New Asymptotic Profiles of Nonstationnary Solutions of the Navier–Stokes System

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Abstract

We show that solutions $u(x,t)$ of the non-stationnary incompressible Navier–Stokes system in $\mathbb{R}^d$ ($d \geq 2$) starting from mild decaying data a behave as $|x| \to \infty$ as a potential field:

$$u(x,t) = e^{t\Delta}a(x) + \gamma_d \nabla_x \left(\sum_{h,k} \frac{\delta_{h,k}\,|x|^2 - dx_h x_k}{d|x|^{d+2}} K_{h,k}(t) \right) + \mathfrak{o}\left(\frac{1}{|x|^{d+1}}\right) \qquad \text{(i)}$$

where γ_d is a constant and $K_{h,k} = \int_0^t (u_h|u_k)_{L^2}$ is the energy matrix of the flow.

We deduce that, for well localized data, and for small t and large enough $|x|$,

$$c\,t\,|x|^{-(d+1)} \leq |u(x,t)| \leq c'\,t\,|x|^{-(d+1)}, \qquad \text{(ii)}$$

where the lower bound holds on the complementary of a set of directions, of arbitrary small measure on $\mathbb{S}^{d-1}$. We also obtain new lower bounds for the large time decay of the weighted-L^p norms, extending previous results of Schonbek, Miyakawa, Bae and Jin.

Nouveaux profils asymptotiques
des solutions non-stationnaires de Navier-Stokes

On montre que la solution $u(x,t)$ de l'équation de Navier–Stokes incompressible dans $\mathbb{R}^d$ ($d \geq 2$) issue d'une donnée de Cauchy générique et modérément décroissante a se comporte, pour $|x| \to \infty$, comme un écoulement potentiel donné par la formule (i) ; γ_d est une constante et $K_{h,k} = \int_0^t (u_h|u_k)_{L^2}$ est la matrice d'énergie de l'écoulement.

On en déduit que, si la donnée est bien localisée, le champ de vitesse vérifie (ii) pour t suffisament petit et $|x|$ assez grand. La borne inférieure est valable sur le complémentaire d'un ensemble de directions, de mesure arbitrairement petite dans $\mathbb{S}^{d-1}$. On obtient aussi de nouvelles bornes inférieures du taux de décroissance en temps grand des moments de la solution dans L^p qui étendent des résultats antérieurs de Schonbek, Miyakawa, Bae and Jin.

Keywords: Asymptotic behavior at infinity. Decay. Upper bound estimates. Lower bound estimates. Mild solutions to the Navier-Stokes system. Asymptotic separation of variables. Peetre weight.
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1 Introduction

Let a be a divergence-free vector field in $\mathbb{R}^d$ $(d \geq 2)$. We consider the Cauchy problem for the Navier–Stokes equations :

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u = -\nabla p, \\ \operatorname{div} u = 0, \\ u(x,0) = a(x). \end{cases} \tag{NS}$$

The unknowns are the velocity field $u = (u_1, \dots, u_d)$ and the pressure p. The problem has to be solved on $\mathbb{R}^d \times [0, +\infty)$ or at least on $\mathbb{R}^d \times [0, T)$ for some $T > 0$.

Because of their parabolic nature, the Navier–Stokes equations feature an infinite-speed propagation effect in the space variable. This phenomenon is usually described by the fact that compactly supported initial data give rise to solutions which immediately have non-compact support. On the other hand, because of the pressure, which can be eliminated from the equations only by applying a non-local operator, the solutions of the Navier–Stokes equations have quite a different behavior as $|x| \to \infty$ from that of solutions of non-linear heat equations.

The main purpose of this paper is to study such asymptotic behavior. For example, we address the following problem. Assume that, at the beginning of the evolution, the fluid is at rest outside a bounded region (say, $a \in C^\infty_\sigma(\mathbb{R}^d)$, the space of smooth, solenoidal and compactly supported vector fields). *At which velocity will the fluid particles that are situated far from that region start to move ?*

We will obtain sharp answers to this and related questions by constructing new asymptotic profiles of solutions to (NS), predicting the pointwise behavior of u as $|x| \to +\infty$.

A few asymptotic profiles of solutions to the Navier–Stokes equations in the whole space are, in fact, already known. For example, F. Planchon [22], studied self-similar profiles. However, his results cannot be used in the case of initial data decaying at infinity faster than $|x|^{-1}$, since the only possible self-similar profile, in this case, would be the zero function. For faster decaying data, the asymptotic profiles of A. Carpio [6], Y. Fujigaki, T. Miyakawa [8], Miyakawa, Schonbek [21], T. Gallay, E. Wayne [11] and Cannone, He, Karch [5] provide valuable information about the large-time behavior of the velocity field. However, in all these works the asymptotics are obtained by computing some spatial norms of expressions involving the solution. The limitation of this approach is that most of the information on the pointwise behavior of the velocity field is lost.

Our method is different, and consists in proving that, asymptotically, the flow behaves as a linear combination of functions of separate variables x and t.

Our profiles imply that, without external forces, the flow associated with decaying initial data behaves at infinity as a potential field, with a generalized Bernouilli formula relating the pressure to the energy matrix $(u_h|u_k)_{L^2}$ of the flow. This illustrates the fact that the spatial behavior at infinity of the flow is almost time-independent, contrary to the temporal asymptotic, which is known to be influenced by spatial decay.

Notations

1. We denote by L^∞_ϑ the space of all measurable functions (or vector fields) f on $\mathbb{R}^d$, such that :

$$\|f\|_{L^\infty_\vartheta} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} (1 + |x|)^\vartheta \, |f(x)| < +\infty.$$

The space $\mathcal{C}_w\left([0,T);L^\infty_\vartheta\right)$ is made of functions $u(x,t)$ such that $u(t)\in L^\infty_\vartheta$ for all $t\geq 0$ and

$$\begin{cases} \lim\limits_{t'\to t}\|u(t')-u(t)\|_{L^\infty_\theta}=0 & \text{if } t>0,\\ u(t)\underset{t\to 0}{\rightharpoonup} u(0) & \text{in the distributional sense.}\end{cases}$$

2. For positive ϕ, the notation $f(x,t)=\mathcal{O}_t\left(\phi(x)^{-1}\right)$ means that $|\phi(x)f(x,t)|\leq C_t$, for some function $t\mapsto C_t$, possibly growing as $t\to\infty$, but locally bounded.
3. The solution of the heat equation is

$$e^{t\Delta}a(x)=(4\pi t)^{-d/2}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4t}}\,a(y)\,dy.$$

4. We also adopt the standard Kronecker symbol: $\delta_{i,j}=1$ if $i=j$, and $\delta_{i,j}=0$ otherwise.

Our starting point is the following well known result (see [19], chapter 25).

Theorem 1.1 *Let $d\geq 2$. There exists a constant $\gamma>0$ such that for any divergence-free vector field $a\in L^\infty(\mathbb{R}^d)$, one can find*

$$T\geq\gamma\min\left\{1;\|a\|_{L^\infty}^{-2}\right\}$$

and a unique mild solution $u\in\mathcal{C}_w\left([0,T];L^\infty\right)$ of (NS). *This solution u is smooth for $t>0$. Moreover, if a belongs to L^∞_ϑ for some $\vartheta\geq 0$, then we also have :*

$$u\in\mathcal{C}_w\left([0,T];L^\infty_{\underline{\vartheta}}\right)$$

with $\underline{\vartheta}=\min\{\vartheta\,;\,d+1\}$.

This conclusion can be restated in a slighlty different way (see also [24, Proposition 3]) :

$$u(x,t)=e^{t\Delta}a+\mathcal{O}_t\left((1+|x|)^{-\min\{2\vartheta\,;\,d+1\}}\right)$$

on $[0,T]\times\mathbb{R}^d$.

Asymptotic behavior of local solutions

We can now state our first main result. Let us introduce the energy matrices :

$$\mathcal{E}_{h,k}(t)=\int_{\mathbb{R}^d}(u_hu_k)(y,t)\,dy\qquad\text{and}\qquad K_{h,k}(t)=\int_0^t\int_{\mathbb{R}^d}(u_hu_k)(y,s)\,dy\,ds. \tag{1}$$

The following theorem describes the asymptotic profile of local solutions.

Theorem 1.2 *For $\vartheta>\frac{d+1}{2}$ and an initial datum $a\in L^\infty_\vartheta$, let $u\in\mathcal{C}_w\left([0,T];L^\infty_{\underline{\vartheta}}\right)$ be the solution of* (NS) *given by the preceding theorem. The following profile holds for $|x|\to+\infty$:*

$$u(x,t)=e^{t\Delta}a(x)+\nabla\Pi(x,t)+\mathcal{O}_t\left(|x|^{-\min\{2\vartheta\,;\,d+2\}}\right) \tag{2}$$

where $\Pi(x,t)$ is given by :

$$\Pi(x,t)=\gamma_d\sum_{h,k}\left(\frac{\delta_{h,k}}{d\,|x|^d}-\frac{x_hx_k}{|x|^{d+2}}\right)\cdot K_{h,k}(t) \tag{3}$$

and $\gamma_d = \pi^{-d/2}\,\Gamma(\frac{d+2}{2})$. If, moreover, the first and second order derivatives of a belong to L^∞_ϑ, then there exists a constant p_0 such that the following profile holds for $t>0$:

$$p(x,t) = p_0 - \gamma_d \sum_{h,k} \left(\frac{\delta_{h,k}}{d\,|x|^d} - \frac{x_h x_k}{|x|^{d+2}} \right) \cdot \mathcal{E}_{h,k}(t) + \mathcal{O}_t\left(|x|^{-\min\{2\vartheta-1\,;\,d+1\}} \right) \tag{4}$$

Remark 1.3 This theorem essentially says that, for *mild decaying* data (this is the meaning of the assumption $a \in L^\infty_\vartheta$, with $\vartheta > \frac{d+1}{2}$),

$$u(x,t) \sim e^{t\Delta}a(x) + \nabla\Pi(x,t), \qquad \text{as } |x| \to \infty.$$

In other words, the flow behaves at infinity as the solution of the heat equation plus a potential field at infinity. In particular, if follows that for *fast decaying* data (*i.e.* when $\vartheta > d+1$, we simply have

$$u(x,t) \sim \nabla\Pi(x,t), \qquad \text{as } |x| \to \infty,$$

since the linear evolution can be included in the lower order terms.

Theorem 1.2 does not cover the case of *slowly decaying* data (*i.e.*, the case $\vartheta \le \frac{d+1}{2}$). The spatial asymptotic of those slowly decaying solutions (including self-similar solutions) has a different structure and cannot be constructed with the same method. We should consider it in an independent paper.

Remark 1.4 The decay of the remainder in (2) cannot exceed $|x|^{-d-2}$. Indeed, (NS) being invariant by translation, the choice of the origin is arbitrary and one can easily check that

$$\nabla\Pi(x-x_0,t) - \nabla\Pi(x,t)$$

decays at infinity as $|x|^{-d-2}$ if $\Pi \not\equiv 0$ and $x_0 \neq 0$.

