Research activity and projects

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The following is a shortened english version of my Habilitation thesis, "Propriétés qualitatives de solutions de quelques équations paraboliques semi-linéaires" *(December 2010, 76 pages, in french). The full Habilitation thesis and my papers are available at the following URL:*

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Introduction

My main research interests lie in the field of Partial Differential Equations, especially evolution nonlinear equations of parabolic type. The models I have been studying mainly arise from Fluid Mechanics and Geophysics, or from Mathematical Biology and Probability theory. Much of my work is devoted to a better understanding of the asymptotic properties of the solutions: the study of their behavior for large time and in the far-field under different perspectives (energy dissipation, time decay or growth, asymptotic profiles, selfsimilarity, spatial spreading, localization issues. . .), as well as the asymptotic analysis when a physical parameter inside the models goes to zero, represent my most important contribution.

It should be stressed that the Navier–Stokes equations constitute an outstanding challenge for all these issues. Not surprisingly, these equations take an important place in my work. A couple of my papers [12, 13], deal with functional analysis (in particular, with the theory of multipliers in Sobolev spaces) or harmonic analysis and approximation theory, in connection with wavelet bases. These two contributions to Analysis are quite independent from the rest of my scientific production, though they are not completely unrelated.

Indeed harmonic analysis is present throughout my work: the classical Fourier analysis (for example, the *Fourier splitting* method) often yields to sharp results on the long time behavior. The Littlewood-Paley analysis is a powerful tool for obtaining existence results. Moreover, the use of a large panel of function spaces (some of them considered as somewhat exhotic in the PDE community until recently) provides valuable information on the solutions. For example, Besov spaces with negative regularity can be used to measure their oscillating character, weighed spaces tell us about their localization; Morrey and Lorentz spaces are a good setting where to study self-similarity, etc. In addition, many physical models have a non-local nature: a perturbation of a quantity in some region of the space affects the whole system even far from that region. We usually attack these models by recasting the initial PDE as a pseudodifferential equation. The theory of singular integrals, well-suited for the study of non-local phenomena, will also play an important role.

Below, I will describe a selection of my results.

1 The Navier–Stokes equations

The following subsections present my main contributions to the study of the Navier–Stokes equations. In the case of a viscous incompressible fluid with constant density the motion can be described as follows:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + f \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0. \end{cases} \tag{NS}$$

Here $u = (u_1, \ldots, u_d)$ is the velocity field, $d \geq 2$, and the scalar p is the (unknown) pressure. The fluid is assumed here to fill the whole space $\mathbb{R}^d$. When more general domains are considered, suitable boundary conditions must be added to (NS). The vector field $f = (f_1, \ldots, f_d)$ is the given external forcing.

For simplicity, the physical parameters have been set equal to one. A suitable rescaling of the equations allows us to make this simplification.

The most important results go back to Leray in the thirties. Global weak solutions to (NS) do exist, but their uniqueness is known only in dimension two. When $d = 2$, solutions cannot develop singularities in finite time. In higher dimension $d \geq 3$ the situation is more open: global smooth solutions exist provided the data satisfy some smallness assumption (for example, a condition of the form $\|u_0\|_{L^d} < \epsilon$). But it is not known if one can get rid of any smallness condition in order to prove that solutions remain regular for all time.

1.1 The Role of the Symmetry in incompressible flows

[15] L. Brandolese
Space-time decay of Navier–Stokes flows invariant under rotations,
Math. Annal. **329**, 685--706 (2004).

[18] L. Brandolese, Y. Meyer
On the instantaneous spreading for the Navier–Stokes system in the whole space.
Cont. Optim. Calc. Var. **8**. **88**, 273--285 (2002).

Thoughout this subsection we assume $f \equiv 0$. We will undertake the study of non-zero external forces later on.

Because of their parabolic nature, the Navier–Stokes equations feature an infinite-speed propagation effect in the space variable. This phenomenon is usually described by the fact that compactly supported initial data give rise to solutions which immediately have non-compact support. On the other hand, because of the pressure, which can be eliminated from the equations only applying a singular integral operator, the solutions of the Navier–Stokes equations behave quite differently as $|x| \to \infty$ from those of the non-linear heat equations.

In 1993, S. Dobrokhotov and A. Schafarevich raised the problem of the existence of solutions (u_1, u_2, u_3) well-localized in $\mathbb{R}^3$ (for example, rapidly decaying as $|x| \to \infty$). Their question was motivated by the observation that all known solutions that were known at that time (for $d = 3$) were decaying at infinity at very slow rates. They also noticed[1] that an apparent obstruction arises

[1] S. Y. Dobrokhotov, A. I. Shafarevich, *Parametrix and asymptotic behavior of localized solutions of Navier-Stokes equations in* $\mathbf{R}^3$ *that are linearized on a smooth flow*, Math. Notes, **51**, (1992), 47–54. S. Y. Dobrokhotov, A. I. Shafarevich, *Some integral identities and remarks on the decay at infinity of the solutions to the Navier-Stokes equations in the entire space*, Russian J. Math. Phys., **2**, (1994), 133–135.

when one tries to construct solutions such that $\partial_t u$, u and $|x|\,p$ simultaneously decay fast at infinity. Another motivation was provided by an important question on the energy dissipation of the flow: indeed it has been understood for some time that there is a relationship between the spatial decay and the time decay of the energy $\|u(t)\|_2^2$.

In my Ph.D dissertation I discovered that the symmetries of the flow affect in an essential way the localization properties of the solution and its energy dissipation. First of all, the absence of symmetry turns out to be an obstruction for constructing fast decaying solution: indeed, we have the following result, obtained in collaboration with Yves Meyer (see [18] for a more a precise statement)

Any strong solution u of the Navier–Stokes equations in $\mathbb{R}^d$ (with $f \equiv 0$) such that

$$\lim_{R\to\infty} R \int_{R<|x|<2R} |u(x,t)|\,dx = 0, \qquad 0 \le t \le T, \tag{1}$$

must satisfy orthogonality conditions: for all $0 \le t \le T$, the components $u_1(\cdot,t),\ldots,u_d(\cdot,t)$ of the velocity field must be orthogonal with respect to the $L^2(\mathbb{R}^d)$-inner product..

In particular, if we want a solution such that $u(x,t) = o(|x|^{-d-1})$ at infinity, *even only in a weak sense*, then we must start with initial data u_0 satisfying the above orthogonality relation. Usually, obstructions to the spatial localization of the flow were expressed in terms of the pressure (for exemple, it is straightforward to see that for generic solution one cannot have $p \in L^1(\mathbb{R}^d)$). The main novelty of [18] is that the localization assumption only involves the velocity field (and not its derivatives), and is optimal.

The above conclusion should be compared with a theorem by T. Miyakawa (2000), that essentially says that an initial decay of the form $u_0(x) \overset{|x|\to\infty}{=} O(|x|^{-d-1})$ is conserved during the evolution.

