Field theory approach to off critical SLE_2 and SLE_4

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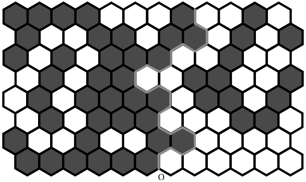
Renormalization: algebraic, geometric and probabilistic aspects

Based on work in collaboration with M. Bauer & D. Bernard

Outline

Introduction

Percolation

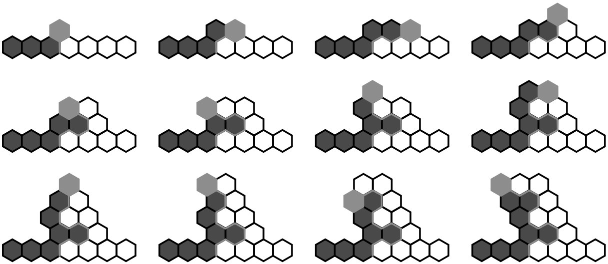


Each site is colored:
White with probability p
Black with probability $1 - p$

The model is critical on the honeycomb lattice at $p = 1/2$

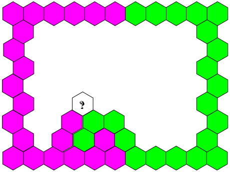
- We want to look at the properties of the interface

Interface as a growth process



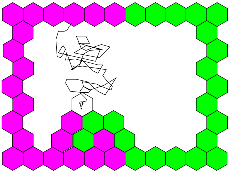
- At each step choose the color of the site with Bernoulli distribution.

Harmonic explorator

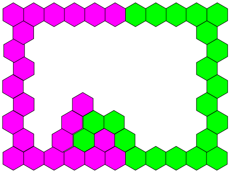


New rule for choosing the new color.

Start a random walk from "?" and stop it when it touches the boundary



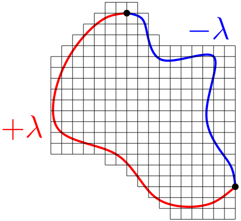
$\longrightarrow$



Gaussian Free Field

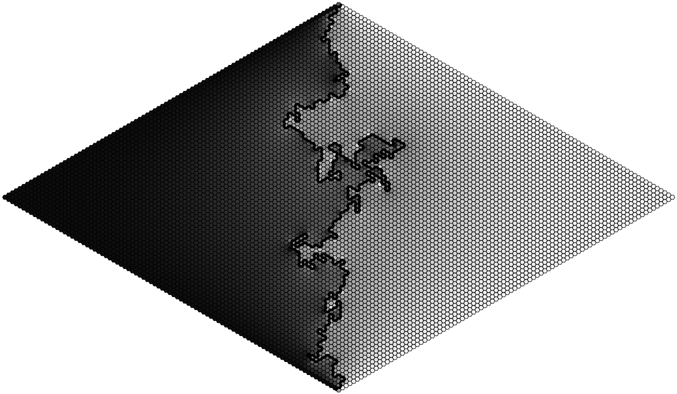
- On each site i of the exagonal lattice there is a variable $h(i)$
- Boltzmann weight $W(\{h(i)\}) = \exp(-\frac{1}{2}\sum_{\langle i,j\rangle}(h(i) - h(j))^2$.

Boundary conditions $\pm\lambda$



- Consider an affinization and look at the Zero Level Line

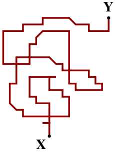
Gaussian Free Field



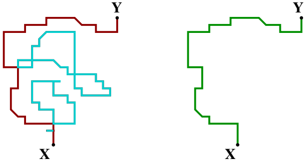
For $\lambda = \sqrt{2}$ the continuum limit of Harmonic explorator and GFF Zero level line are statistically the same SLE_4 [Schramm, Sheffield '03].

Loop Erased Random Walks [Lawler]

Let us consider a Random Walk w from X to Y



Erase the loops in an ordered way; call $\mathcal{L}(w)$ the Loop Erasure of w



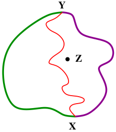
Boltzmann weight of a simple walk

$$\omega_{\mathcal{L}}(\gamma) = \sum_{w|\mathcal{L}(w)=\gamma} \omega(p) \qquad \omega(p) = \mu^{|p|}$$

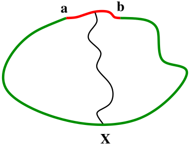
Relations with Uniform Spanning Trees [Pemantle '91, Wilson '96]
Continuum limit SLE_2[Schramm '99]

A few questions we can ask?