Even if $u(x,t)$ develops a singularity in finite time, the potential field in (2) will remain uniformly bounded away from the origin :

$$|\nabla\Pi(x,t)| \le C\,\|a\|_{L^2}\, t|x|^{-d-1}.$$

However, the above result provides no information about the singularity itself, nor does it prevent it from appearing : as long as the solution is smooth, the remainder of (2) compensates for the singularity at the origin of $\nabla\Pi(x,t)$.

Remark 1.5 The above profile for the pressure has some analogy with Bernoulli's formula for potential flows :

$$p = p_0 + \frac{1}{2}\rho U^2.$$

Such a formula holds rigorously for the stationary Euler equation with no external force, but this identity can also be useful when dealing with high Reynolds flows around aerodynamical bodies (see, *e.g.*, the description of the Prandtl laminarity theory in [14, Chapter 9]).

The asymptotic profiles of Theorem 1.2 are meaningful when the leading term does not vanish identically. It turns out that this is the case for generic solutions. Indeed, the next result provides a necessary and sufficient condition for $\nabla\Pi$ to be identically zero.

Proposition 1.6 *Let u as in Theorem 1.2 and $K=(K_{h,k})$. For any $t \in [0,T]$, the homogeneous function $x \mapsto \nabla\Pi(x,t)$ vanishes identically on $\mathbb{R}^d$ if and only if the matrix $K(t)$ is proportional to the identity matrix, i.e.*

$$\forall\, h,k \in \{1,\dots,d\}, \qquad K_{h,k}(t) = \alpha(t)\,\delta_{h,k} \tag{5}$$

with $\alpha = \frac{1}{d}\,\mathrm{Tr}\,K$.

This shows that $\nabla\Pi$ does not vanish for generic flows. Conditions (5) also occur in the paper of T. Miyakawa and M. Schonbek [21]. It is shown therein that a high decay rate of the energy of the flow for large time is essentially equivalent to (5) holding in the limit $t \to +\infty$.

Such orthogonality relations can also be described in terms of vanishing moments of the vorticity $\omega = \operatorname{curl} u$ of the flow (see, *e.g.* [11], [12]). Focusing on the vorticity, in fact, has crucial advantages in the study of the large time behavior of solutions, especially in the two-timensional case. We refer *e.g.* to the recent work of Gallay and Wayne [13] on the global stability results of vortex solutions.

On the other hand, the large space behavior of the vorticity is less interesting than that of the velocity field. This can be shown by taking the $\operatorname{curl}(\cdot)$ operator term-by-term in formula (2): the term $\operatorname{curl}(\nabla\Pi)$ identically vanishes. The physical interpretation of this remark is the following: if we start with an initial datum with localized vorticity, then the vorticity will remain localized as far as the solution exists (this fact, of course, was already known).

In principle, it would be possible to extend formula (2) and to write a higher-order asymptotic for u as $|x| \to \infty$. The above observation allows us to predict that all the higher-order terms of the expansion of the velocity field must be curl-free. Otherwise, there would be a limitation on the decay rate of ω as $|x| \to \infty$ and this would contredict *e.g.* the results of [12], [18]. In other words, all the higher-order terms of the expansion of u should be gradients.

Large time asymptotics

Under a suitable smallness assumption such as

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |x|\,|a(x)| \le \varepsilon_0,$$

one can take $T = +\infty$ in Theorem 1.1 (see, *e.g.*, [20]). Moreover, the localization property of the flow persists uniformly. One has :

$$|u(x,t)| \le C\,(1+|x|)^{-\alpha}\,(1+t)^{-\beta/2}$$

for any $\alpha, \beta \ge 0$ such that $\alpha + \beta \le \min\{\vartheta\,;\, d+1\}$. When $\vartheta = d$ or $d+1$, one needs the additional assumption that the above estimate already holds for $e^{t\Delta}a$ (see [1]). In particular, these estimates imply that, for large t :

$$\|u\|^2_{L^2([0,t],L^2)} \le \begin{cases} C, & \text{if } \vartheta > \frac{d+2}{2} \\ C_\epsilon\, t^{-\vartheta+\frac{d+2}{2}+\epsilon}, & \text{if } \vartheta \le \frac{d+2}{2}, \end{cases}$$

for all $\epsilon > 0$ (this bound also holds for $\epsilon = 0$, $\vartheta \ne \frac{d+2}{2}$, but we will not use this fact).

We can now give our asymptotic profile for global solutions.

Theorem 1.7 *Given $\vartheta > \frac{d+1}{2}$, let $u(x,t)$ be a solution of* (NS) *on $\mathbb{R}_+ \times \mathbb{R}^d$ such that*

$$|u(x,t)| \le C_0\,(1+|x|)^{-\alpha}\,(1+t)^{-\beta/2} \tag{6}$$

for any $\alpha, \beta \ge 0$ such that $\alpha + \beta \le \vartheta$. Then,

$$u(x,t) = e^{t\Delta}a(x) + \nabla\Pi(x,t) + |x|^{-d-1}E\left(\frac{x}{\sqrt{t+1}}\,;\,t\right) + \mathcal{R}(x,t) \tag{7}$$

with the following estimates :

$$|E(x,t)| \le Ce^{-c|x|^2}\,\|u\|^2_{L^2([0,t],L^2)} \tag{8}$$

and, for any $0 \le \alpha \le \min\{1, \vartheta - \frac{d+1}{2}\}$*, and all* $t \ge 1$,

$$
\begin{aligned}
&|\mathcal{R}(x,t)| \le C_\alpha\, |x|^{-d-1-\alpha}\, t^{-\frac{1}{2}+\frac{\alpha}{2}}, &&\textit{if } \vartheta > \tfrac{d+3}{2},\\
&|\mathcal{R}(x,t)| \le C_{\alpha,\epsilon}\, |x|^{-d-1-\alpha}\, t^{\frac{d+2+\alpha}{2}-\vartheta+\epsilon}, &&\textit{if } \tfrac{d+1}{2} < \vartheta \le \tfrac{d+3}{2}.
\end{aligned}
\tag{9}
$$

Due to the form of the remainder terms, it seems impossible to obtain a description of the pointwise behavior of u for large but fixed $|x|$, and $t \to \infty$. Conclusion (7) is interesting only for (x,t) such that $|x| \ge C\sqrt{t+1}$. For those points, this profile provides more information than those in [8] or [5] (on the other hand, our assumptions are necessarily more stringent).

Applications

For smooth and fast decaying initial data, according to Theorem 1.2 one has

$$|u(x,t)| \le C_t\,(1+|x|)^{-(d+1)}. \tag{10}$$

Theorem 1.2 allows us to answer the more subtle problem of the validity of the corresponding *lower bound* to (10).

A first difficulty is the following: the upper bound ensures that $u(\cdot,t)$ is integrable, so that the divergence-free condition implies

$$\forall t > 0, \quad \forall j \in \{1,\dots,d\}, \qquad \int_{\mathbb{R}^d} u_j(x,t)\,dx = 0.$$

In particular, since u is smooth for $t > 0$, no uniform lower bound by a given positive function can hold. However, non-uniform and anisotropic lower bounds do hold, even if the initial data is rapidly decreasing.

More precisely, for generic flows (i.e. if we exclude flows with special symmetries) starting from fast decaying data, we will prove that for some small $t_0 > 0$ (depending only on the initial datum), and for $j = 1,\dots,d$,

$$c\,t\,|x|^{-(d+1)} \le |u_j(x,t)| \le c'\,t\,|x|^{-(d+1)}, \qquad c, c' > 0 \tag{11}$$

for all $t \in (0,t_0]$ and all $|x| \ge C/\sqrt{t}$, with x outside a small set of exceptional directions, along which the decay can be faster. In other words, the constant c in (11) is independent of t or $|x|$, but does depend on the direction $x/|x|$ (see Theorem 3.1 below for a more precise statement). For example, we will see that in dimension two the exceptional set is made of at most *six directions*. The remarkable fact is that the above lower bound holds *e.g.* for compactly supported data (that is, even without assuming that $|a(x)| \ge c(1+|x|)^{-d-1}$). In particular, this allows us to improve the previously known results (see *e.g.*, [3], [19]) on the instantaneous spatial spreading property of highly localized flows.

For generic global strong solutions, Theorem 1.7 implies various lower bounds. More precisely, starting from a fast decaying initial datum, we get, for all $0 \le \alpha \le d+1$, and large t :

$$\|u(t)\|_{L^\infty_\alpha} \ge c\,t^{-(d+1-\alpha)/2}. \tag{12}$$

This result is a converse to Miyakawa's property (6).

In the same spirit, Theorem 1.7 can be applied to estimate the decay of the moments of the solutions : for all $1 \le p < \infty$ and $\alpha \ge 0$ such that

$$\alpha + \frac{d}{p} < d+1, \tag{13}$$

we obtain, for large t,

$$\| (1+|x|)^\alpha \, u(t)\|_{L^p} \geq c\, t^{-\frac{1}{2}(d+1-\alpha-\frac{d}{p})}. \tag{14}$$

This lower bound seemed to be known only in a few particular cases (namely, $p = 2$ and $0 \leq \alpha \leq 2$, see [23], [12], [4], or $1 \leq p \leq \infty$ and $\alpha = 0$, see [8]). The corresponding upper bounds to (14), starting with the work of M. E. Schonbek, have been studied by many authors. See [18] for a quite general result.

In some sense, the restriction (13) on the parameters could be removed, since (11) implies that for generic solutions, one has

$$\| (1+|x|)^\alpha \, u(t)\|_{L^p} = \infty$$

whenever $\alpha + \frac{d}{p} \geq d+1$ (see also [3]).

Our results can also be applied to the study of the anisotropic decay of the velocity field. In the whole space, we show that not too stringent anisotropic assumptions on the decay of the data will be conserved by the flow. We also show that, if the initial data is well localized in $\mathbb{R}^d$, then the flow decays faster than $(1+|x|)^{-(d+1)}$ as soon as one component does. This prevents localized flows in $\mathbb{R}^d$ from having a really anisotropic decay.

The situation can be different in other unbounded domains. For example, we will briefly discuss the case of the half plane $x_d > 0$, with Neumann boundary conditions, and show that, in this case, generic flows have a genuinely anisotropic decay.

