nabla)\mathbf{u}^N + \nabla p^N = 0, \quad t>0, \\ &\nabla \cdot \mathbf{u}^N = 0, \qquad \mathbf{u}^N(0) = \mathbf{u}_0, \end{aligned} \tag{3.1}$$

where U^N is a retarded mollification of $\mathbf{u}^N$.

We recall that the retarded mollification U^N of $\mathbf{u}^N$ is defined by

$$U^N(x,t) = \delta^{-4} \int\int \psi(y/\delta, s/\delta)\widetilde{\mathbf{u}}^N(x-y, t-s)\,\mathrm{d}y\,\mathrm{d}s, \quad \delta = N^{-1},$$

where ψ is a smooth function with $\psi \ge 0$,

$$\int\int \psi\,\mathrm{d}x\,\mathrm{d}t = 1, \quad \text{and} \quad \operatorname{supp}\psi \subset \{(x,t) : |x|^2 \le t,\ 1 < t < 2\},$$

and $\widetilde{\mathbf{u}}^N$ is the zero-extension of the function $\mathbf{u}^N$ which is originally defined for $t \ge 0$. (Refer to [8,13,6].) One easily verifies that

$$\int_0^t \left(\int |U^N(x,s)|^2 \mathrm{d}x\right)^{1/2} \mathrm{d}s \le C \int_0^t \left(\int |\mathbf{u}^N(x,s)|^2 \mathrm{d}x\right)^{1/2} \mathrm{d}s.$$

In the following, we write $U = U^N$, $\mathbf{u} = \mathbf{u}^N$ and $p = p^N$ for simplicity. For $\mathbb{R}^2$, we have that $U = \mathbf{u}$ is the unique weak solution to (1.1). In case $n \ge 3$ for $\mathbb{R}^n$, the estimates derived below are uniform in N, hence the desired results are obtained through passage to the limit $N \to \infty$.

Let $\mathbf{v}$ be defined in (2.9). Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$, then $\mathbf{w}$ satisfies

$$\begin{aligned} &\operatorname{div} \mathbf{w} = 0 \\ &\frac{\partial}{\partial t}\mathbf{w} - \Delta \mathbf{w} = -(U \cdot \nabla)\mathbf{u} - \nabla p, \quad t > 0. \end{aligned} \tag{3.2}$$

These equations are the same for (3.4) in [5], so that we may obtain the following theorem.

Theorem 3.1. *Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$, $(1+|x|^2)\mathbf{u}_0 \in L^1(\mathbb{R}^3)$ and $\nabla \cdot \mathbf{u}_0 = 0$. Let $\mathbf{v}$ be a solution of heat equation defined by (2.9) under the assumption that $(b_{kl}) \ne \mathbf{0}$ or $(c^0_{kl}) \ne (c\delta_{kl})$ for any c. Let $\mathbf{u}(\cdot,t)$ be a solution to (3.1). Then, for any small $\delta > 0$, there $T_1 > 0$ such that*

$$\|\omega \mathbf{w}(t)\|^2 \le \delta(1+t)^{\frac{3}{2}-\frac{3}{r}}$$

for all $t > T_1$.

For the proof, we provide the lemmas, but we will skip those proofs, since the proofs are the same to those in [5] and those in [2]. For brevity, we denote by $\omega = (1+|x|^2)^{1/2}$.

Lemma 3.2. *There exist C such that for $t > 0$*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \omega^2|\mathbf{w}(x,t)|^2\mathrm{d}x + \int_{\mathbb{R}^3}\omega^2|\nabla \mathbf{w}(x,t)|^2\mathrm{d}x \\ &\quad \le C\left(\|\mathbf{w}(t)\|^2 + \|\omega\nabla\mathbf{v}\|_\infty^2 + \|\nabla \mathbf{v}\|^2\right) + C\left(\|U\|^2 + \|\nabla U\|^2\right)\|\omega\mathbf{w}\|^2. \end{aligned}$$

Denote X and Y by

$$X = (1+t)^4\|\omega\mathbf{w}(t)\|^2, \quad \text{and} \quad Y = (1+t)^4\|\omega\nabla\mathbf{w}(t)\|^2.$$