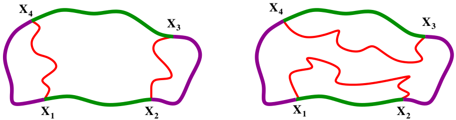
- Left passage



- Fractal dimension $\sim$ passage through a ball.
- Boundary hitting probabilities



- Crossing formulae
- Topology of interfaces



Definition of SLE

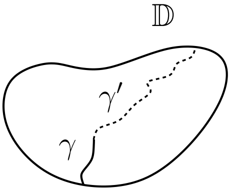
Two key properties of these models

- Domain Markov Property (often valid already in the discrete model)
- Conformal Invariance (valid in the continuum limit)

Domain Markov Property

Suppose we know part of our interface: what is the law of the remaining of the curve?

The conditioned law is the same as the one in the cut domain



$$P_{\mathbb{D}}[\gamma'|\gamma] = P_{\mathbb{D}\setminus\gamma}[\gamma]$$

Conformal Invariance: preliminaries

Conformal invariance is a statement about transport of probabilities under conformal mappings.

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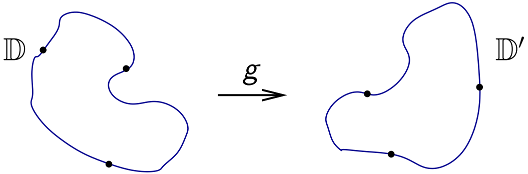
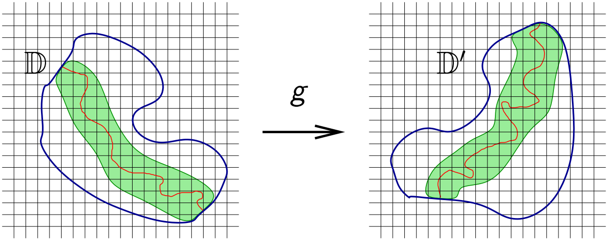
Consider two simply connected domains $\mathbb{D}$ and $\mathbb{D}'$:
Riemann Mapping Theorem:
$\exists!$ a conformal map between $g(z)$ from $\mathbb{D}$ to $\mathbb{D}'$ mapping the three marked points.



Conformal Invariance

The underlying lattice is **NOT** transformed under the mapping $g(z)$



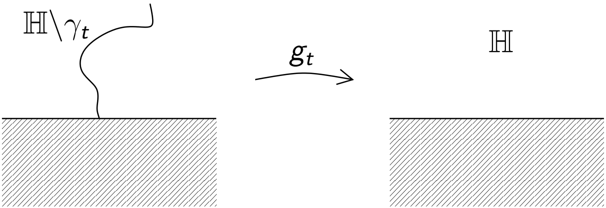
The statement of Conformal Invariance

$$P_{\mathbb{D}}[\gamma] = P_{g(\mathbb{D})=\mathbb{D}'}[g(\gamma)]$$

Löwner equation

Idea: code the shape of the curve by a uniformizing map.

Chordal Löwner equation $\mathbb{H}\backslash\gamma_t \rightarrow \mathbb{H}$



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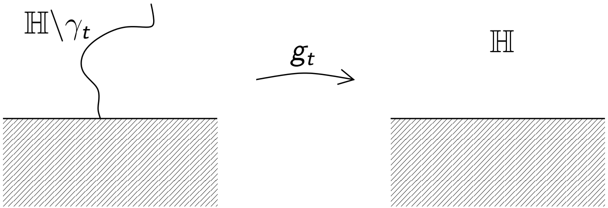

- Hydrodynamical normalization + time parametrization:

$$g_t(z) = z + \frac{2t}{z} + O(1/z^2)$$

Löwner equation

$g_t(z)$ satisfies the equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \xi_t}$$

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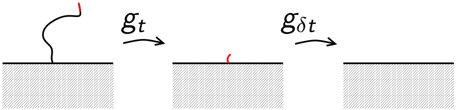
Sketch of the derivation: composition of infinitesimal maps



$$g_{\delta t}(z) = \sqrt{z^2 + 4\delta t} + O(\delta t^2)$$

$$g_{t+\delta t}(z) = g_{\delta t}(g_t(z) - \xi_t) + \xi_t$$

Schramm's idea [Schramm '99]

- If the curve γ is random then its measure induces a probability measure on the driving function ξ_t.

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  Domain Markov Property + Conformal invariance

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We are left with a single parameter κ
$\kappa = 6$: percolation; $\kappa = 4$: Harmonic explorator; $\kappa = 2$: LERW;
$\kappa = 8$: UST; $\kappa = 3$: Ising (spin cluster); $\kappa = 8/3$ SAW.