The asymptotic separation of variables method

The proof of (2) relies on a new, simple method that is a sort of "asymptotic separation of variables". We can summarize it as follows: one starts writing the Navier–Stokes equation in the usual integral form

$$u(t) = e^{t\Delta}a - \int_0^t e^{(t-s)\Delta}\, \mathbb{P}\, \mathrm{div}(u \otimes u)(s)\, ds, \tag{15}$$

where $\mathbb{P}$ is the Leray-Hopf projector onto the divergence-free vector fields :

$$\mathbb{P}f = f - \nabla\Delta^{-1}(\mathrm{div}\, f).$$

Then we use a classical decomposition of the nonlinear term (see *e.g.*, [23], [3])

$$(u\otimes u)(x,t) = \left(\int_{\mathbb{R}^d} (u\otimes u)(y,t)\, dy\right) g(x) + v(x,t),$$

where g denotes the standard gaussian function, and v is defined through this formula.

Since $\displaystyle\int_{\mathbb{R}^d} v(x,t)\, dx = 0$, the function $e^{t\Delta}\, \mathbb{P}\, \mathrm{div}\, v$ behaves at infinity better than the previous non-linearity $e^{t\Delta}\mathbb{P}\ \mathrm{div}(u\otimes u)$: its contribution can be included in the remainder terms. The next step consists in observing that the kernel of $e^{t\Delta}\mathbb{P}$ div behaves, as $|x| \to \infty$, as a time independent homogeneous tensor $H(x)$. Then we show that by applying $e^{(t-s)\Delta}\mathbb{P}$ div to a matrix of the form $\mathcal{E}(s)g$, where the coefficients of $\mathcal{E}(s)$ depend only on time, we get $H(x)\cdot\mathcal{E}(s)$, plus some lower order terms. A time integration then yields a principal part for the velocity field of the form $H(x)\cdot K(t)$, as $|x|\to\infty$. An explicit computation of this product provides the expression for $\Pi(x,t)$ in (2).

We point out that the above strategy is not specific to the Navier–Stokes equations, but can be adapted to obtain the spatial asymptotics for more general models. What one essentially needs for its application are sufficiently explicit expressions (or sharp estimates) for the kernels of the operators involved.

Structure of the article

Our main results are Theorem 1.2, its companion Theorem 1.7, and Theorem 3.1. Corollary 3.6 also has some interest, since it extends a few results in the existing literature and its proof is very short. This paper is organized as follows. The proof of Theorem 1.2 and Theorem 1.7 is contained in Sections 2.4–2.5, after we have prepared some preliminary estimates. In Section 2.6 we establish Proposition 1.6, in a slightly more complete form. The remaining part of the paper is devoted to applications : in Section 3.1 we give a precise statement and a proof of (11). Section 3.2 contains the proof of (12) and (14). The last sections deal with the anisotropic decay of solutions.

2 Proof of the main results

Let us now focus on the proof of the above results.

We shall use the following notations for the kernel of the convolution operator $e^{t\Delta}\mathbb{P}\,\mathrm{div}$:

$$F_{j;h,k}(x,t) = \int_{\mathbb{R}^d} ie^{-t|\xi|^2+ix\cdot\xi}\left(\frac{1}{2}\left[\xi_h\delta_{j,k}+\xi_k\delta_{j,h}\right] - \frac{\xi_j\xi_h\xi_k}{|\xi|^2}\right)\frac{d\xi}{(2\pi)^d}.$$

According to (15), the j^{th} component of (NS) can therefore be written as

$$u_j(t) = e^{t\Delta}a_j - \sum_{h=1}^{d}\sum_{k=1}^{d}\int_0^t F_{j;h,k}(t-s) * (u_hu_k)(s)\,ds. \tag{16}$$

This kernel is related to the standard gaussian function $g(x) = (4\pi)^{-d/2}e^{-|x|^2/4}$ in the following way. One has :

$$F_{j;h,k}(x,t) = F^{(1)}_{j;h,k}(x,t) + F^{(2)}_{j;h,k}(x,t)$$

with

$$F^{(1)}_{j;h,k}(x,t) = \frac{1}{2}\left[(\partial_h g_t)\delta_{j,k} + (\partial_k g_t)\delta_{j,h}\right], \qquad F^{(2)}_{j;h,k}(x,t) = \int_t^\infty \partial_j\partial_h\partial_k g_s(x)\,ds$$

Note that $F^{(1)}_{j;h,k} = F^{(1)}_{j;k,h}$ and $F^{(2)}_{j;h,k} = F^{(2)}_{j;k,h}$ accordingly to the fact that only the symmetrical kernel has a physical meaning; $g_t(x) = t^{-d/2}g(x/\sqrt{t})$ is the fundamental solution of the heat equation.

2.1 Some elementary computations on F

We shall need time-independent asymptotics of F, valid in the region where $|x|^2 \gg t$.

Lemma 2.1 *There exist two positive constants C and c that depend only on d, and a family of smooth functions $\Psi_{j;h,k}$ satisfying*

$$|\Psi_{j;h,k}(x)| + |\nabla\Psi_{j;h,k}(x)| \le Ce^{-c|x|^2} \tag{17a}$$

such that :

$$F_{j;h,k}(x,t) = \gamma_d\left(\frac{\sigma_{j,h,k}(x)}{|x|^{d+2}} - (d+2)\frac{x_jx_hx_k}{|x|^{d+4}}\right) + |x|^{-(d+1)}\Psi_{j;h,k}\left(\frac{x}{\sqrt{t}}\right) \tag{17b}$$

with $\gamma_d = \pi^{-d/2}\,\Gamma(\frac{d+2}{2})$ and $\sigma_{j,h,k}(x) = \delta_{j,h}x_k + \delta_{j,k}x_h + \delta_{h,k}x_j$.

Remark 2.2 Note that $x = 0$ is not a singular value of F ; indeed, $F_{j;h,k}$ is a $\mathcal{C}^\infty$ function on $\mathbb{R}^d \times]0; +\infty[$ and one may immediately check on the Fourier transform that $F_{j;h,k}(0,t) = 0$. Actually, for $|x|^2 \leq t$, the following computations also imply that :

$$|F_{j;h,k}(x)| \leq C\, \frac{\delta_{j,h}|x_k| + \delta_{j,k}|x_h| + \delta_{h,k}|x_j|}{t^{(d+2)/2}} + \mathcal{O}\left(\frac{|x|^2}{t^{(d+3)/2}}\right).$$

Remark 2.3 In Theorem 1.7, one has :

$$E_j(y,t) = -\sum_{h,k} \int_0^t \int_{\mathbb{R}^d} (u_h u_k)(z, t-s)\, \Psi_{j;h,k}\left(\sqrt{\frac{t+1}{s+1}}\, y\right) dz\, ds$$

with the $\Psi_{j;h,k}$ given by this lemma.

Proof. For all indices j, h, k (distinct or not) in $\{1, \ldots, d\}$, one has :

$$F^{(1)}_{j;h,k}(x,t) = -\frac{\delta_{j,k}\, x_h + \delta_{j,h}\, x_k}{4(4\pi)^{d/2}\, t^{(d+2)/2}} e^{-|x|^2/4t},$$

thus $F^{(1)}_{j;h,k}(x,t) = |x|^{-(d+1)}\, \Psi^{(1)}_{j;h,k}\left(x/\sqrt{t}\right)$, with

$$\Psi^{(1)}_{j;h,k}(x) = -2^{-d-1}\pi^{-d/2}(\delta_{j,k}\, x_h + \delta_{j,h}\, x_k)|x|^{d+1} e^{-|x|^2/4}.$$

Let us introduce $\sigma_{j,h,k}(x) = \delta_{j,h}x_k + \delta_{j,k}x_h + \delta_{h,k}x_j$. One also has :

$$F^{(2)}_{j;h,k}(x,t) = \int_t^\infty \left(\frac{\sigma_{j,h,k}(x)}{(2s)^2} - \frac{x_j x_h x_k}{(2s)^3}\right) g_s(x)\, ds.$$

The change of variable $\lambda = |x|/\sqrt{4s}$ gives $g_s(x) = \pi^{-d/2}|x|^{-d}\lambda^d e^{-\lambda^2}$, and therefore :

$$F^{(2)}_{j;h,k}(x,t) = 2\pi^{-d/2} \int_0^{|x|/\sqrt{4t}} \left(\frac{\sigma_{j,h,k}(x)}{|x|^{d+2}}\lambda^{d+1} - \frac{2x_j x_h x_k}{|x|^{d+4}}\lambda^{d+3}\right) e^{-\lambda^2}\, d\lambda.$$

The following formula provides information when $A = |x|/\sqrt{t} \gg 1$:

$$\int_0^A \lambda^{d+n} e^{-\lambda^2} d\lambda = \frac{1}{2}\Gamma\left(\frac{d+n+1}{2}\right) - \int_A^\infty \lambda^{d+n} e^{-\lambda^2} d\lambda.$$

This leads to :

$$\pi^{d/2} F^{(2)}_{j;h,k}(x,t) = \frac{\sigma_{j,h,k}(x)}{|x|^{d+2}}\,\Gamma\left(\frac{d+2}{2}\right) - \frac{2x_j x_h x_k}{|x|^{d+4}}\,\Gamma\left(\frac{d+4}{2}\right) + |x|^{-(d+1)}\,\Psi^{(2)}_{j;h,k}\left(\frac{x}{\sqrt{t}}\right)$$

with

$$\Psi^{(2)}_{j;h,k}(x) = -\frac{2\sigma_{j,h,k}(x)}{|x|}\int_{|x|/2}^\infty \lambda^{d+1} e^{-\lambda^2} d\lambda + \frac{4x_j x_h x_k}{|x|^3}\int_{|x|/2}^\infty \lambda^{d+3} e^{-\lambda^2} d\lambda.$$

Conclusion (17b) follows immediately from the well known formula $\Gamma(z+1) = z\Gamma(z)$. The bounds on $\Psi_{j;h,k}$ and its derivatives are also obvious.

□

The second valuable property of F is that the convolution with the standard gaussian function is equivalent to a shift in time.

Lemma 2.4 *For all $t > 0$ and $x \in \mathbb{R}^d$, one has :*

$$\big(F_{j;h,k}(\cdot,t) * g\big)(x) = F_{j;h,k}(x, t+1). \tag{18}$$

Proof. Since $(g_t)_{t\geq 0}$ is a convolution semi-group, *i.e.* $g_t * g = g_{t+1}$, one has :

$$F^{(1)}_{j;h,k}(\cdot,t) * g = \frac{1}{2}\left[(\partial_h g_t)\delta_{j,k} + (\partial_k g_t)\delta_{j,h}\right] * g = F^{(1)}_{j;h,k}(\cdot, t+1)$$

and $F^{(2)}_{j;h,k}(\cdot,t) * g = \displaystyle\int_t^\infty \partial_j\partial_h\partial_k g_{s+1}(x)\,ds = \int_{t+1}^\infty \partial_j\partial_h\partial_k g_s(x)\,ds = F^{(2)}_{j;h,k}(\cdot,t+1).$

□

Let us finally recall a classical estimate of the L^1 norm of the kernel.

Lemma 2.5 *There exists a constant $C > 0$ such that*

$$\forall t > 0, \qquad \|F(\cdot,t)\|_{L^1} \leq C\,t^{-1/2}. \tag{19}$$

Proof. This follows from (17b).