Starting with initial data satisfying orthogonality relations is necessary (but not sufficient) for obtaining well localized flows, since such relations are not invariant under the Navier–Stokes flow. Finding a sufficient condition is more subtle. However, putting suitable symmetry conditions I could construct, in my Ph.D (see also [18]), the **first example of solution in** $\mathbb{R}^3$ for which orthogonality persists, and which is **strongly localized in space**. The corresponding initial datum is

$$u_0(x_1,x_2,x_3) = \begin{pmatrix} x_1(x_3^2 - x_2^2) \\ x_2(x_1^2 - x_3^2) \\ x_3(x_2^2 - x_1^2) \end{pmatrix} e^{-|x|^2}, \qquad x \in \mathbb{R}^3. \tag{2}$$

It should be noted that well localized flows in $\mathbb{R}^2$ were known since a long time. Indeed, in the case $d = 2$, there are flows with radial vorticity (the so called "Beltrami flows"), such that the velocity field belongs to the Schwartz class. The main difficulty for their extension to higher dimensions was an apparent topological obstruction: the conservation of the mass (the divergence-free condition for the velocity field) seemed to forbid three-dimensional flows to have a fast decay. What I realized, is that the topological obstruction is no longer an obstacle if one considers flows invariant under *discrete* subgroups of $O(3)$.

As only few exact, physically reasonable, solutions of Navier–Stokes are known, several specialists considered my examples of flows interesting in their own sake.

This study later culminated with the paper [15]. Therein, I succeded in providing the **complete classification of the sharp decay rates**, in space or in time, of the solutions, depending on their *symmetry group*, in dimension two and three.

The most important results can be summarized as follows (we assume for simplicity that the initial datum is in the Schwartz class. Recall also that $f \equiv 0$):

The solutions of the Navier–Stokes equations in $\mathbb{R}^2$ that are invariant under the action of the cyclic group of order k decay at infinity as $\mathcal{O}(|x|^{-k-1})$; they also decay as $\mathcal{O}(t^{-(k+1)/2})$ if they are invariant under the dihedral group of order $2k$.

In dimension three, the (small) strong solutions decay at infinity as $\mathcal{O}(|x|^{-5})$ and $\mathcal{O}(t^{-5/2})$, $\mathcal{O}(|x|^{-6})$ and $\mathcal{O}(t^{-3})$, or $\mathcal{O}(|x|^{-8})$ and $\mathcal{O}(t^{-4})$ if they are invariant, respectively, under the action of the complete symmetry group of the tetrahedron, of the cube (or of the octahedron), of the dodecahedron (or of the icosahedron).

The other subgroups of $O(3)$ are also considered in [15].

1.2 Energy dissipation for Navier–Stokes flows

[17] L. Brandolese,
Asymptotic behavior of the energy and pointwise estimates for solutions to the Navier-Stokes equations, Rev. Mat. Iberoamericana **20**, 223-256 (2004)

The famous Acta Mathematica paper by Jean Leray in 1934, where he gave the construction of weak solutions, ends with the following question:

- *" I do not know if the energy $\|u(t)\|_2^2$ goes necessarily to 0 as $t \to \infty$. "*

His question was affimatively answered in 1984 indepentently by K. Masuda and T. Kato. Soon after, M. E. Schonbek was able to provide the first explicit decay rates for the energy. Her results were later extended and improved by many authors. The best result in this direction is due to M. Wiegner in 1987[2]: for $0 \le \alpha \le (d+2)/2$, if the solution of the heat equation $e^{t\Delta}u_0$ satisfies $\|e^{t\Delta}u_0\|_2^2 \le C(1+t)^{-\alpha}$, then for a weak solution u of (NS) starting from u_0 one has $\|u(t)\|_2^2 \le C'(1+t)^{-\alpha}$ as well.

This theorem by Wiegner raises the following question:

- *Do solutions of (NS) such that $\|u(t)\|_2^2$ dissipates faster than $t^{-(d+2)/2}$ as $t \to \infty$ exist ?*

The positive answer in the special case $d = 2$ was given by A. Majda and M. Schonbek in 1991.[3] **The case $d \ge 3$, however, was left open**. Some progress was made later by T. Miyakawa and M. Schonbek,[4] who described a class of solutions with possibly fast decay of the energy. However, these authors could not establish whether their class of solutions was vacuous or not, when $d \ge 3$.

I answered these questions in [20, 17]. Therein, using suitable symmetries, I was able to construct explicit examples of solutions verifying the conditions of Miyakawa and Schonbek. I also computed the exact energy decay rates of such solutions, which is indeed much faster than predicted by Wiegner's result. Soon after, Thierry Gallay and E. Wayne[5] proved with a different

[2] M. Wiegner, *Decay results for weak solutions of the Navier–Stokes equations in $\mathbb{R}^n$*, J. London Math. Soc. **35** (1987) 303–313.

[3] M. E. Schonbek, *Lower Bounds of Rates of Decay for Solutions to the Navier-Stokes Equations*, J. of the Amer. Math. Soc., **4**, N. 3, 423–449 (1991).

[4] T. Miyakawa, M.E, Schonbek, *On optimal decay for weak solutions to the Navier–Stokes equations in $\mathbb{R}^n$*, Mathematica Bohemica **126** (2000), 443–445.

[5] *Long-time asymptotics of the Navier-Stokes and vorticity equations on $\mathbb{R}^3$*, Phil. Trans Roy. Soc. Lond., **360**, 2155–2188 (2002).

method that in fact there are solutions that decay at arbitrary fast rates in the L^2-norm. Their approach, however, does not lead to explicit examples.

1.3 How fast does the motion propagate in a fluid ?

[3] H.-O. Bae, L. Brandolese
On the effect of external forces on incompressible fluid motions at large distances,
Ann. Univ. Ferrara (2009).

[9] L. Brandolese, F. Vigneron
New asymptotic profiles for the nonstationnary incompressible Navier–Stokes equations,
J. Math. Pures Appl. **88**, 64--86 (2007)

Assume that, at the beginning of the evolution, the fluid is at rest outside a bounded region (say, $u_0 \in C^\infty_{0,\sigma}(\mathbb{R}^d)$, the space of smooth, solenoidal and compactly supported vector fields).

- *At which velocity the fluid particles that are situated far from that region will start to move ?*

More in general, the problem that we would like to address is the following: measuring how fast the motion does propagate inside a fluid.