From Lemma 3.2, there is $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}X + Y \le C\left(\|U\|^2 + \|\nabla U\|^2\right)X + 4(1+t)^3\|\omega\mathbf{w}\|^2 + C(1+t)^4\left[\|\mathbf{w}(t)\|^2 + \|\omega\nabla\mathbf{v}\|_\infty^2 + \|\nabla\mathbf{v}\|^2\right]. \tag{3.3}$$

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Lemma 3.3. *For each $R > 0$, there is a constant C such that*

$$\int_{\mathbb{R}^3} \omega^2 |\mathbf{w}(t)|^2 \mathrm{d}x \le \frac{C}{R^2} \int_{\mathbb{R}^3} \omega^2 |\nabla \mathbf{w}(x)|^2 \mathrm{d}x + C\|\mathbf{w}\|_{L^2}^2 + C \int_{|\xi| \le R} |\nabla_\xi \widehat{\mathbf{w}}(\xi)|^2 \mathrm{d}\xi.$$

Take $R = \frac{1}{\sqrt{\epsilon(1+t)}}$, for small enough $\epsilon > 0$. Apply Lemma 3.3 to (3.3), then there is $C_2 > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} X + C_2 Y &\le C\left(\|U\|^2 + \|\nabla U\|^2\right) X + C(1+t)^4 \left[\|\mathbf{w}(t)\|^2 + \|\omega \nabla \mathbf{v}\|_\infty^2 + \|\nabla \mathbf{v}\|^2\right] \\ &\quad + C(1+t)^3 \int_{|\xi| \le (\epsilon(1+t))^{-1/2}} |\nabla_\xi \widehat{\mathbf{w}}(\xi)|^2 \mathrm{d}\xi. \end{aligned} \tag{3.4}$$

Lemma 3.4. *For each $\epsilon > 0$, there is C such that*

$$\int_{|\xi| \le (\epsilon(1+t))^{-1/2}} |\nabla_\xi \widehat{\mathbf{w}}(\xi)|^2 \mathrm{d}\xi \le C(1+t)^{-5/2}.$$

Q2 **Proof** (*Proof of* Theorem 3.1). Apply Lemma 3.4 to the inequality (3.4) to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} X + C_2 Y &\le C(1+t)^4 \left[\|\mathbf{w}\|^2 + \|\omega \nabla \mathbf{v}\|_\infty^2 + \|\nabla \mathbf{v}\|^2\right] + C(1+t)^{1/2} + C\left(\|U\|^2 + \|\nabla U\|^2\right) X \\ &= I_1 + C(1+t)^{1/2} + I_2 X.\end{aligned}$$

Solving the above inequality for $t > T$, we have the inequality

$$X(t) \le C I_3(T) X(T) + \int_T^t I_3(s)(I_1 + (1+s)^{1/2}) \mathrm{d}s,$$

where $I_3(s) = \mathrm{e}^{\int_s^t I_2(\tau)\mathrm{d}\tau}$. There is C_3 independent of t such that $I_3(t) \le C_3$ since

$$\int_s^t I_2(\tau) \mathrm{d}\tau \le C \int_0^\infty \left(\|\nabla \mathbf{u}\|^2 + \|\mathbf{u}\|^2\right)(\tau) \mathrm{d}\tau < \infty.$$

From the result in [17], for any $\beta > 0$ given, there is a large time T_1 such that

$$\|\mathbf{w}(t)\|^2 \le \beta(1+t)^{-5/2} \quad \text{for } t > T_1.$$

From the estimates of heat equation, we may obtain that

$$\|\nabla \mathbf{v}(t)\| \le C(1+t)^{\frac{1}{4} - \frac{3}{2r}}, \qquad \|\omega \nabla \mathbf{v}(t)\|_\infty \le C(1+t)^{-\frac{3}{2r}}.$$

So, for $\beta > 0$ given,

$$I_1(t) \le \beta C(1+t)^{\frac{9}{2} - \frac{3}{r}} + C(1+t)^{4 - \frac{3}{r}} \quad \text{for } t > T_1,$$

hence,

$$\int_{T_1}^t I_3(s)\left(I_1 + (1+s)^{1/2}\right) \mathrm{d}s \le \beta C(1+t)^{\frac{11}{2} - \frac{3}{r}} + C(1+t)^{5 - \frac{3}{r}} \quad \text{for } t > T_1.$$