□

2.2 Decomposition of the non-linear term

Theorems 1.2 and 1.7 rely on a suitable decomposition of the non-linear term. A similar decomposition has been previously used by M. Schonbek [23] to prove lower bounds on the large-time decay of the L^2-norm of the flow. This part of the computations is common to both proofs.

Let us first explain the decomposition on a gaussian non-linearity. If g denotes the standard gaussian function and g_t the fundamental solution of the heat equation, one sets :

$$g_t^2(x) = \left(\int_{\mathbb{R}^d} g_t(y)^2 dy\right) g + \triangle(x,t).$$

The remainder $\triangle(x,t)$ has a mean value of zero :

$$\int_{\mathbb{R}^d} \triangle(x,t)dx = 0.$$

For fixed $x \in \mathbb{R}^d$, this approximation scheme behaves badly if $t \to 0$ or $t \to +\infty$; indeed, a simple computation leads to

$$\triangle(x,t) = \left\{1 - \left(\frac{t}{2}\right)^{d/2} e^{(2-t)|x|^2/4t}\right\} g_t^2(x).$$

But this computation also shows that such approximation scheme is satisfactory at least when

$$t \simeq 2 \quad \text{or} \quad |x|^2 \simeq 2d\,\frac{t}{t-2}\ln\frac{t}{2}\,,$$

i.e. when $\triangle(x,t)$ is close to zero. We now perform a similar decomposition for the non-linearity in (NS).

Now let a be a L^∞_ϑ divergence-free vector field and $u \in \mathcal{C}_w\big([0,T]; L^\infty_{\underline{\vartheta}}\big)$ be the solution of (NS) given by Theorem 1.1, starting from these initial data. Recall that $\underline{\vartheta} = \min\{\vartheta; d+1\}$. Since $\vartheta > \frac{d}{2}$, one has $L^\infty_{\underline{\vartheta}} \subset L^2$ and hence the energy matrix

$$\mathcal{E}_{h,k}(t) = \int_{\mathbb{R}^d} (u_h u_k)(y,t)\, dy$$

is well defined. Consistently with the preceding approximation scheme, let us define $v_{h,k}$ by

$$(u_h u_k)(x,t) = \mathcal{E}_{h,k}(t) g(x) + v_{h,k}(x,t). \tag{20}$$

Thanks to Lemma 2.4, the integral equation (16) is hence equivalent to :

$$u_j(t) = e^{t\Delta} a_j - \sum_{h,k} \int_0^t \mathcal{E}_{h,k}(s)\, F_{j;h,k}(t+1-s)\, ds - \sum_{h,k} \int_0^t v_{h,k}(s) * F_{j;h,k}(t-s)\, ds.$$

The time-independent asymptotic (17b) of the kernel $F_{j;h,k}$ now leads to

$$u_j(x,t) = e^{t\Delta} a_j(x) + \frac{P_j(x,t)}{|x|^{d+4}} + R_j(x,t), \tag{21a}$$

where P_j is given by

$$P_j(x,t) = \gamma_d \sum_{h,k} \big((d+2) x_j x_h x_k - |x|^2 \sigma_{j,h,k}(x)\big)\, K_{h,k}(t), \tag{21b}$$

with $\sigma_{j,h,k}(x) = \delta_{j,h} x_k + \delta_{j,k} x_h + \delta_{h,k} x_j$. The remainder

$$R_j(x,t) = -\sum_{h,k} \Big(R^{(1)}_{j;h,k}(x,t) + R^{(2)}_{j;h,k}(x,t) \Big)$$

is given by :

$$R^{(1)}_{j;h,k}(x,t) = |x|^{-(d+1)} \int_0^t \mathcal{E}_{h,k}(t-s)\, \Psi_{j;h,k}\left(\frac{x}{\sqrt{s+1}}\right) ds, \tag{21c}$$

$$R^{(2)}_{j;h,k}(x,t) = \int_0^t v_{h,k}(s) * F_{j;h,k}(t-s)\, ds. \tag{21d}$$

The functions $\Psi_{j;h,k}$ are given by (17b).

Remark 2.6 The above remainder is not small when $|x| \le \sqrt{t}$. As the solution $u(x,t)$ is smooth at least for small $t > 0$, the homogeneous polynomial and the remainder have to behave in exactly anti-symmetrical ways when $|x| \to 0$. The same compensation also occurs for a.e. $x \in \mathbb{R}^d$ when $t \to 0$.

The polynomial profile $\widetilde{P}(x,t) = |x|^{-d-4} P(x,t)$ has no vorticity, *i.e.* the matrix

$$\operatorname{rot} \widetilde{P} = \big(\partial_i \widetilde{P}_j - \partial_j \widetilde{P}_i\big)_{i,j}$$

is identically zero. This means that the polynomial profile is a gradient vector field. In fact, one may check immediately that :

$$\frac{P(x,t)}{|x|^{d+4}} = \nabla \Pi \qquad \text{with} \qquad \Pi(x,t) = \gamma_d \left(\frac{\operatorname{Tr} K(t)}{d\,|x|^d} - \sum_{h,k} \frac{x_h x_k}{|x|^{d+2}} \cdot K_{h,k}(t) \right). \tag{22}$$

2.3 General bounds of the remainder terms $R_{j;h,k}$

Let us now compute some upper bounds of the remainder terms. This second part of the proof is also shared by Theorem 1.2 and 1.7.

Bound of $R^{(1)}_{j;h,k}(x,t)$. The bound (17a) gives :

$$\Psi_{j;h,k}\Big(\frac{x}{\sqrt{4(s+1)}}\Big) \le C \exp\Big(-\frac{c\,|x|^2}{4(s+1)}\Big),$$

hence

$$|R^{(1)}_{j;h,k}(x,t)| \le C\,|x|^{-d-1} \exp\Big(-\frac{c\,|x|^2}{4(t+1)}\Big) \int_0^t \|u(s)\|^2_{L^2}\, ds. \tag{23}$$

Bound of $R^{(2)}_{j;h,k}(x,t)$. Since $\int_{\mathbb{R}^d} v_{h,k}(x,s)\,dx = 0$, the second remainder can also be written :

$$R^{(2)}_{j;h,k}(x,t) = \int_0^t \int_{\mathbb{R}^d} v_{h,k}(y,s)\ \big(F_{j;h,k}(x-y,t-s) - F_{j;h,k}(x,t-s)\big)\, ds\, dy.$$

The Taylor formula gives :

$$\begin{aligned}|R^{(2)}_{j;h,k}(x,t)| \le& \int_0^t \int_{|y|\le|x|/2} |y|\,|v_{h,k}(y,s)| \sup_{|z|\le|x|/2} |\nabla F_{j;h,k}(x+z,t-s)|\, ds\, dy\\ &+ \int_0^t \Big(\int_{|y|\ge|x|/2} |v_{h,k}(y,s)|\, dy\Big) |F_{j;h,k}(x,t-s)|\, ds\\ &+ \int_0^t \int_{|y|\ge|x|/2} |v_{h,k}(y,s)|\,|F_{j;h,k}(x-y,t-s)|\, ds\, dy.\end{aligned} \tag{24}$$

Thanks to (17a)–(17b), one has $|\nabla F_{j;h,k}(x,t)| \le C|x|^{-(d+2)}$ uniformly for $t>0$. Applying (19) as well, we get :

$$\begin{aligned}|R^{(2)}_{j;h,k}(x,t)| \le C\bigg(&\Big(\int_0^t\int_{|y|\le|x|/2} |y|\,|v_{h,k}(y,s)|\, dy\, ds\Big)|x|^{-(d+2)}\\ &+ \Big(\int_0^t\int_{|y|\ge|x|/2} |v_{h,k}(y,s)|\, dy\, ds\Big)|x|^{-(d+1)}\\ &+ \int_0^t (t-s)^{-1/2} \sup_{|y|\ge|x|/2} |v_{h,k}(y,s)|\, ds.\end{aligned} \tag{25}$$

To conclude the proofs of Theorem 1.2 and 1.7, we shall now use the assumptions on u to estimate (23) and (25).

2.4 Local-in-time solutions. Proof of Theorem 1.2

The goal of this section is to get upper bounds of the above remainders that provide valuable information for short time. In particular, in view of the proof of the lower bounds (11), it is of interest to have information on the behavior as $t \to 0$ of the last term appearing in the right-hand side of (2).

The remainder $R^{(1)}_{j;h,k}$ satisfies

$$|R^{(1)}_{j;h,k}(x,t)| \le \frac{C\,(t+1)^{1/2}}{|x|^{\min\{2\vartheta\,;\,d+2\}}} \int_0^t \|u(s)\|^2_{L^2}\, ds. \tag{26}$$

Indeed, if $\frac{d+1}{2} < \vartheta \le \frac{d+2}{2}$, one has

$$\exp\left(-\frac{c\,|x|^2}{4(t+1)}\right) \le C'|x|^{d+1-2\vartheta}(t+1)^{\vartheta-\frac{d+1}{2}}$$

and if $\vartheta \ge \frac{d+2}{2}$, one also has

$$\exp\left(-\frac{c\,|x|^2}{4(t+1)}\right) \le C'|x|^{-1}(t+1)^{1/2}.$$

In both cases, our estimates can blow up as $t \to \infty$, but not faster than $(1+t)^{1/2}$.

To deal with the remainder $R^{(2)}_{j;h,k}$, one may notice that the definition of v implies :

$$|v_{h,k}(y,s)| \le |u(y,s)|^2 + \|u(s)\|_2^2\, g(y) \le C(1+|y|)^{-2\underline{\vartheta}}\, \|u(s)\|^2_{L^\infty_{\underline{\vartheta}}}$$

with $\underline{\vartheta} = \min\{\vartheta\,;\,d+1\}$. Therefore, since $2\underline{\vartheta} > d+1$:

$$\begin{aligned}|R^{(2)}_{j;h,k}(x,t)| \le C\,|x|^{-d-2} &\int_0^t \|u(s)\|^2_{L^\infty_{\underline{\vartheta}}}\, ds \\ &+ |x|^{-1-2\underline{\vartheta}} \left(\int_0^t \|u(s)\|^2_{L^\infty_{\underline{\vartheta}}}\, ds\right) \\ &+ (1+|x|)^{-2\underline{\vartheta}} \int_0^t \|u(s)\|^2_{L^\infty_{\underline{\vartheta}}}\, (t-s)^{-1/2}\, ds,\end{aligned}$$

and hence for $|x| \ge 1$:

$$|R^{(2)}_{j;h,k}(x,t)| \le \frac{C\,(t+\sqrt{t})}{|x|^{\min\{2\vartheta\,;\,d+2\}}}\,\sup_{s\le t}\|u(s)\|^2_{L^\infty_{\underline{\vartheta}}}\,. \tag{27}$$

This ends the proof of (2). To obtain an asymptotic profile for the pressure, we need the following simple result on the localization of the derivatives.