The main results of the paper [9], is the construction of **new asymptotic profiles, describing the pointwise behavior** of the fluid in the parabolic region $|x| \gg \sqrt{t}$ (the extension to nonzero external forces is done in [3]). Roughly speaking, such profiles tell us that, for flows emanating from localized data in $\mathbb{R}^d$, one has

$$u(x,t) = \mathcal{K}(x) \int_0^t\!\!\int f(y,s)\,dy\,ds \,+\, \text{l.o.t.} \tag{3}$$

for all $t > 0$ such that the strong solution u is defined, and for all $|x| \gg R(t)$, with $R(t) > 0$. Here, $\mathcal{K}(x)$ is the matrix of the second order derivatives of the fundamental solution of the Laplacian in $\mathbb{R}^d$. Hence, $\mathcal{K}_{j,k}(x)$ are explicitely known homogeneneous functions of degree $-d$.

Thus, the far-field behavior of the velocity essentially depends on the mean of the external force. In the case $\int_0^t\!\int f = 0$ (this includes the important case of the free NS equations, *i.e.*, $f \equiv 0$), the behavior for $|x| \gg R(t)$ is quite different (see [9]):

$$u(x,t) = \nabla\mathcal{K}(x) \colon \int_0^t\!\!\int (u\otimes u)(y,s)\,dy\,ds \,+\, \text{l.o.t.} \tag{4}$$

Formula (4) was obtained in collaboration with F. Vigneron. The interesting feature here is that u behaves at infinity like a potential field. Moreover, it is somewhat striking that an information on the mean of the space-time of the velocity is enough to describe the behavior at *all points at large distances*.

The proof of (3)-(4) relies on a method that is a sort of "asymptotic separation of variables". It consists in proving that, asymptotically, the flow behaves like a linear combination of functions of separate variables, of the form $\phi(t)\psi(x)$. This is made possible through a fine analysis of the kernel of the singular operators involved in the integral formulation of (NS).

An application of our asymptotic profile (3) is the sharp pointwise estimate of the form

$$ct|x|^{-d} \le |u(x,t)| \le c't|x|^{-d}, \tag{5}$$

valid for $0 < t < t_0$ small enough and $|x| \geq R(t)$. The constant c in (5) in strictly positive providef $\int_0^t\!\int f \neq 0$.

Such pointwise estimates should be compared to those valid in the case of the Navier–Stokes equations *without forcing* studied in [9], where the behavior of u is such that

$$Ct|x|^{-d-1} \leq |u(x,t)| \leq C't|x|^{-d-1}. \tag{6}$$

Here one has to put a suitable *non-symmetry assumption* on the initial data in order to guarantee that $C > 0$. In addition, the lower bounds are non isotropic: they are in fact valid along almost all directions in $\mathbb{R}^d$ (indeed, C is independent on $|x|$ but depends on the angles $x/|x|$).

It is worth observing that even if f is small and well localized (say, compactly supported in space-time) but with non-zero mean, then the velocity of the fluid particles at *all times* $t > 0$ and in *all points* x outside balls $B(0, R(t))$ with large radii, is considerably faster than in the case of the forcing-free Navier–Stokes equations. This contrasts with the asymptotic properties of the solution of other semilinear parabolic equations, where localized external forces do not affect the behavior of the solution as $|x| \to \infty$: only the large time behavior is influenced by f.

Estimates (6) can be applied, *e.g.*, to the case of flows arising from compactly supported initial data. In such case, they describe how fast fluid particles do start their motion in the far field. This answers the question addressed at the beginning of this section.

Starting with the work of T. Miyakawa, many authors derived pointwise *upper bound* estimates for u. The validity of the more subtle *pointwise lower bounds* in (5)-(6) seems to us remarkable. They give a **complete understanding on the spatial spreading effects** dictated by the incompressibility constraint. Before, lower bounds were known to be valid only for spatial norms for the velocity, like the energy norm $\|u(\cdot,t)\|_2$.

Future developements. The simplicity of the formula of our asymptotic profile is due to the fact that we started with well localized data. It is reasonable to expect that the least u_0 is localized, the more complicated is the structure of the asymptotic profiles. The exact relationship between this structure of the asymptotics and the data deserves to be studied in detail.

On the other hand, there are many interesting models arising from fluid mechanics and geophysics for which the problem of constructing asymptotic profiles is still open (quasi-geostrophic equations, density-dependent Navier–Stokes, etc.). I have more or less advanced cooperations with a few colleagues (M. Schonbek, C. Bjorland, M. Paicu, etc.) in this direction.

Moreover, it would be important to understand in which way the different boundary conditions affect the behavior at the spatial infinity. We have partial results in the case of the half-space with F. Vigneron, H.-O. Bae and B.-J. Jin, that are the starting point of a more substantial study in the case of the exterior domain.

Many results of [9] have not been extended yet to more general domains. In particular, it would be interesting to study the influence of the different boundary conditions on the propagation of the motion in the far field. It would also be interesting to extend these results to more general models in fluid mechanics. The study of incompressible density dependent Navier–Stokes equations is under way.

1.4 Concentration-diffusion effects in incompressible viscous flow

[5] L. Brandolese

Concentration-diffusion effects in incompressible viscous flows
`Indiana Univ. Math. J.` **58**`, 789--806 (2009).`

One of the most important questions in Partial Differential Equations is to know whether a finite energy, and initially smooth, nonstationnary Navier–Stokes flow will always remain regular during its evolution, or can develop a singularity in finite time.

As a first step toward the understanding of possible blow-up mechanisms, it can be interesting to exhibit examples of smooth and decaying initial data such that, even if the corresponding solutions remain regular for all time, "something strange" happens around a given point (x_0, t_0) in space-time. This is problem that I addressed in [5].

The main result of [5] is the **construction of** a class of (smooth) **solutions** to the incompressible Navier–Stokes equations such that, in the absence of any external forcing ($f \equiv 0$), **the motion of the fluid particles tends to be more and more concentrated** around x_0, as the time t approaches t_0. This corresponds precisely to the qualitative behavior that one would expect in the presence of a singularity, even though such "concentration of the motion" is not strong enough to imply their formation.

More precisely, given a finite sequence of times $0 < t_1 < \cdots < t_N$:

We construct an example of a smooth solution of the free nonstationnary Navier–Stokes equations in $\mathbb{R}^d$, $d = 2, 3$, such that: (i) The velocity field $u(x, t)$ is spatially poorly localized at the beginning of the evolution but tends to concentrate until, as the time t approaches t_1, it becomes well-localized. (ii) Then u spreads out again after t_1, and such concentration-diffusion phenomena are later reproduced near the instants t_2, t_3, ...

Here, a flow is said to be "highly concentrated" at the time t when $|u(x, t)| \le C_t(1 + |x|)^{-M}$, for some large exponent $M > 0$. It is said to be "spatially diffused" or to "spread out" at the time t', if $|u(x, t')| \ge C_{t'}(1 + |x|)^{-m}$ for some small $m > 0$. What it matters, of course, is that $m < M$.