Some results

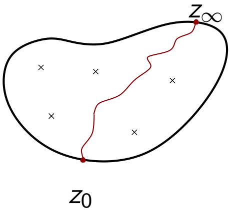
- It is always possible to define a continous curve $\gamma_t = \lim_{z\to 0} g_t^{-1}(z+\xi_t)$ [Rhode & Schramm]
- There are three phases [RS]
  - $\kappa \leq 4$: the curve γ_t is simple.
  - $4 < \kappa < 8$: the curve γ_t can touch itself.
  - $\kappa \geq 8$: the curve γ_t is space filling.
- The fractal dimension of γ_t is $\min\{1+\kappa/8, 2\}$ [Beffara].
- Proof of Mandelbrot's conjecture. The boundary of 5 independent Brownian Excursions is the same as the boundary of 8 independent $\mathrm{SLE}_{8/3}$ starting from the same point, its dimension is $4/3$.

SLE/CFT correspondence

SLE/CFT correspondence I [Bernard, Bauer]

In Boundary Conformal Field theory

$$\langle \mathcal{O} \rangle_{\mathbb{D}} = \frac{\langle \mathcal{O} \text{ BCOs} \rangle_{\mathbb{D}}}{\langle \text{BCOs} \rangle_{\mathbb{D}}} = \frac{\langle \mathcal{O} \psi(z_0)\psi(z_\infty) \rangle_{\mathbb{D}}}{\langle \psi(z_0)\psi(z_\infty) \rangle_{\mathbb{D}}}$$



The ψs are boundary Primary Fields

$$\langle \mathcal{O} \psi(z_0)\psi(z_\infty) \rangle_{\mathbb{D}} = |g'(z_0)|^h |g'(z_\infty)|^h \langle {}^g\mathcal{O} \psi(g(z_0))\psi(g(z_\infty)) \rangle_{g(\mathbb{D})}$$

The Martingale Condition

$$\langle \mathcal{O} \rangle_{\mathbb{D}\setminus\gamma_t} = \frac{\langle {}^{g_t}\mathcal{O} \psi(g_t(z_0))\psi(g_t(z_\infty)) \rangle_{\mathbb{D}}}{\langle \psi(g_t(z_0))\psi(g_t(z_\infty)) \rangle_{\mathbb{D}}} = \text{It is a (local) martingale}$$

imposes conditions on the fields ψ

SLE/CFT correspondence II

- The conformal weigth of ψ is related to κ:

$$h_{1,2} = \frac{6-\kappa}{2\kappa}$$

- The correlation functions satisfy a differential equation which says that they must be degenerate at level -2

This fixes the value of the central charge

$$c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$$

consistent with the previous identification of the statistical models: Percolation & SAW $c=0$, GFF $c=1$, Ising $c=1/2$, LERW & UST $c=-2$.

Off critical SLE

Naive considerations I

- We want to show how the off-critical measure on random curves is related to the critical one.
- One can still use the Löwner map: how does the the law of the driving function get modified?
- At scales smaller than the correlation length (but large w.r.t. the lattice spacing) the curve should look like a critical SLE

$$\text{"}\lim_{\lambda\to 0^+}\frac{1}{\lambda}(\xi_{s+\lambda^2 t}-\xi_s)\sim\sqrt{\kappa}B_t\text{"}$$

- Does something even stronger hold?

$$d\xi_t=\sqrt{\kappa}B_t+F_t dt \qquad ?$$

Naive considerations II

The probability of a piece of interface in a discrete model out of criticality is

$$\mathbb{P}[\gamma_t] = \frac{\sum_{\rho|\gamma_t} w_\rho}{\sum_\rho w_\rho} = \frac{Z_{\mathbb{D}}[\gamma_t]}{Z_D} = M_t \ \mathbb{P}^{(c)}[\gamma_t]$$

with

$$M_t = \frac{Z_{\mathbb{D}}[\gamma_t]/Z^c_{\mathbb{D}}[\gamma_t]}{Z_D/Z^c_{\mathbb{D}}}$$

By construction this is a martingale (= conserved in average).

Assume for the moment that it remains well defined in the continuum limit (need some precisation).

Girsanov's theorem

If B_t is a Brownian Motion w.r.t. $\mathbb{P}^c$, then how does it get modified by $\mathbb{P} = M_t \mathbb{P}^c$?

It gets a drift term

$$dB_t = d\tilde{B}_t + F_t dt$$

$\tilde{B}_t$ is Brownian Motion w.r.t. the off critical measure $\mathbb{P}$, while

$$M_t^{-1} dM_t = F_t dB_t$$

We shall apply this to the driving function of our favourite SLE, which is proportional to a Brownian Motion.

How to compute M_t and drift: field theory I

$$M_t \sim \frac{Z_{\mathbb{D}}[\gamma_t]}{Z^c_{\mathbb{D}}[\gamma_t]} = e^{E_{\mathbb{D}}[\gamma_t] - E^c_{\mathbb{D}}[\gamma_t]} \frac{Z_{\mathbb{D}\setminus\gamma_t}}{Z^c_{\mathbb{D}\setminus\gamma_t}}$$

R.G. arguments tells us when $E_{\mathbb{D}}[\gamma_t]$ & $E^c_{\mathbb{D}}[\gamma_t]$ are irrelevant: this holds for LERW and GFF.