Proposition 2.7 *Given $u \in L^\infty([0,T];L^\infty_\vartheta)$ a solution of the Navier-Stokes system with Cauchy datum $a = u(0)$ and $0 \le \vartheta \le d+1$. If, for some index i, one has $\partial_i a \in L^\infty_\vartheta$, then :*

$$\partial_i u \in L^\infty([0,T];L^\infty_\vartheta). \tag{28}$$

If, moreover, $\partial_i a \in L^\infty_\vartheta$ and $\partial_i\partial_j a \in L^\infty_\vartheta$ holds for all $i,j \in \{1,\dots,d\}$, then

$$\partial_i\partial_j u \in L^\infty([0,T];L^\infty_\vartheta) \tag{29}$$

and

$$t^{1/2}\partial_t u \in L^\infty([0,T];L^\infty_\vartheta). \tag{30}$$

Proof. Let us first deal with the first order spatial derivatives. Taking the i-th derivative in (16) leads to the affine fixed point problem :

$$\partial_i u = \Theta(\partial_i u)$$

with $\Theta = (\Theta_1,\dots,\Theta_d)$ and

$$\Theta_j w = e^{t\Delta}(\partial_i a_j) - 2\sum_{h,k}\int_0^t F_{j;h,k}(t-s) * (u_h w_k)(s)\, ds.$$

Proposition 3 of [24] implies that Θ is a continuous operator on $X = L^\infty([0,T_0];L^\infty_\vartheta)$, $0 \le T_0 < T$ and that

$$\left\|\Theta(w-w')\right\|_X \le C_0\, T_0^{\frac12} \sup_{t\in[0,T]} \|u(t)\|_{L^\infty_\vartheta} \left\|w-w'\right\|_X .$$

One may therefore choose $T_0>0$ such that Θ is a contraction of the Banach space X. Its only fixed point $w=\partial_i u$ belongs therefore to this function space. The same argument also holds on $[T_0,2T_0]$,... and leads finally to $\partial_i u \in L^\infty([0,T];L^\infty_\vartheta)$.

Conclusion (29) also follows from the contraction mapping theorem in a similar way.

For the time derivative, the starting point is again an identity that directly follows from (16), namely,

$$\partial_t u = \tilde\Theta(\partial_t u),$$

with

$$\tilde\Theta_j(w) = \Delta a_j - \sum_{h,k} F_{j;h,k}(t) * (a_h a_k) - 2\sum_{h,k}\int_0^t F_{j;h,k}(s) * (u_h w_k)(t-s)\,ds.$$

The Banach space we deal with is

$$Y = \{w\,;\, t^{1/2}\,\|w(t)\|_{L^\infty_\vartheta} \in L^\infty([0,T_0])\}.$$

Proposition 3 of [24] implies now that

$$-\Delta a_j + \sum_{h,k} F_{j;h,k}(t) * (a_h a_k) \in Y$$

and

$$\left\|\tilde\Theta(w-w')\right\|_Y \le \pi C_0\, T_0^{\frac12} \sup_{t\in[0,T]} \|u(t)\|_{L^\infty_\vartheta} \left\|w-w'\right\|_Y .$$

Here, we have used the fact that :

$$\forall t>0, \qquad \int_0^t \frac{ds}{\sqrt{s(t-s)}} = \pi.$$

The conclusion now follows in the same lines as above. □

We can now establish (4) : The pressure is defined up to an arbitrary function of t by :

$$-\nabla p = (\partial_t - \Delta)u + \operatorname{div}(u\otimes u).$$

Let us now replace u by the its profile given by (21a), that is $u = e^{t\Delta}a + \nabla\Pi + R$. One gets :

$$-\nabla p = \nabla(\partial_t-\Delta)\Pi + (\partial_t-\Delta)R + \operatorname{div}(u\otimes u).$$

This yields :

$$p(x,t) = p_0 - \gamma_d\left(\frac{\operatorname{Tr}\mathcal{E}(t)}{d\,|x|^d} - \sum_{h,k}\frac{x_h x_k}{|x|^{d+2}}\cdot\mathcal{E}_{h,k}(t)\right) + q(x,t), \tag{31}$$

where the remainder term $q(x,t)$ satisfies

$$-\nabla q = -\nabla\Delta\Pi + (\partial_t-\Delta)R + \operatorname{div}(u\otimes u). \tag{32}$$

Let us show that, for all $t>0$, we have $\nabla q = \mathcal{O}_t(|x|^{-\min\{2\vartheta,d+2\}})$.

– We obviously have $\nabla\Delta\Pi = \mathcal{O}_t(|x|^{-\min\{2\vartheta,d+2\}})$, since the left hand side is a homogeneous function of degree $-d-3$ which is smooth for $x \neq 0$.

– The term $\mathrm{div}(u \otimes u) = (u \cdot \nabla)u$ belongs uniformly to $L^\infty_{2\vartheta}$ because of (28).

– The remainder is the sum of two terms: one checks immediately that $(\partial_t - \Delta)R^{(1)}$ is exponentially decaying as $|x| \to \infty$. The second term is

$$(\partial_t - \Delta)R^{(2)}_{j;h,k}(x,t) = v_{h,k}(0) * F_{j;h,k}(t) + \int_0^t (\partial_t - \Delta)v_{h,k}(t-s) * F_{j;h,k}(s)\,ds$$

where $v_{h,k}(x,t) = u_h u_k - \mathcal{E}_{h,k}(t)\,g(x)$ and $t^{1/2}\partial_t v_{h,k}$ belongs to $L^\infty([0,T]; L^\infty_{2\vartheta})$.

We now use again that $\displaystyle\int v_{h,k}(0)\,dx = 0$: if we apply Lemma 2.5, the computation (25) shows that the first term is bounded in $t^{-1/2}L^\infty([0,T]; L^\infty_{\min\{2\vartheta;d+2\}})$. On the other hand, $\partial_t v_{h,k}$ and $\Delta v_{h,k}$ also have a vanishing integral. Therefore, using the estimates on the space-time derivatives provided by Proposition 2.7 shows that the second term belongs to $L^\infty([0,T]; L^\infty_{\min\{2\vartheta;d+2\}})$.

Hence we get $\nabla q \in \mathcal{O}_t(|x|^{-\min\{2\vartheta,d+2\}})$. Our last step is the following elementary estimate :

Lemma 2.8 *Let $\alpha > 1$ and $f \in C^1(\mathbb{R}^d)$ such that $\nabla f \in L^\infty_\alpha$. Then there is a constant c such that $f - c \in L^\infty_{\alpha-1}$.*

Proof. For any $\omega \in \mathbb{R}^d$, $|\omega| = 1$, let $\ell_\omega \equiv \lim_{r\to\infty} f(r\omega) = f(0) + \displaystyle\int_0^\infty \nabla f(s\omega)\cdot\omega\,ds$. If $\tilde\omega$ is another point of the unit sphere then for all $r > 0$ we have

$$|\ell_\omega - \ell_{\tilde\omega}| \le \int_r^\infty |\nabla f(s\omega)|\,ds + Cr \sup_{|x|\ge r} |\nabla f(x)| + \int_r^\infty |\nabla f(s\tilde\omega)|\,ds.$$

Letting $r \to \infty$ we get that $c \equiv \ell_\omega$ is independent of ω. But

$$|f(r\omega) - c| \le \int_r^\infty |\nabla f(s\omega)|\,ds \le C(1+r)^{-\alpha+1}$$

and the conclusion follows. $\square$

The standard properties of strong solutions imply that $q(x,t)$ is smooth for $x \neq 0$. Applying this lemma (for fixed t, $0 < t < T$), with $f(x) = \chi(x)q(x,t)$, where χ is a smooth function such that $\chi(x) \equiv 0$ for $|x| \le r$ and $\chi \equiv 1$ for $|x| \ge r'$ for some $0 < r < r'$ implies that, $q(x,t) = c + \mathcal{O}_t(|x|^{-\min\{2\vartheta-1,d+1\}})$. This completes the proof of Theorem 1.2. $\square$

For later use, let us note explicitly that if $R(x,t) = \mathcal{O}_t(|x|^{-\min\{2\vartheta,d+2\}})$ denotes the last term in the right hand side of (2), then we have proved that, for all $0 \le t \le T$,

$$|R(x,t)| \le \frac{C(\sqrt{t}+t)\,\|a\|_{L^\infty_\vartheta}}{|x|^{\min\{2\vartheta,d+2\}}}. \tag{33}$$

2.5 Global-in-time solutions. Proof of Theorem 1.7

Let us now focus on long time asymptotics. Let u be a global solution satisfying (6). Going back to (21a), we see that the profile (7) holds with

$$E_j(y,t) = -\sum_{h,k} \int_0^t \int_{\mathbb{R}^d} (u_h u_k)(y,t-s)\, \Psi_{j;h,k}\left(\sqrt{\frac{t+1}{s+1}}\, y\right) dz\, ds$$

and

$$\mathcal{R}_j(x,t) = -\sum_{h,k} R^{(2)}_{j;h,k}.$$

The bound (17a) immediately implies (8). To prove (9), we start by observing that assumption (6) implies :

$$\|u(t)\|^2_{L^2} \le C_\epsilon\,(1+t)^{-\left(\vartheta-\frac{d}{2}-\epsilon\right)}$$

for any $\epsilon>0$ and therefore, letting $v=(v_{h,k})$,

$$|v(y,s)| \le |u(y,s)|^2 + \|u(s)\|_2^2\, g(y) \le C\,(1+|y|)^{-2\alpha}\,(1+s)^{-(\vartheta-\alpha)}$$

for $d/2<\alpha\le\vartheta$.

A consequence of (17b) is that, for all $0\le\beta\le d+1$ and $0\le\gamma\le 1$

$$\begin{aligned}
&|F_{j;h,k}(x,t)| \le C|x|^{-\beta} t^{-(d+1-\beta)/2},\\
&|\nabla F_{j;h,k}(x,t)| \le |x|^{-(d+1+\gamma)} t^{-(1-\gamma)/2}.
\end{aligned}$$

Therefore, coming back to (24) we get

$$\begin{aligned}
|\mathcal{R}(x,t)| \le{}& C\biggl(\int_0^t\int_{|y|\le|x|/2} (1+|y|)^{1-2\alpha}(1+s)^{-\vartheta+\alpha}(t-s)^{-\frac12+\frac{\gamma}{2}}\,dy\,ds\biggr)|x|^{-(d+1+\gamma)}\\
&+C\biggl(\int_0^t\int_{|y|\ge|x|/2}(1+|y|)^{-2\alpha}(1+s)^{-\vartheta+\alpha}|x|^{-\beta}(t-s)^{-\frac{d+1-\beta}{2}}\,dy\,ds\biggr)\\
&+C\int_0^t (t-s)^{-1/2}(1+|x|)^{-2\alpha}(1+s)^{-\vartheta+\alpha}\,ds,
\end{aligned}\tag{34}$$

where, in the last integral, we have also used Lemma 2.5.