The main tool is a combination of an analyticity argument, consisting in writing the solution of Navier–Stokes as an absolute convergent series in a well chosen function space, and of an asymptotic formula obtained in [9]. This allows us to reduce the construction of our solution to that of an initial data u_0, such that the function

$$t \mapsto \int_0^t \int (e^{s\Delta} u_{0,1}\, e^{s\Delta} u_{0,2})(x, s)\, dx\, ds.$$

changes sign around $t_1, t_2, \ldots$ (here $e^{t\Delta}$ is the heat kernel). This last problem is finally solved applying the Fourier transform and some elementary linear algebra.

1.5 Stability of stationary flows

`[2] C. Bjorland, L. Brandolese, D. Iftimie, M. E. Schonbek,`
L^p-solutions of the steady-state Navier–Stokes equations with rough external forces`,`
`Comm. Part. Diff. Equ.,` **36**`, 216--246, 2011.`

In this subsection we deal with (NS) in the whole $\mathbb{R}^3$, with a time-independent forcing term $f = f(x)$. In this setting, one can look for a stationary solution $U = U(x)$ of the system. Our work, in collaboration with C. Bjorland, D. Iftimie and M. Schonbek, can be viewed as a continuation of those of Cannone and Karch[6] and Kozono and Yamazaki.[7] One of the goal of [2], is to provide *necessary and sufficient* conditions on f implying that a stationary solution U belongs to some given function space (before, previous authors focused only on sufficient conditions). For exemple, provided the $L^{3,\infty}$-norm of $\Delta^{-1}f$ is small enough, then U will belong to L^p if and only if f belongs to the Bessel potential space $\dot{H}_p^{-2}$. Another important issue, is the stability of these solutions. For this, we essentially prove the following (omitting a couple of technical assumptions):

If $\|\Delta^{-1}f\|_{L^{3,\infty}} < \epsilon$ then there exists a unique stationary solution $U(x) \in L^{3,\infty}(\mathbb{R}^3)$ of the Navier–Stokes equations

$$\nabla \cdot (U \otimes U) + \nabla P = \Delta U + f.$$

Let $u_0 \in L^{3,\infty}$, with $\limsup_{R\to 0} R\,|\{|u_0| > R\}|^{1/3} < \epsilon$. Let also $u(x,t)$ be a given global solution of the Cauchy problem (NS) starting with u_0, and such that $u \in L^\infty_{loc}(\mathbb{R}^+, L^{3,\infty})$. Then u becomes small in $L^{3,\infty}$ as $t \to \infty$, and converges weakly to U.

This result differs from the stability results for stationary solutions of Navier–Stokes available in the literature: indeed, previously, it was only known that small stationary solutions U were stable in the class of *small* non-stationary solutions u. On the other hand, we proved that **they attract also** *large* **non-stationary solutions** u. Indeed, no matter how large u is at the beginning of the evolution, we prove that after some time u becomes small in $L^{3,\infty}$. The reason is that we do not need to restrict the size of $\|u_0\|_{L^{3,\infty}}$: only a portion of the $L^{3,\infty}$-norm needs to be small.

The proof consists in proving that one has $u_0 = v_0 + w_0$, with $v_0 \in L^2$ and w_0 small in $L^{3,\infty}$. Next one finds a global solution w of the Cauchy problem with datum w_0 and writes energy estimates for the equation satisfied by the difference $v = u - w$. A similar idea was used also by I. Gallagher, D. Iftimie and F. Planchon (Ann. Inst. Fourier, 2004), in the particular case $f \equiv 0$ and $U \equiv 0$.

Future developements. A few questions on the stability remains open. For example, is it possible to relax the smallness condition on $\|\Delta^{-1}f\|_{L^{3,\infty}}$? A more general smallness assumptions might involve homogeneous Morrey–Campanato norms, but the analysis of the singular integral operators involved seems harder in such spaces.

Another interesting **open problem** related to our study is the global **stability of Landau solutions**. These are axisymmetric stationary solutions in $\mathbb{R}^3$, smooth outside the origin, and homogeneous of degree -1, whose existence has been known for a very long time. Some progress has been very recently made by V. Šverák and G. Karch. This is an active research subject.

2 Chemotaxis and other systems of interacting particles

[4] P. Biler, L. Brandolese
On the parabolic elliptic limit of the doubly parabolic Keller–Segel system modelling chemotaxis
Studia Math. **193**, 241--261 (2009).

[6] M. Cannone and G. Karch, *Smooth or singular solutions to the Navier-Stokes system?*, J. Differential Equations, **197**, (2004), 247–274. M. Cannone, G. Karch, *About the regularized Navier-Stokes equations*, J. Math. Fluid Mech., **7**, (2005), 1–28.

[7] H. Kozono, M. Yamazaki, *The stability of small stationary solutions in Morrey spaces of the Navier-Stokes equation*, Indiana Univ. Math. J., **44**, (1995), 1307–1336. H. Kozono, , M. Yamazaki, *Exterior problem for the Navier-Stokes equations, existence, uniqueness and stability of stationary solutions*, in: Theory of the Navier-Stokes equations, Ser. Adv. Math. Appl. Sci., **47**, World Sci. Publ., River Edge, NJ, (1998), 86–98,

`[11] P. Biler, L. Brandolese`
Global existence and finite time blow up for some models of interacting particles
`Colloq. Math.` **`106,`** `293--303 (2006).`

In collaboration with Piotr Biler, we studied, in [4], [11], two related nonlinear parabolic systems which are frequently used as models for chemotactic phenomena, including the aggregation of microorganisms caused by a chemoattractant, *i.e.* a chemical whose concentration gradient governs the oriented movement of those microorganisms. The parabolic character of the systems comes from the diffusion described by the Laplacians. The three-dimensional version of the system (PE) below is also used in astrophysics as a model of the evolution of a cloud of self-gravitating particles in the mean field approximation.

The first one is the classical parabolic-elliptic Keller–Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \varphi), & \\ \Delta \varphi + u = 0, & x \in \mathbb{R}^2,\ t > 0, \\ u(0) = u_0. & \end{cases} \tag{PE}$$

Here, $u = u(x,t)$, $\varphi = \varphi(x,t)$ are either functions or suitable (tempered) distributions. When $u \geq 0$, $\varphi \geq 0$, they may be interpreted as concentrations (densities) of microorganisms and chemicals, respectively.

The second one is the parabolic-parabolic system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \varphi), & \\ \tau \partial_t \varphi = \Delta \varphi + u, & x \in \mathbb{R}^2,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = 0, & \end{cases} \tag{PP}$$

where $\tau > 0$ is a fixed parameter. Each of these models can be considered as a single nonlinear parabolic equation for u with a *nonlocal* (either in x or in (x,t)) nonlinearity, since the term $\nabla\varphi$ can be expressed as a linear integral operator acting on u. In the latter model, the variations of the concentration φ are governed by the linear nonhomogeneous heat equation, and therefore are slower than in the former system, where the response of φ to the variations of u is instantaneous.