Use field theory to compute $\frac{Z_{\mathbb{D}\setminus\gamma_t}}{Z^c_{\mathbb{D}\setminus\gamma_t}}$

- SLE_4 Massive free field

$$S_{FF}(\varphi) = \int \frac{dz^2}{8\pi} \partial\varphi\bar{\partial}\varphi + m^2\varphi^2$$

- SLE_2 Massive symplectic fermions

$$S_{sf}(\chi^{\pm}) = \int dz^2 (4\partial\chi^+\bar{\partial}\chi^- + m^2\chi^+\chi^-)$$

How to compute M_t and drift II: gaussian free field

$$M_t = \frac{\int [D\varphi]_{DBC} e^{\int \frac{dz^2}{8\pi} \partial\varphi\bar{\partial}\varphi + m^2\varphi^2}}{\int [D\varphi]_{DBC} e^{\int \frac{dz^2}{8\pi} \partial\varphi\bar{\partial}\varphi}} = e^{-\int \frac{dz^2}{8\pi} m^2 \phi_t \Phi_t^{[m]}} \left[\frac{\det[-\Delta + m^2]_{\mathbb{H}_t}}{\det[-\Delta]_{\mathbb{H}_t}} \right]^{-1/}$$

$$\Delta\phi_t = 0, \qquad \Delta\Phi_t - m^2\Phi_t = 0$$

plus discontinuos Dirichlet BCs in $\mathbb{H}_t$.
The rigorous definition of the Laplacian determinant is through $\zeta-$function regularization.

Task: Prove that this is a martingale.

The tricky part is to compute the derivative of the determinants ratio. Recast it in such a way one can use Hadamard formula.

How to compute M_t and drift II: gaussian free field

The result for the drift is

$$F_t = -2\sqrt{2}\int \frac{dz^2}{4\pi} m^2 \Phi_t(z)\theta_t(z)$$

$\theta_t(z) = -\Im\mathrm{m}\frac{2}{g_t(z)-\xi_t}$ is related to the Poisson kernel in $\mathbb{H}_t$

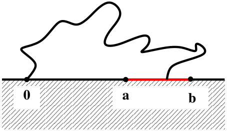
Byproduct:

- Decomposition of the field

$$\mathbb{E}[\langle e^{J_+\star\varphi}\rangle_{\mathbb{H}_+}\langle e^{J_-\star\varphi}\rangle_{\mathbb{H}_-}] = \langle e^{J\star\varphi}\rangle_{\mathbb{H}}$$

Drift for LERW

We consider so called Dipolar SLE on $\mathbb{H}$



- Critical driving function

$$d\xi_t^c = \sqrt{2}dB_t + F^c_{t,[a,b]}dt$$

- Girsanov's martingale

$$M_t = \left[\frac{\det[-\Delta + m^2]_{\mathbb{H}_t}}{\det[-\Delta]_{\mathbb{H}_t}}\right] \frac{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x)\rangle^m_{\mathbb{H}_t}}{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x)\rangle^{m=0}_{\mathbb{H}_t}}$$

$\psi^{\pm}(x) = \lim_{\delta\to 0} \delta^{-1}\chi^{\pm}(x + i\delta)$

- Off critical drift

$$F_{t,[a,b]} = 2\partial_{\xi_t} \log\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x)\rangle^m_{\mathbb{H}_t}$$

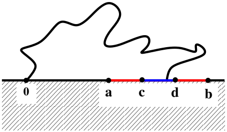
An application: boundary hitting probability

- Consistency check: ratios of correlation functions of symplectic fermions must be martingales

$$\frac{\langle \psi^+(\gamma_t) \prod_{j=1}^{N+1} \chi^-(z_j) \prod_{j=1}^{N} \chi^N(z_j) \rangle}{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle^m_{\mathbb{H}_t}}$$

- Boundary Hitting Probability

$$P_{[a,b]}^{[c,d]} = \frac{\langle \psi^+(\gamma_t) \int_c^d dx \psi^-(x) \rangle^m_{\mathbb{H}_t}}{\langle \psi^+(\gamma_t) \int_a^b dx \psi^-(x) \rangle^m_{\mathbb{H}_t}}$$



Conclusions & open problems

- Conclusions
  - SLE techniques to describe interfaces in 2D models & relation to CFT approach.
  - Girsanov's theorem to describe SLEs out of criticality.
  - Application to SLE_2 and SLE_4: computation of the drift.
- Perspectives
  - Computation of other probabilities for LERW (left passage)
  - Off criticality for other values of κ and other kinds of perturbations (e.g. integrable perturbations): much more ambitious.