Let us call I_1, I_2 and I_3 the three terms of the right-hand side. To estimate I_1, we fix a small $\epsilon>0$ and choose $\alpha=\frac{d+1}{2}+\epsilon$. Then we write $I_1=I_{1,1}+I_{1,2}$, where these two terms are obtained by splitting the integral $\int_0^t$ into $\int_0^{t/2}$ and $\int_{t/2}^t$. Then we have, for all $t\ge1$,

$$I_{1,1}\le C_\gamma\,|x|^{-(d+1+\gamma)}\cdot\begin{cases} t^{-\frac12+\frac{\gamma}{2}}, & \text{if } \vartheta>\frac{d+3}{2}\\ t^{\frac{d+2+\gamma}{2}-\vartheta} & \text{if } \frac{d+1}{2}<\vartheta\le\frac{d+3}{2}\end{cases}$$

and

$$I_{1,2}\le C_{\gamma,\epsilon}\,|x|^{-(d+1+\gamma)}t^{\frac{d+2+\gamma}{2}+\epsilon-\vartheta}.$$

Thus,

$$I_1\le C_{\gamma,\epsilon}\,|x|^{-(d+1+\gamma)}\cdot\begin{cases} t^{-\frac12+\frac{\alpha}{2}}, & \text{if } \vartheta>\frac{d+3}{2},\\ t^{\frac{d+2+\alpha}{2}-\vartheta+\epsilon}, & \text{if } \frac{d+1}{2}<\vartheta<\frac{d+3}{2},\end{cases}\tag{35}$$

To estimate I_2, we choose again $\alpha=\frac{d+1}{2}+\epsilon$ and $\beta=d+\gamma$. Then the same argument as before shows that I_2 can be bounded as in (35). To estimate I_3, we take $\alpha=\frac{d+1+\gamma}{2}$. This choice shows that I_3 is also bounded by the function on the right hand side of (35). Summing all these bounds completes the proof of Theorem 1.7.

□

2.6 Criterion for the vanishing of $\nabla\Pi$ – Proof of Proposition 1.6

We now prove Proposition 1.6, which we restate in a more complete form.

Proposition 2.9 *For any real matrix $K = (K_{h,k})$, let us define a family of homogeneous polynomials by*

$$Q_j(x) = \sum_{h,k} \left(|x|^2 \sigma_{j,h,k}(x) - (d+2) x_j x_h x_k\right) K_{h,k}. \tag{36}$$

The following assertions are equivalent :

1. *The matrix is proportional to the identity matrix, i.e.*
$$\forall h, k \in \{1, \dots, d\}, \qquad K_{h,k} = \alpha\, \delta_{h,k} \tag{37}$$
with $\alpha = \frac{1}{d} \operatorname{Tr} K$.
2. *$Q_j \equiv 0$ for all indices $j \in \{1, \dots, d\}$.*
3. *There exists an index $j \in \{1, \dots, d\}$ such that $Q_j \equiv 0$.*
4. *There exists an index $j \in \{1, \dots, d\}$ such that $\partial_j Q_j \equiv 0$.*

Putting the terms $x_j x_\ell^2$ in factor in (36), one gets the following expression for the j^{th} component of Q :

$$\begin{aligned} Q_j(x) = x_j \sum_{\ell=1}^{d} \left\{ \operatorname{Tr} K - d K_{\ell,\ell} + 2(K_{j,j} - K_{\ell,\ell}) \right\} x_\ell^2 \\ + 2|x|^2 \tilde{K}(\mathbb{E}_j, x) - (d+2) x_j \tilde{K}(x,x) \end{aligned}$$

where $\mathbb{E}_i = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the canonical basis of $\mathbb{R}^d$ and $\tilde{K}$ is the bilinear form defined by the non-diagonal coefficients of K :

$$\tilde{K}(u,v) = \sum_{h \neq k} K_{h,k} u_h v_k.$$

Relations (37) express the fact that the matrix $K = (K_{h,k})_{1 \le h,k \le d}$ is a scalar multiple of the identity matrix. In such a case, one can immediately check on the previous expression that $Q_j(x) = 0$.

Let us prove conversly that $\partial_j Q_j \equiv 0$ implies $K = \alpha \operatorname{Id}$. One has :

$$\begin{aligned} \partial_j Q_j(x) = \sum_{\ell=1}^{d} \left\{ (1 + 2\delta_{j,l})(\operatorname{Tr} K - d K_{\ell,\ell}) + 2(K_{j,j} - K_{\ell,\ell}) \right\} x_\ell^2 \\ - 2d x_j \tilde{K}(\mathbb{E}_j, x) - (d+2) \tilde{K}(x,x). \end{aligned}$$

The fact that $\partial_j Q_j(\mathbb{E}_i) = 0$ for all i implies

$$\forall \ell \in \{1, \dots, d\}, \qquad (1 + 2\delta_{j,l})(\operatorname{Tr} K - d K_{\ell,\ell}) + 2(K_{j,j} - K_{\ell,\ell}) = 0$$

and hence $K_{i,i} = \frac{1}{d} \operatorname{Tr} K$ $(i = 1, \dots, d)$, *i.e.* all the diagonal entries of K are equal. Therefore :

$$\partial_j Q_j(x) = -2d x_j \tilde{K}(\mathbb{E}_j, x) - (d+2) \tilde{K}(x,x)$$

and this expression should vanish identically. A new derivation with respect to x_j gives

$$\partial_j^2 Q_j = -4(d-1) \tilde{K}(\mathbb{E}_j, x) = 0,$$

i.e. $\tilde{K}(\mathbb{E}_j, x) = 0$ as $d \ge 2$, and hence $\tilde{K}(x,x) \equiv 0$. This proves that the matrix K is a scalar multiple of the identity matrix.

□

3 Applications

Let us now explore a few consequences of the above results.

3.1 Instantaneous spreading property

It is a consequence of the result of [3] that, if the components of the initial data have no special symmetries, then the corresponding solution $u(x,t)$ of (NS) satisfies

$$\liminf_{R\to\infty} R \int_{R\le|x|\le 2R} |u(x,t)|\,dx > 0\,, \tag{38}$$

for $t > 0$ belonging at least to a sequence of points t_k converging to zero as $k \to \infty$. In particular for those t, one has :

$$\int_{\mathbb{R}^d} |x|\,|u(x,t)|\,dx = +\infty \qquad \text{and} \qquad \int_{\mathbb{R}^d} |x|^{d+2}|u(x,t)|^2\,dx = +\infty.$$

(See also, *e.g.*, [19], Theorem 25.2). The precise assumption guaranteeing (38) is the non-orthogonality of the components with respect to the L^2-inner product, *i.e.* one can find $j \neq k$ in $\{1,\dots,d\}$ such that

$$\int_{\mathbb{R}^d} a_j(x)\,a_k(x)\,dx \neq 0 \tag{39a}$$

or such that

$$\int_{\mathbb{R}^d} a_j^2(x)\,dx \neq \int_{\mathbb{R}^d} a_k^2(x)\,dx. \tag{39b}$$

Even if (38) already explains that the limitation $\vartheta \le d+1$ in Theorem 1.1 is optimal for generic flows, such a condition does not provide much information on the pointwise decay of u, as $|x| \to \infty$.

Theorem 1.2 and Proposition 1.6 not only provide such information, but also allow us to give a simpler proof of these facts. The instantaneous spreading property is fully described by the following result.

Theorem 3.1 *For $\vartheta > d+1$, let $a \in L^\infty_\vartheta$ be a divergence-free vector field. Let u be the corresponding solution of* (NS) *in $\mathcal{C}_w\big([0,T]; L^\infty_{d+1}\big)$. For $0 < t \le T$, we set*

$$\kappa_t = \max\{1, t^{-1/2}, t^{-1/(\vartheta-d-1)}\}.$$

1. *There is a constant $c > 0$ such that, for $0 < t \le T$ and $|x| \ge c\kappa_t$:*
$$|u(x,t)| \le c\,t\,|x|^{-(d+1)}. \tag{40}$$
2. *Conversely, if* (39) *holds for a couple of indices (j,k), then there exists $t_0 \in (0,T]$ and a constant $c' > 0$ such that for all $0 < t \le t_0$ and all x in a conic neighborhood of the x_j or x_k axis, with $|x| \ge c\kappa_t$:*
$$|u_j(x,t)| \ge c'\,t\,|x|^{-(d+1)}. \tag{41}$$
3. *Actually, if* (39) *holds, the lower bound* (41) *holds in almost all directions : the set*
$$\Sigma = \left\{ \sigma \in \mathbb{S}^{d-1}\,;\, \liminf_{\substack{t\to 0^+\\ |x|\to\infty\\ x\in\mathbb{R}\sigma}} \Big(t^{-1}\,|x|^{d+1}\,|u_j(x,t)|\Big) = 0 \right\} \tag{42}$$
is a closed subset of the sphere $\mathbb{S}^{d-1}$, of measure zero.

Remark 3.2 The non-orthogonality assumption (39) cannot be removed. Indeed, [3] contains an example of symmetric and highly localized flow for which (38) and (41) break down. For a further understanding of how symmetries usually lead to highly localized flows, one may also refer to [16] or [2].

The previous results show that, for generic flows, *each component* of the velocity field instantaneously spreads out in *any direction*. In particular, one has the following corollary.

Corollary 3.3 *Under the same assumptions as in Theorem 3.1 there exist a time $t_0 > 0$ and two positive constants c_0, C such that for all $t \in (0, t_0]$:*

$$\int_{R\le|x|\le 2R} |u_j(x,t)|\,dx \ge \frac{c_0 t}{R} \tag{43}$$

for all $R \ge C\, t^{-1/\min\{2\,;\,\vartheta-d-1\}}$. In particular, for $j,k = 1,\ldots,d$,

$$\int_{\mathbb{R}^d} |x_k|^{\vartheta p} |u_j(x,t)|^p\,dx = +\infty, \tag{44}$$

as soon as $1 \le p < +\infty$ and $\vartheta + \frac{d}{p} \ge d+1$.

Remark 3.4 The spreading property (44) should be compared with F. Vigneron's result [24], according to which

$$\int_{\mathbb{R}^d} (1+|x|)^{\vartheta p} |u(x,t)|^p\,dx \le C_t \int_{\mathbb{R}^d} (1+|x|)^{\vartheta p} |a(x)|^p\,dx$$

as long as the solution exists in $L^p(\mathbb{R}^d)$, provided that $p > d$ and $\vartheta + \frac{d}{p} < d+1$.