The theory of the system (PE) is relatively well developed. One of the most intriguing properties of (PE), considered for *positive* and *integrable* solutions u, is the existence of a threshold value 8π of mass $M \equiv \int u(x,t)\,dx$, as shown in the pioneering work of Jäger and Luckhaus and in more recent contributions by Biler, Blanchet, Carrillo, Dolbeault, Masmoudi, Perthame and many others. Namely, if $u_0 \geq 0$ is such that $\int u_0(x)\,dx > 8\pi$, then any regular, positive solution u of (PE) cannot be global in time.

The doubly parabolic system (PP) has been a bit less studied. For instance, Calvez and Corriaz (2008) showed that if for the initial data u_0 one has $M < 8\pi$, then positive solutions are global in time. However, $M \leq 8\pi$ is no longer a *necessary* condition for the existence of global in time solutions. Usual proofs of blow up for (PE) involve calculations of the second order moments of a solution and then symmetrization. These methods seem do not work for (PP), hence another approach is needed to show a blow up for that system.

2.1 Convergence of the parabolic-parabolic to the parabolic-elliptic system

A nice result by Raczyński (Asympt. Anal. 2008) shows that the solutions of the systems (PP) and (PE) enjoy a kind of stability property as $\tau \to 0$: solutions of (PP) converge in a suitable sense to those of (PE). It had been an old question raised by J. J. L. Velázquez and D. Wrzosek. However, Raczyński result was obtained for suitably small solutions in quite a big functional space of pseudomeasures. In addition, the exact physical meaning of the convergence result in the pseudomeasure norm is not so clear.

Our main result in [4] establishes that **the convergence as $\tau \to 0$ of solutions** u^τ of the system (PP) to the corresponding solutions u of (PE) holds true in a much larger class of spaces. Indeed, any *shift invariant spaces of local measures*, in the sense defined by Lemarié-Rieusset [8], is suitable. An advantage of our approach is that it allows to obtain the convergence results in more natural norms, like, *e.g.*, the $L^\infty_t(L^1_x)$ or the $L^\infty_{t,x}$ norms.

Our proof requires a smallness condition on the quantity $\operatorname{ess\,sup}_{x\in\mathbb{R}^2,\,t>0}(t+|x|)^2|e^{t\Delta}u_0(x)|$. This allows the initial mass to be large. However, we do not know if the convergence result remains true if one removes *any* smallness assumption.

2.2 Finite time blow up

Models describing chemotaxis ($d=2$) or gravitational interaction ($d\geq 2$) feature concentration phenomena which may eventually lead to a collapse of solutions. These phenomena manifest by the formation of singularities of solutions like weak convergence either to Dirac point masses (in $\mathbb{R}^2$) or to unbounded functions $\sim |x|^{-2}$ (in $\mathbb{R}^3$).

The purpose of our paper [11] is to show that a very different kind of finite time blow up can occur for solutions of (PE) or (PP): nonpositive (in fact, **complex-valued**) and oscillating **solutions can explode** (below, $\|\cdot\|_{\dot B^{s,q}_p}$ is the Besov norm):

There exists $u_0\in\mathcal{S}_0(\mathbb{R}^d)$ (the space of functions belonging to the Schwartz class, with vanishing moments of all order), such that the corresponding solution u of (PE) or (PP) blows up in finite time: there exists $t^*>0$ such that $\|u(t)\|_{\dot B^{s,q}_p}\to\infty$ as $t\to t^*$ for all $s\in\mathbb{R}$, $1\leq p,q\leq\infty$. In particular, any Lebesque, Sobolev and Triebel-Lizorkin norm becomes infinite at the time t^*.

The proof consists in constructing a cascade of lower bounds for the Fourier transform $\widehat{u}$, by iterating Duhamel formula. This idea came to from a paper by Montgomery-Smith[9], who analysed a scalar toy model for the Navier–Stokes system.

Our exploding solution u is in fact complex-valued since its Fourier transform enjoys some *positivity and nonsymmetry* properties. Even though these solutions have no straightforward physical/biological interpretation, our analysis gives some understanding on how the nonlinearities interact. For example, it shows the limitations of the semigroup method, and on other methods relying on size estimates, in solving the open problem of the global existence of solutions to (PP).

Very recently, using the positivity of the Fourier transform in suitable fequency cones led Li and Sinai to their celebrated construction[10] of complex valued solutions of the Navier–Stokes equations that blow up in finite time. The spirit of their construction is similar to that of [11], even though it is technically much more demanding, due to non-scalar nature of the Navier–Stokes system.

[8] P.G. Lemarié-Rieusset, *Recent developements in the Navier–Stokes problem*, Chapman & Hall, CRC Press Boca Raton (2002)

[9] S. Montgomery-Smith, *Finite time blow up for a Navier–Stokes like equations*, Proc. Amer. Math. Soc. **129**, N. 10 , 3025–3029 (2001)

[10] D. Li, Y. Sinai, *Blow ups of complex valued solutions of the Navier–Stokes system and renormalization group method,* J. Eur. Math. Soc., **10**, 2008

Future developements. The method that we used in [11] is effective for a class of nonlinear heat equations. This can be a very interesting **research program**, since it could bridge an old gap between, on one hand, a blow up result of J. Ball, H.A. Levine and L. Payne[11] and global well–posedness results in the case of large but *fast oscillating data*. The oscillations are measured by the size of the data in some Besov of Triebel-Lizorkin norms with negative regularity. Specifically, it seems hopeful to make further progress toward a conjecture formulated by Yves Meyer[12].

3 Selfsimilarity

[7] L. Brandolese,
Fine properties of self-similar solutions of the Navier–Stokes equations,
Arch. Rational Mech. Anal. **192**, N.3, 375--401 (2009)

[8] L. Brandolese, G. Karch,
Far-field asymptotics of solutions to convection equations with anomalous diffusion,
J. Evol. Equ. **8**, 307--326 (2008).

When a physical model has some scaling invariance, it is natural to look for self-similar solutions. These are of the form $u = u_\lambda$ for all $\lambda > 0$, where $u_\lambda(x,t) = \lambda^\alpha u(\lambda x, \lambda^\beta t)$. A classical method for constructing such solutions consists in writing $u = t^{-\alpha/\beta}U(t^{-1/\beta}x)$, where $U(x) = u(x,1)$ is the self-similar profile. Next one eliminates the time variable and solves an elliptic equation in U. It turns out, however, that in most cases this equation is too complicated, that can be solved only in some particular cases (additional radial symmetries, very special nonlinearities, etc.).

More recently, Giga and Miyakawa[13] proposed a new method, later popularised and extended by Cannone, Meyer and Planchon[14], for obtaining self-similar solutions. Their idea consists in studying the well-posedness of the Cauchy problem in function spaces that are large enough to contain homogeneous functions (this excludes *e.g.* Lebesgue spaces, thus motivating the use of rougher spaces): if one can start with an initial data homogeneous of degree $-\alpha$, then the solution obtained by fixed point will be self-similar. This method is usually quite simple to apply, and it yields to existence results previously out of reach. However, it usually provides only few information on the self-similar solution itself.