Therefore, we have the inequality for $t > T_1$

$$\begin{aligned} X(t) &\le C_3 X(T_1) + C_4 \beta (1+t)^{\frac{11}{2} - \frac{3}{r}} + C_5 (1+t)^{5 - \frac{3}{r}} \\ &\le C_6 + C_4 \beta (1+t)^{\frac{11}{2} - \frac{3}{r}} + C_5 (1+t)^{5 - \frac{3}{r}}. \end{aligned}$$

Notice that C_4, C_5 and C_6 depend on T_1. Therefore,

$$\|\omega \mathbf{w}(t)\|^2 \le C_6 (1+t)^{-4} + C_4 \beta (1+t)^{\frac{3}{2} - \frac{3}{r}} + C_5 (1+t)^{1 - \frac{3}{r}}.$$

Fix β_0 to be small enough. Now, take β small that $C_4 \beta \le \frac{\beta_0 C_0}{4}$, T_2 $(>T_1)$ large enough that $C_6(1+T_2)^{-\frac{11}{2} + \frac{3}{r}} \le \frac{C_0 \beta_0}{4}$, and $C_5(1+T_2)^{-\frac{1}{2}} \le \frac{C_0 \beta_0}{4}$, then we have

$$\|\omega \mathbf{w}(t)\|^2 \le C_0 \beta_0 (1+t)^{\frac{3}{2} - \frac{3}{r}} \quad \text{for } t > T_2,$$

which completes the proof. □

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Considering that for $t > T_2$, and for $\beta_0 < 1$

$$\|\omega \mathbf{u}(t)\|^2 \geq \|\omega \mathbf{v}(t)\|^2 - \|\omega \mathbf{w}(t)\|^2,$$

we obtain the following theorem. The upper bound is obtained easily by $\|\omega \mathbf{u}(t)\|^2 \leq \|\omega \mathbf{v}(t)\|^2 + \|\omega \mathbf{w}(t)\|^2$.

Theorem 3.5. *Let $\nabla \cdot \mathbf{u}_0 = 0$ and $\int_{\mathbb{R}^3}(1+|x|^2)|\mathbf{u}_0(x)|dx < \infty$ is integrable, and $\mathbf{f}$ satisfies (2.2)–(2.6) for $\gamma = 1$. Then, there exists a weak solution $\mathbf{u}$ of the Navier–Stokes equations (1.1) with initial data $\mathbf{u}_0$ such that there are positive constants M_0, M_1 satisfying the following: if $\beta_3 > 1$ and $\beta_4 > 1/2$, then*

$$M_0 t^{-\frac{1}{4}} \leq \left(\int_{\mathbb{R}^3}(1+|x|^2)|\mathbf{u}(x,t)|^2 dx\right)^{1/2} \leq M_1 t^{-\frac{1}{4}}$$

for sufficiently large $t > 0$ if $(b_{jk}) \neq 0$, where $b_{jk} \equiv \int y_k a_j(y)dy$.

If $0 \leq \beta_3 = -2 + \frac{3}{r} < 1$ with $1 < r \leq \frac{6}{5}$, and if $\beta_4 > \frac{3}{2r} - 1$, then we have

$$M_0 t^{\frac{5}{4}-\frac{3}{2r}} \leq \left(\int (1+|x|^2)|\mathbf{u}(x,t)|^2 dx\right)^{1/2} \leq M_1 t^{\frac{5}{4}-\frac{3}{2r}}.$$

If $\beta_3 = 1$ with $1 < r < \frac{6}{5}$, and if $\beta_4 > 1/2$, then we have that for large $t > 0$,

$$M_0 t^{-\frac{1}{4}} \ln(1+t) \leq \left(\int (1+|x|^2)|\mathbf{u}(x,t)|^2 dx\right)^{1/2} \leq M_1 t^{\frac{5}{4}-\frac{3}{2r}}.$$

We note that for $\gamma < n\left(\frac{1}{r} - \frac{1}{2}\right)$, combining the estimates in the previous section and those in [6] and in [2], we also obtain the upper bounds easily

$$\|\omega^\gamma \mathbf{u}\| \leq C t^{\frac{\gamma}{2} - \frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}.$$

Uncited references

[11]. Q3

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