Proof. First, let us establish the upper bound (40). Theorem 1.2, together with (33), immediately implies, for $|x| \ge 1$:

$$|u(x,t) - e^{t\Delta}a(x)| \le C\,t\,|x|^{-(d+1)} + C\sqrt{t}\,|x|^{-(d+2)} \le C\,t\,|x|^{-(d+1)},$$

for all $x \in \mathbb{R}^d$ such that $|x| \ge t^{-1/2}$. Moreover, if $|x| \ge (2t^{-1})^{1/(\vartheta-d-1)}$, one has :

$$|e^{t\Delta}a(x)| \le C|x|^{-\vartheta} \le \frac{C}{2} t|x|^{-d-1}. \tag{45}$$

This proves (40). Let us now focus on the lower bound (41) of $|u_j(x,t)|$.

Let $j \ne k$ such that $\alpha \equiv \int_{\mathbb{R}^d} (a_j a_k)\,dx \ne 0$. Then, for some $t_0 > 0$, possibly depending on a, and all $0 < t \le t_0$, we have

$$|K_{j,k}(t)| \ge \frac{|\alpha| t}{2}.$$

Let $\epsilon > 0$ and let $\Gamma_k = \{x\,;\, |x_r| < \epsilon |x_k| \;\; (r \ne k)\}$ be a conical neighborhood of the x_k-axis. Recalling (21b) we get, for ϵ small and R large enough :

$$\forall x \in \Gamma_k, \qquad |x| \ge R \quad \Longrightarrow \quad |P_j(x,t)| \ge \frac{|\alpha| t}{3} |x_k|^3.$$

Using (21a), (33) and the first of (45) now leads, for large enough $|x|$ and $x \in \Gamma_k$, to

$$|u(x,t)| \ge |u_j(x,t)| \ge \frac{|\alpha| t}{4} |x_k|^{-(d+1)} \ge \frac{|\alpha| t}{4} |x|^{-(d+1)}$$

and (41) follows in this case.

In the second case we choose $j \neq k$ such that $\int_{\mathbb{R}^d} a_j^2\,dx \neq \int_{\mathbb{R}^d} a_k^2\,dx$. Then we have

$$d\int_{\mathbb{R}^d} a_j^2\,dx \neq \sum_{m=1}^d \int_{\mathbb{R}^2} a_m^2\,dx$$

(otherwise, $d\int a_k^2\,dx \neq \sum_{m=1}^d \int_{\mathbb{R}^d} a_m^2\,dx$, and we should exchange j with k). Let us set

$$\beta = \sum_{m=1}^d \int_{\mathbb{R}^d} a_m^2\,dx - d\int_{\mathbb{R}^d} a_j^2\,dx.$$

Arguing as before, we see that there exists a conic neighborhood Γ_j of the x_j axis, such that for all $x \in \Gamma_j$ and $|x|$ large enough we have

$$|u(x,t)| \geq |u_j(x,t)| \geq \frac{|\beta| t}{4}|x_j|^{-(d+1)} \geq \frac{|\beta| t}{4}|x|^{-(d+1)}.$$

Then (41) follows in this second case as well.

Let us now prove the last statement of Theorem 3.1. The map $s \mapsto \displaystyle\int_{\mathbb{R}^d}(u_h u_k)(x,s)\,dx$ is continuous. Therefore, (21b) implies that

$$\forall x \in \mathbb{R}^d, \qquad \lim_{t\to 0}\frac{1}{t}P(x,t) = \mathcal{P}(x)$$

where $\mathcal{P} = (\mathcal{P}_1, \dots \mathcal{P}_d)$ is given by

$$\mathcal{P}_j(x) = \gamma_d \sum_{h,k}\left(\int_{\mathbb{R}^d} a_h a_k\right)\left((d+2)x_j x_h x_k - |x|^2 \sigma_{j,h,k}(x)\right),$$

which is a homogeneous polynomial of degree exactly three. According to Proposition 1.6, the assumption (39) means that $\mathcal{P}_j \not\equiv 0$ for all $j = 1, \dots, d$.

The convergence of $t^{-1}\,P(x,t)$ to $\mathcal{P}(x)$ is uniform when x belongs to the (compact) unit sphere. Let us define the following dense open subsets of $\mathbb{S}^{d-1}$:

$$\Omega_j = \{\omega \in \mathbb{S}^{d-1} \colon \mathcal{P}_j(\omega) \neq 0\}.$$

Given $\omega \in \Omega_j$, let us define $T_\omega > 0$ as the supremum of $t \leq T'$ such that

$$\frac{1}{t}|P_j(\omega,t)| \geq \frac{1}{2}|\mathcal{P}_j(\omega)|.$$

Also let $c_\omega = \frac{1}{4}|\mathcal{P}_j(\omega)|$.

From (21a), (33) and the obvious estimate $|e^{t\Delta}a(x)| \leq C|x|^{-\vartheta}$, we get, for $\omega = x/|x| \in \Omega_j$:

$$|u_j(x,t)| \geq 2c_\omega\, t\, |x|^{-(d+1)} - C\, t^{1/2}|x|^{-(d+2)} - C|x|^{-\vartheta} \geq c_\omega\, t\, |x|^{-(d+1)}$$

for all $0 \leq t \leq T_\omega$ and $|x| \geq C_\omega\, t^{-\vartheta^*}$. The complement of Ω_j is an algebraic surface and therefore the set $\Sigma = \mathbb{S}^{d-1}\backslash\Omega_j$ has measure zero in $\mathbb{S}^{d-1}$.

Finally, (43) and the corollary follow immediately from (41) and the fact that a function bounded from below, at infinity, by $|x|^{-d-1}$ does not belong to any weighted Lebesgue space $L^p(\mathbb{R}^d, (1+|x|)^{\vartheta p}\,dx)$ when $1 \leq p < \infty$ and $\vartheta + \frac{d}{p} \geq d+1$.

$\square$

Remark 3.5 The set of exceptional directions where, at some time $t > 0$, our lower bound (41) breaks down corresponds to the roots of the polynomial $P(\cdot, t)$ on the sphere $\mathbb{S}^{d-1}$. In dimension two we have a more precise description of such a set : it is made of at most *six directions*. This is due to the fact that a homogeneous polynomial in $\mathbb{R}^2$ of degree exactly three has at most six zeros on $\mathbb{S}^1$, as is easily checked by passing to polar coordinates. In principle, this conclusion is valid only during a short time interval $[0, t_0]$. Indeed we cannot exclude that after some time there are flows featuring some kind of creation of symmetry: it may happen that, for some time $t_1 > t_0$, $\int u_1(x, t_1)^2 \, dx = \int u_2(x, t_1)^2 \, dx$ and $\int (u_1 u_2)(x, t_1) \, dx = 0$, in a such way that the velocity field first instantaneously spreads out but then recovers a good localization at time t_1. Subsequently, the flow would remain localized, or spread out again, depending on whether the components u_1 and u_2 remain orthogonal or not, after t_1. However, no example of such a somewhat pathological flow is known so far.

3.2 Lower bounds of solutions in weighted spaces

Let us establish a few consequences of Theorem 1.7. Throughout this section we suppose $t \geq 1$. For well localized data, *e.g.*, when $a \in L^\infty_\vartheta$ with $\vartheta > \frac{d+2}{2}$, the limit

$$\nabla\Pi_\infty(x) \equiv \lim_{t\to\infty} \nabla\Pi(x, t)$$

is well defined. In this case, a consequence of (7) is that, for some $\beta > 0$,

$$\Big| u(x, t) - e^{t\Delta}a(x) - \nabla\Pi_\infty(x) \Big| \leq C|x|^{-d-1} e^{-|x|^2/(t+1)} + C|x|^{-d-1}\, t^{-\beta}. \tag{46}$$

This leads us to introduce, for all $A > 0$, the region

$$\mathcal{D}_A(t) = \{x \in \mathbb{R}^d \,;\, |x|^2 \geq A(t+1)\}.$$

Since, generically, $\nabla\Pi_\infty \not\equiv 0$, several lower bounds for the large time behavior of u can be obtained as an easy consequence of (46). For example, if we introduce the weighted norm

$$\|f\|_{L^p_\alpha} = \left(\int |f(x)|^p (1 + |x|)^{p\alpha} \, dx \right)^{1/p},$$

then, taking $A > 0$ large enough, for all $1 \leq p < \infty$ and $\alpha \geq 0$, such that

$$\alpha + \frac{d}{p} < d + 1, \tag{47}$$

we get, for all $t > 0$ large enough,

$$\begin{aligned} \big\| u(t) - e^{t\Delta}a \big\|^p_{L^p_\alpha} &\geq \int_{\mathcal{D}_A(t)} |u(x, t) - e^{t\Delta}a(x)|^p (1 + |x|)^{p\alpha} \, dx \\ &\geq \frac{1}{2} \int_{\mathcal{D}_A(t)} |\nabla\Pi_\infty(x)|^p (1 + |x|)^{p\alpha} \, dx \\ &\geq C\,(At)^{-\frac{p}{2}(d+1-\alpha-\frac{d}{p})}. \end{aligned} \tag{48}$$

If the datum is highly oscillating (for example, if the Fourier transform of a satisfies some suitable vanishing condition at the origin) then the L^p_ϑ norms of $e^{t\Delta}a$ decay faster as $t \to \infty$ than the right hand side of (48). Then, (48) will be in fact a lower bound for $\|u(t)\|^p_{L^p_\alpha}$ in this case. A similar conclusion remains true if we drop this assumption on the oscillations and start with a datum that is simply well localized. Indeed, we have the following :

Corollary 3.6 *Let u be as in Theorem 1.7, starting from $a \in L^\infty_\vartheta$, with $\vartheta > d+1$. We also assume that $\nabla\Pi_\infty \not\equiv 0$. Then there exist $t_0 > 0$ and a constant $c > 0$ such that, for all $1 \le p < \infty$ and $\alpha \ge 0$, satisfying* (47)*, we have, for all $t \ge t_0$:*

$$\|u(t)\|_{L^p_\alpha} \ge c\,t^{-\frac{1}{2}(d+1-\alpha-\frac{d}{p})}. \tag{49}$$

Moreover, for all $0 \le \alpha \le d+1$ and all $t \ge t_0$:

$$\|u(t)\|_{L^\infty_\alpha} \ge c\,t^{-\frac{1}{2}(d+1-\alpha)}. \tag{50}$$

The lower bound (49) was already known for $p = 2$ and $0 \le \alpha \le 2$ (see, *e.g.*, [23], [4]), or $1 \le p \le \infty$ and $\alpha = 0$ (see [8]). The decay profile (6), under the assumption of Corollary 3.6, immediately implies the (slightly weaker) upper bound $\|u(t)\|_{L^p_\alpha} \le c_\epsilon\, t^{-\frac{1}{2}(d+1-\alpha-\frac{d}{p}-\epsilon)}$ for all $\epsilon > 0$. In fact, the "sharp" upper bound (*i.e.* the bound with $\epsilon = 0$) has been obtained, at least for $p \ge 2$ and with some additional restrictions on α, by many authors (see [18] and the references therein).