My **main contribution** to this subject, [7], [8], is a simple device that can be used to derive **explicit relations**, valid asymptotically, **between the datum** $u_0(x)$ **and the profile** $U(x)$. Such device consists in formulae describing the behavior at infinity of a class of bilinear operators, from information on the behavior at infinity of the arguments. The simplest instance is the *asymptotic formula for convolution integrals*: $f * g(x) = \big(\int f\big)g + \big(\int g\big)f\, +$ l.o.t. as $|x| \to \infty$. Of course, the validity such kind of formulae requires rather stringent assumptions on the factors f and g (the above formula is obviously wrong, *e.g.*, for $f = g = e^{-|x|^2}$). In the application, usually f is the kernel of some singular integral operator and g is a nonlinear functional of the unknown solution. The main issue, in each specific situation, is to obtain enough *a priori* information on the solution

[11] J. Ball, *Remarks on blow-up and non-existence theorems for nonlinear parabolic evolution equations*, Quart. J. Math. Oxford Ser. **28**, 473–486, 1977

[12] Y. Meyer, *Oscillating patterns in some nonlinear evolution equations* in *Mathematical foundation of turbulent viscous flows*, 101–187, Lecture Notes in Math., 1871, Springer, Berlin, 2006. (See section 2)

[13] Y. Giga, T. Miyakawa, *Navier–Stokes Flow in* $\mathbb{R}^3$ *with Measures as Initial Vorticity and Morrey Spaces*, Comm. Part. Diff. Eq. **14**, 5, 577–618 (1989).

[14] M. Cannone, Y. Meyer, F. Planchon, *Solutions auto-simlaires des équations de Navier–Stokes,* In: Séminaire X-EDP, Centre de Mathématiques, Ecole polytechnique, 1993-94.

in order to guarantee the validity the needed asymptotic formula.

The next subsections illustrate two concrete applications.

3.1 Convection-diffusion equation with anomalous diffusion

In collaboration with G. Karch, [8], we studied the initial value problem for the a class of multidimensional conservation laws with anomalous diffusion. An instance, is given *e.g.* by the system

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha/2} u + b \cdot \nabla(u|u|^{(\alpha-1)/d}) = 0 \\ u(x,0) = u_0, \end{cases} \qquad x \in \mathbb{R}^d, \tag{7}$$

where $1 < \alpha < 2$ and $b \in \mathbb{R}^d$.

Linear evolution problems involving fractional Laplacian describing *the anomalous diffusion* (or α-stable Lévy diffusion) have been extensively studied in the mathematical and physical literature

An important motivation for the above and related models is the probabilistic interpretation of nonlinear evolution problems with an anomalous diffusion, obtained recently by Jourdain, Méléard, and Woyczyński[15]. These authors considered a class of nonlinear integro-differential equations involving a fractional power of the Laplacian and a nonlocal quadratic nonlinearity represented by a singular integral operator. They associated with the equation a nonlinear singular diffusion and proved *propagation of chaos* to the law of this diffusion for the related interacting particle systems. In particular, due to the probabilistic origin of (7)-(7), the function $u(\cdot,t)$ should be interpreted as the density of a probability distribution for every $t > 0$, if the initial datum is so.

Biler, Karch and Woyczyński[16] constructed a family of self-similar solution $u_M = u_M(x,t)$ of equation (7) starting from the homogeneous measure $u_0 = M\delta_0$. Those functions satisfy the scaling relation

$$u_M(x,t) = t^{d/\alpha} U_M(xt^{-1/\alpha}) \quad \text{where} \quad U_M(x) = u_M(x,1) \tag{8}$$

for all $x \in \mathbb{R}^d$ and $t > 0$.

In [8] we established a comparison principle between the self-similar solution u_M and the fundamental solution $p_\alpha(x,t)$ of the linear equation $\partial_t v + (-\Delta)^{\alpha/2} v = 0$. Namely, under suitable conditions on the parameters we obtained that

$$0 \le u_M(x,t) \le C p_\alpha(x,t)$$

for some $C > 0$.

Moreover, letting $U_M(x) = u_M(x,1)$ and $P_\alpha(x) = p_\alpha(x,1)$, we computed the exact behavior as $|x| \to \infty$ of the self-similar profile U_M:

$$U_M(x) = M P_\alpha(x) + \frac{c_1(\alpha,d)\,\alpha^2}{\alpha+1}\, \|U_M\|_{\widetilde{q}}^{\widetilde{q}} \frac{b \cdot x}{|x|^{\alpha+d+2}} + \text{l.o.t.} \tag{9}$$

Here, $\widetilde{q} = 1 + (\alpha-1)/d$. Moreover, $c_1(\alpha,d) = 2\pi\alpha 2^{\alpha-1}\pi^{-(d+4)/2}\sin(\alpha\pi/2)\Gamma\big(\frac{\alpha+d+2}{2}\big)\Gamma\big(\frac{\alpha}{2}\big)$. This result is thus specific to the fractional nature of the diffusion operator: in the limit case $\alpha = 2$,

[15] B. Jourdain, S. Méléard and W. Woyczyński, *A probabilistic approach for nonlinear equations involving the fractional Laplacian and singular operator*, Potential Analysis **23** (2005), 55–81

[16] P. Biler, G. Karch, W. A. Woyczyński, *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws*, Ann. Inst. Henri Poincaré, Analyse non-linéaire, **18** (2001), 613–637.

which corresponds to the usual Laplacian, $c_1(\alpha, d)$ boils down to zero. The estimates for the lower order terms can be computed explicitly. The asymptotic expansion of solutions in (9) can be viewed as the nonlinear counterpart of the classical asymptotic developements for $|x| \to \infty$ of the fundamental solution $p_\alpha(x,t)$ due to Kolokoltsov[17].

3.2 Fine properties of self-similar solutions of Navier–Stokes

Here we consider the self-similar solutions $u(x,t) = \frac{1}{\sqrt{t}} U(\frac{x}{\sqrt{t}})$ of the Navier–Stokes system that were constructed by Cannone, Meyer and Planchon in 1994, starting with suitably small initial data u_0 homogeneous of degree -1.

In [8], assuming that u_0 is sufficiently smooth on the sphere, we established the formula

$$U(x) = u_0(x) - \log(|x|) \frac{\mathcal{Q}(x) \colon A}{|x|^6} + \text{l.o.t.} \qquad \text{as } |x| \to \infty.$$

Here $A = (A_{h,k})$ is the 2×2 matrix with entries $A_{h,k} = \int_{\mathbb{S}^1} u_{0,h} u_{0,k}$ and $\mathcal{Q} = \mathcal{Q}_{j,h,k}$ is a homogeneous polynomial of degree three (its explicit expression is obtained by computing the third-order derivatives of the fundamental solution of the Laplacian in $\mathbb{R}^2$).

For $d = 3$, we obtained the formula

$$U(x) = u_0(x) + \Delta u_0(x) - \mathbb{P}\nabla \cdot (a \otimes a) - \frac{\mathcal{Q}(x) \colon B}{|x|^7} + \text{l.o.t.} \qquad \text{as } |x| \to \infty \tag{10}$$

for a $d \times d$ constant real matrix $B = (B_{h,k})$ depending on u_0. Here $\mathbb{P} = \mathrm{Id} - \nabla(\Delta)^{-1}\mathrm{div}$ is the Leray-Hopf projector onto the divergence-free vector fields.

The (non)-presence of the logarithmic factor in these formulae is due to the remarkable cancellations hidden inside the kernel of Leray's projector.

Future developements There are scaling invariant models, for which no non-trivial self-similar solutions is known to exist. It is the case, *e.g.*, for the supercritical **quasi-geostrophic equations**, a very popular model in Geophysics. The search of these solutions and the study of their properties is a part of our future research program.

4 Other models in fluid mechanics

4.1 The Boussinesq system

[1] L. Brandolese, M. E. Schonbek,
Large time decay and growth for a viscous Boussinesq system,
Trans. Amer. Math. Soc. (to appear).

In this section we address the problem of the heat transfer inside viscous incompressible flows in the whole space $\mathbb{R}^3$. Accordingly with the Boussinesq approximation, we neglect the variations of the density in the continuity equation and the local heat source due to the viscous dissipation. We rather take into account the variations of the temperature by putting an additional vertical buoyancy force term in the equation of the fluid motion.

[17] V. Kolokoltsov, *Symmetric stable laws and stable-like jump-diffusions*, Proc. London Math. Soc. **80** (2000), 725–768.

This leads us to the Cauchy problem for the Boussinesq system (we set the physical constants equal to one):

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = \Delta\theta \\ \partial_t u + u\cdot\nabla u + \nabla p = \Delta u + \theta e_3 \\ \nabla\cdot u = 0 \\ u|_{t=0} = u_0,\ \ \theta|_{t=0} = \theta_0. \end{cases} \qquad x\in\mathbb{R}^3, t\in\mathbb{R}_+ \tag{B}$$

Here $u\colon \mathbb{R}^3\times\mathbb{R}^+\to\mathbb{R}^3$ is the velocity field. The scalar fields $p\colon \mathbb{R}^3\times\mathbb{R}^+\to\mathbb{R}$ and $\theta\colon \mathbb{R}^3\times\mathbb{R}^+\to\mathbb{R}$ denote respectively the pressure and the temperature of the fluid. Moreover, $e_3=(0,0,1)$. The existence of global weak solutions for L^2 data, as well as that of strong solutions in the case of small data is well known for this system.

An **important conclusion** of in my paper with M. Schonbek [1], is that for a large class of solutions, one has $\|u(t)\|_2^2\to+\infty$ for $t\to\infty$. In other words, **the kinetic energy of the fluid grows to infinity** for large time. This strikingly constrasts with the Navier–Stokes equations, that correspond to the case $\theta\equiv 0$ in (B), for which it is well known that $\|u(t)\|_2\to 0$ for $t\to\infty$. More precisely we establish the validity of the estimates

$$c(1+t)^{1/4}\le\|u(t)\|_2\le C(1+t)^{1/4}, \qquad t\gg 1 \tag{11}$$

provided the initial temperature θ_0 has non-zero mean. This unexpected result illustrates the **physical limitations** of the above **Boussinesq approximation** (despite the large number of papers using it), at least for the study of the heat convection inside fluids filling domains where Poincaré's inequality is not available, such as the whole space.

In fact, we prove that $L^p(\mathbb{R}^3)$-norms of strong solutions u grow large at infinity if and only if $1\le p<3$. Our work is thus complementary to that of Karch and Prioux (Proc. Amer. Math. Soc., 2009) who obtained stability results in L^p, for $p\ge 3$. We will also deal with weighted spaces, so let us introduce

$$\|f\|_{L^p_r}=\left(\int |f(x)|^p(1+|x|)^{pr}\,dx\right)^{1/p}.$$

In [1] we established the following:

Let (u,θ) be the mild solution of (B), starting from a sufficiently small (in a suitable norm) and decaying datum (θ_0,u_0). Then, for all r,p such that $r\ge 0$, $1\le p<\infty$, $r+\frac{3}{p}<3$, and t large enough, we have

$$\phi(|m_0|)\,\bigl(1+t\bigr)^{\frac12(r+\frac3p-1)}\le\|u(t)\|_{L^p_r}\le C'\bigl(1+t\bigr)^{\frac12(r+\frac3p-1)}. \tag{12}$$

Here, $m_0=\int\theta_0$ and $\phi\colon\mathbb{R}^+\to\mathbb{R}$ is a continuous function such that $\phi(0)=0$ and $\phi(\sigma)>0$ if $\sigma>0$. Moreover, when $r+\frac3p\ge 3$, and $m_0\neq 0$, we have

$$\|u(t)\|_{L^p_r}=+\infty, \qquad \forall\, t>0.$$

In the case $\int\theta_0=0$ the lower bound brakes down. In this case, we establish the improved estimates

$$\phi(\widetilde{\boldsymbol{m}})\,\bigl(1+t\bigr)^{\frac12(r+\frac3p-2)}\le\|u(t)\|_{L^p_r}\le C'\bigl(1+t\bigr)^{\frac12(r+\frac3p-2)},$$

valid now in the extended range $r+\frac3p<\min\{a,4\}$. Here, $\widetilde{\boldsymbol{m}}=\liminf_{t\to\infty}\frac1t\left|\int_0^t\!\int y\theta(y,s)\,dy\,ds\right|$. Notice that in the second case, the energy does not blow up.

Future developements. A few variants of the Boussinesq system are also of interest. For exemple, one can remove the diffusion term $\Delta\theta$ in the equation of the temperature. We expect that some of the results of [1] remain valid in this case. This is a work progress. On the other hand, the questions that we addressed for (B) could be asked in the case of a bouyancy force $\simeq \theta(x)\nabla(\frac{1}{|x|})$. In other words, the gravity is no longer assumed to be constant inside the domain. This is a more realistic assumption when working in the whole space.

4.2 Magnetohydrodynamics

[10] `L. Brandolese, F. Vigneron,`
On the localization of the velocity and the magnetic field in the MHD equations,
`Proc. Edinburgh Math. Soc.` **137 A**, `475--495 (2007).`

The magnetohydrodynamics (MHD) equations are a well-known model in plasma physics, describing the interactions between a magnetic field and a fluid made of moving electrically charged particles. A common example of application is the design of tokamaks: the purpose of these machines is to confine a plasma in a region, with a density and a temperature large enough to entertain thermonuclear fusion reactions. This can be achieved, at least during a small time interval, by applying strong magnetic fields. The MHD is also the standard model for the study of the dynamics of the solar corona.