Proof. By our assumptions, $a \in L^1(\mathbb{R}^d)$ and $\operatorname{div} a = 0$. Thus, $\displaystyle\int a(y)\,dy = 0$. A direct computation (using the same method as in the proof of (24)) then yields

$$|e^{t\Delta}a(x)| \le C(1+|x|)^{-\vartheta}\,(1+t)^{(\vartheta-d-1)/2}.$$

It then follows that, for $t \ge 1$,

$$\int_{\mathcal{D}_A(t)} |e^{t\Delta}a(x)|^p(1+|x|)^{\alpha p}\,dx \le C\,A^{-\frac{p}{2}(\vartheta-\alpha-\frac{d}{p})}\,t^{-\frac{p}{2}(d+1-\alpha-\frac{d}{p})}.$$

Here the exponent of A is strictly smaller than that of (48). If A is large enough, then a comparison between this inequality and (48) gives (49). The proof of (50) is essentially the same. $\square$

3.3 Flows with anisotropic decay in the whole space

This short section contains a positive and a negative result about flows in $\mathbb{R}^d$ with anisotropic decay at infinity.

Theorem 1.2 implies that (NS) flows may inherit the anisotropic decay properties of the initial data, as long as these properties do not violate the instantaneous spreading limit given by Theorem 3.1.

Proposition 3.7 *Let a be a bounded divergence-free vector field and u the corresponding solution of* (NS) *in $\mathcal{C}_w([0,T);L^\infty)$. Let us also assume that there exists a function m such that*

$$|e^{t\Delta}a(x)| \le C_t(1+|x|)^{-\vartheta}\,m(x)^{-1} \tag{51}$$

with $\frac{d+1}{2} < \vartheta \le d+1$ and $1 \le m(x) \le C(1+|x|)^{d+1-\vartheta}$. Then, for all $T' < T$ there exists a constant $C_{T'}$ (this also might depend on the data) such that :

$$|u(x,t)| \le C_{T'}(1+|x|)^{-\vartheta}\,m(x)^{-1} \tag{52}$$

for all $t \in [0;T']$.

Proof. This is an obvious consequence of (2) and (33). $\square$

Let us give some examples of anisotropic weights satisfying (51). A *Peetre-type weight* is a measurable function $m : \mathbb{R}^d \to [1; +\infty)$ such that

$$\exists C_0 > 0, \quad \forall x, y \in \mathbb{R}^d, \qquad m(x+y) \leq C_0\, m(x)m(y). \tag{53}$$

Common examples are (for $\alpha_i \geq 0$) :

$$m_1(x) = 1 + |x_1|^{\alpha_1} + \ldots + |x_d|^{\alpha_d} \qquad \text{and} \qquad m_2(x) = e^{\alpha|x|}.$$

The class of Peetre-type weights is stable by finite sums and products, translations and orthogonal transforms.

Lemma 3.8 *Let m be a Peetre-type weight such that $m(x) \leq C\exp(c|x|)$ and $T > 0$. Then, there is a constant $C_T > 0$ such that*

$$\left\| m(e^{t\Delta}a) \right\|_{L^\infty} \leq C_T \left\| ma \right\|_{L^\infty}. \tag{54}$$

Proof. It is an elementary computation :

$$\begin{aligned} m(x)\left| e^{t\Delta}a(x) \right| &\leq C_0 \left[(mg_t) * (m|a|) \right](x) \\ &\leq C_0\,(4\pi)^{-d/2} \left\| ma \right\|_{L^\infty} \int_{\mathbb{R}^d} m\big(\sqrt{t}\,y\big)\, e^{-y^2/4}\, dy. \end{aligned}$$

The conclusion follows from the bound $m\big(\sqrt{t}\,y\big) \leq C\exp(c\,T\,|y|)$. $\square$

As a converse to the previous result, the following property implies that highly localized flows cannot decay at infinity in a really anisotropic way.

Proposition 3.9 *Let $a \in L^\infty_{d+1+\varepsilon}$ be a divergence-free vector field with $0 < \varepsilon < 1$, and u the corresponding solution of* (NS) *in $\mathcal{C}_w\left([0,T); L^\infty_{d+1}\right)$. For some $t > 0$, let us assume that there exist an index $j \in \{1, \ldots, d\}$ and a subset $\Sigma \subset \mathbb{S}^{d-1}$ of positive measure such that*

$$\forall \sigma \in \Sigma, \qquad \lim_{\substack{|x| \to +\infty \\ x \in \mathbb{R}\sigma}} |x|^{d+1} u_j(x,t) = 0. \tag{55}$$

Then, there exists a constant $C > 0$ such that

$$|u_k(x,t)| \leq C(1+|x|)^{-(d+1+\varepsilon)} \tag{56}$$

for all $k = 1, \ldots, d$. Moreover, if (55) *holds for a finite time interval $t \in [T_0, T_1]$, then C may be chosen uniformly with respect to t.*

Proof. Our assumptions imply that the polynomial $P_j(x,t)$ identically vanishes. Proposition 1.6 then implies that all the other components of $P(x,t)$ also vanish. Our statement is once again a consequence of (2). $\square$

3.4 Application to the decay in a half-space domain

Our last application of Theorem 1.2 is the study of the decay of solutions of the Navier–Stokes equations in the half space

$$\mathbb{R}^d_+ = \{(x', x_d) \;:\; x' \in \mathbb{R}^{d-1},\, x_d > 0\}.$$

We set $u' = (u_1, \ldots, u_{d-1})$ and $x' = (x_1, \ldots, x_{d-1})$. Let $\{e^{-tA'}\}_{t\geq 0}$ be the semigroup generated by $-A' = \Delta$, in the case of the Neumann boundary conditions :

$$\partial_d u'|_{\partial \mathbb{R}^d_+} = 0, \qquad u_d|_{\partial \mathbb{R}^d_+} = 0. \tag{57}$$

where $\partial_d = \frac{\partial}{\partial x_d}$. The integral formulation of the Navier–Stokes system in $\mathbb{R}^d_+$ is

$$u(t) = e^{-tA'}a - \int_0^t e^{-(t-s)A'}\mathbb{P}\,\mathrm{div}(u\otimes u)(s)\,ds, \tag{58}$$

with $\mathrm{div}\, a = 0$. We refer to [9] for the construction of weak and strong solutions to (58).

We have the following result:

Proposition 3.10 *Assume that $a \in L^\infty_\vartheta(\mathbb{R}^d_+)$, with $\vartheta > \frac{(d+1)}{2}$. Then there exist $T > 0$ and a unique strong solution $u \in \mathcal{C}_w\left([0,T); L^\infty_{\underline{\vartheta}}(\mathbb{R}^d_+)\right)$ of (58). Such a solution satisfies*

$$u(x,t) = e^{t\Delta}a(x) + H(x,t) + \mathcal{O}_t\left(|x|^{-\min\{2\vartheta\,;\,d+2\}}\right), \tag{59}$$

where $H = (H_1, \ldots, H_d)$ is homogeneous of degree $-(d+1)$ for all $t \in [0,T)$, and such that:

$$|H_j(x,t)| \;\leq\; C|x'|\cdot|x|^{-(d+2)}, \qquad (1 \leq j \leq d-1) \tag{60}$$

$$|H_d(x,t)| \;\leq\; C|x_d|\cdot|x|^{-(d+2)}. \tag{61}$$

As an immediate consequence of Proposition 3.10, we obtain the following anisotropic decay estimates (assuming that a is well localized).

$$\begin{aligned}
u_d(x,t) &= \mathcal{O}\big(|x'|^{-(d+2)}\big), \quad u'(x,t) = \mathcal{O}\big(|x'|^{-(d+1)}\big), && \text{when } |x'| \to +\infty,\ x_d \text{ fixed}\\
u_d(x,t) &= \mathcal{O}\big(|x_d|^{-(d+1)}\big), \quad u'(x,t) = \mathcal{O}\big(|x_d|^{-(d+2)}\big), && \text{when } x_d \to +\infty,\ x' \text{ fixed.}
\end{aligned}$$

It is also worth noticing that Proposition 3.9 is not violated as the above decay holds only in a cylindrical region, and not in a conical one.

Proof. This is immediate. Indeed, the study of (58) is reduced to that of (NS) in the following way. If u solves (58), then one can construct a solution of (NS) in the whole $\mathbb{R}^d$, setting

$$\tilde{u}_j(x_1, \ldots, x_{d-1}, -x_d, t) = u_j(x_1, \ldots, x_{d-1}, x_d, t)$$

for $j = 1, \ldots, d-1$ and $\tilde{u}_d(x_1, \ldots, x_{d-1}, -x_d, t) = -u_d(x_1, \ldots, x_{d-1}, x_d, t)$ (see [9]). Then under the assumptions of Proposition 3.10 we can apply (2) to $\tilde{u}$. But the integrals $\tilde{K}_{j,d}(t) \equiv \int_0^t\int_{\mathbb{R}^d}(\tilde{u}_j\tilde{u}_d)(x,s)\,dx\,ds$ vanish, for $j \neq d$. Hence, from (21b) we see that $|P(x,t)|$ is bounded by a function $H(x,t)$ satisfying (60).

□

Conclusions

Theorem 1.2 provides a quite complete answer to the spatial decay problem of solutions of the free Navier–Stokes equations in the whole space, at least for well localized data. It would be interesting to know if some of the results of the present paper can be adapted to flows in other domains.

For example, in the half-space case, the Neumann boundary condition considered in the previous section is not the most interesting one, since it destroys the boundary layer effects. The construction of the asymptotics as in Theorem 1.2, in the case of Dirichlet boundary conditions, would require a careful analysis of Ukai's formula (or its more recent reformulations) for the Stokes semigroup.

In the case of stationary flows, asymptotic profiles have been given, *e.g.*, by F. Haldi and P. Wittwer [15], [26]. Their results model the wake flow beyond an obstacle. However, they do not deal with the obstacle itself, but with a half-plane domain and a technical boundary condition dictated by experimental knowledge.

For the non-stationary equation (NS) in $\mathbb{R}^3\backslash\Omega$ with Dirichlet boundary conditions on $\partial\Omega$, it seems reasonable to expect that anisotropic lower bound estimates for the decay of u should hold, when the net forces exerted by the fluid on the boundary, *i.e.*

$$\int_{\partial\Omega} \big(T[u,p]\cdot\nu\big)(y,t)\,dS_y$$

(where $T_{j,k}[u,p] = \partial_j u_k + \partial_k u_j - \delta_{j,k}p$ and $T[u,p] = (T_{j,k}[u,p])_{j,k}$ is the stress tensor) do not vanish. This last condition, which is motivated by the results of Y. Kozono [17] and C. He, T. Miyakawa [16] on the L^1-summability of solutions, would play, in the exterior domain case, a role equivalent to the non-vanishing criterion given by Proposition 1.6.

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