In the main theorem in [10] **we study the trajectories of the solution in weighted Lebesgue spaces**. More precisely we address the problem of local stability of conditions of the form $(u_0, B_0) \in L^{p_0}_{\omega_0} \times L^{p_1}_{\omega_1}$, where u_0 and B_0 are the initial velocity and magnetic field and ω_1, ω_2 are weight functions. Depending on the values of p_0, p_1 and the growth at infinity of the two weight functions, the solution (u, B) of MHD may, or may not, remain in the weighted space $L^{p_0}_{\omega_0} \times L^{p_1}_{\omega_1}$. In some cases, the solution ceases to belong to a weighted Lebesgue space, but neverthless still remain spatially localized *in a weak sense*, a notion that we made precise in the article.

The precise statement is rather technical, as it involves conditions on the two parameters p_0 and p_1 and the weight functions ω_0 and ω_1, which are coupled in a non-trivial way.

5 Miscellanea

In this section I describe in a succint way the content of my other papers.

5.1 Wavelet bases and approximation theory

[13] `L. Brandolese,`
Poisson kernels and sparse wavelet expansions
`Proc. Amer. Math. Soc. (2005).`

This is a contribution to the approximation theory, motivated by the previous work by DeVore, Kiriazis, Meyer, Peller, Petrushev and others.

A *sparse wavelet expansion* in $\mathbb{R}^n$ is a series of the form

$$f(x) = \sum_{\psi\in F}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} \alpha(j,k)\psi(2^j x - k), \qquad x \in \mathbb{R}^n, \tag{13}$$

where

$$\alpha(j,k) = 2^{nj} \int f(x)\bar{\psi}(2^j x - k)\, dx, \tag{14}$$

F is a finite set of orthogonal wavelets and only few coefficients $|\alpha(j,k)|$ are larger than fixed small thresholds.

Roughly speaking, the result of [13] establishes that a function f is the sum of a **sparse wavelet expansion** if and only if it is the sum of a sparse expansion made of dilated and shifted Poisson kernels. The main application is a new characterization of a family of homogeneous Besov spaces by means of **atomic decompositions** involving poorly localized building blocks.

For example, one has $f \in \cap_{p>0} \dot{B}_p^{-d/p,p}(\mathbb{R}^d)$ if and only if $f(x) = \sum_{n\in\mathbb{N}} c_n P(\lambda_n(x-\beta_n))$, for some sequence $\lambda_n > 0$, $\beta_n \in \mathbb{R}^d$ and a sequence (c_n) such that $|c_n|$ is rapidly decreasing for $n \to \infty$. Here, $P_n(x) = (1+|x|^2)^{-(d+1)/2}$ is the Poisson kernel

One can apply such atomic decompositions to the Navier–Stokes equations. This was in fact one important motivation of our study. Indeed, I established in [16] that, when the initial vorticity $\nabla \times u_0$ is spatially well localized, then $u(\cdot,t) \in \cap_{p>0} \dot{B}_p^{3/p,p}(\mathbb{R}^3)$ uniformly in some time interval $[0,T]$. Hence, the flow can be approximated using a small number of wavelets, or a small number of dilated and shifted Poisson kernels. Sparse representations of solutions are important in numerical approximations.

5.2 Molecules of the Hardy space and pointwise multipliers of Sobolev spaces

[12] L. Brandolese,
Application of homogeneous realized Sobolev spaces to Navier-Stokes,
SIAM J. Math. Anal., **37**, N.2, 673--683 (2005).

The function space $\Delta^{-1}\mathcal{H}^1$, is the space made of all distributions vanishing at infinity and such that their Laplacian belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^3)$. Such space gives an useful insight of solutions of the Navier–Stokes equations. Indeed, when $u \in \Delta^{-1}\mathcal{H}^1$, all the terms that contribute to $\partial_t u$ in (NS) have the same regularity (this is an application of the div-curl theory of Coifman, Lions, Meyer and Semmes).

Moreover, the Hardy space has a very simple structure, due to its well-known atomic decomposition[18]. Hence, solving the equations in $\Delta^{-1}\mathcal{H}^1$ yields a **natural decomposition of the flow into simple "building blocks"**. Furioli and Terraneo[19] successfully studied the converse problem of the evolution of each building block. Their result essentially states that if Δu is a *molecule* of the Hardy space (in a sense close to that of Coifman and Weiss) at the beginning of the evolution, then this property remains true during a certain time interval.

The main contribution in [12] is a new interpretation of their result. We proved that the **persistence of the molecule structure** is closely related to the realization "à la Bourdaud"[20] of the homogeneous Sobolev space $\dot{H}^\delta$. Noticing that the operators involved in the Navier–Stokes equations turn out to be Fourier pointwise multipliers of the realized spaces, we could drop all the technical estimates that were needed to study the evolution of molecules. This provides a shorter, elegant proof, and also a better understanding of Furioli and Terraneo's result.

[18] R. R. Coifman, G. Weiss, *Extentions of Hardy Spaces and their use in Analysis*, Bull. Amer. Mat. Soc., **83**, 569–645 (1977).
[19] G. Furioli, E. Terraneo *Molecules of the Hardy space and the Navier–Stokes equations*, Funkcial Ekvac. **45**, N. 1 141–160 (2002).
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5.3 Atomic decomposition of the vorticity

[16] `L. Brandolese`
Atomic decomposition for the vorticity of a viscous flow
`Math. Nachr. (2004).`

When considering the motion of incompressible fluids, rather than trying to solve the equation of the velocity field u, one can first study the evolution of the vorticity field $\omega = \nabla \times u$ and then recover u from ω via the classical Biot–Savart law. The advantage of this approach is that the vorticity field often features striking geometrical properties (the so called coherent structures). Understanding the stability properties and the motion of these structures inside the fluid flow is one of the more fascinating challenges of fluid mechanics.

In the paper [16] we showed that the vorticity of a viscous flow in $\mathbb{R}^3$ admits an atomic decomposition of the form $\omega(x,t) = \sum_{k=1}^{\infty} \omega_k(x - x_k, t)$, with localized and oscillating building blocks ω_k, if such a property is satisfied at the beginning of the evolution. We also studied the long time behavior of an isolated coherent structure and the special behavior of flows with highly oscillating vorticities.

The proof of this property is quite technical, as it involves the study of the trajectory of the solution in a function space defined through an infinite sequence of subscripts. The main idea is that one chooses the subscripts depending on the position of the coherent structures at the beginning of the evolution. In other words, the function space where we solve the vorticity equation depends on (and is adapted to) the initial data.

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