Connection between the renormalization groups of Stückelberg-Petermann and Wilson

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joint work with Romeo Brunetti
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Outline

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- **Stückelberg - Petermann RG** $\mathcal{R}$ (Causal perturbation theory) Non-uniqueness of S-matrix.
  Change $S \to \hat{S}$ of the renormalization presription can be absorbed in a renormalization of the interaction $V \to Z(V)$:

$$\hat{S}(V) = S(Z(V)) \quad \forall\, V$$

  $\mathcal{R} = \{\text{appearing } Z\}$ is a group - group of finite renormalizations of S.

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  Definition of the effective potential V_Λ at scale Λ: Let V original interaction. Then
  $S_\Lambda(V_\Lambda) = S(V)$ i.e. $V_\Lambda := S_\Lambda^{-1} \circ S\,(V)$

S unknown $\longrightarrow$ One computes V_Λ by solving **flow equation** (Polchinski, Salmhofer, Kopper etc.):

$$\text{Def } V_\Lambda \quad \Rightarrow \quad \frac{d}{d\Lambda} V_\Lambda = F_\Lambda(V_\Lambda \otimes V_\Lambda)$$

where F_Λ is linear and explicitly known.

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$S(V)$ is obtained by intergrating flow equation and computing

$$\lim_{\Lambda \to \infty} S_\Lambda(V_\Lambda)$$

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field configuration space: $C^\infty(\mathbb{M})$ ("off-shell formalism")

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- $\mathcal{F}:\ \frac{\delta^n F}{\delta\varphi^n}$ is a distribution (includes local functionals)
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In $\mathcal{F}, \mathcal{F}_{\mathrm{loc}}$ additional condition on $\mathrm{WF}(\frac{\delta^n F}{\delta\varphi^n})$ which is a microlocal version of translation invariance.

Example for a **local** observable:

$F(\varphi) = \int dx\, f(x)\, L(\varphi(x), \partial\varphi(x), \ldots)$,

$f \in \mathcal{D}$, $L \in \mathcal{C}^\infty$ does not need to be a polynomial.

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Poisson algebra of free fields

$\Delta_R\,,\, \Delta_A$: retarded, advanced propagator of KG-operator
In terms of $\Delta = \Delta_R - \Delta_A$ (commutator function) one defines Poisson bracket and obtains **Poisson algebra of free fields**.

$$\{F, G\} \stackrel{\text{def}}{=} \int dxdy\, \frac{\delta F}{\delta\varphi(x)}\, \Delta(x-y)\, \frac{\delta G}{\delta\varphi(y)}$$

Definition of $\star_p$ (product with propagator p):

Let $p \in \mathcal{S}'(\mathbb{M})$ with suitable properties which depend on whether the functionals F and G are non-local ($F, G \in \mathcal{F}_0$) or not ($F, G \in \mathcal{F}$)

$$F \star_p G := \sum_{n\geq 0} \frac{\hbar^n}{n!} \int dx_1 ... dy_1 ... \frac{\delta^n F}{\delta\varphi(x_1)...\delta\varphi(x_n)}$$

$$p(x_1 - y_1)...p(x_n - y_n) \frac{\delta^n G}{\delta\varphi(y_1)...\delta\varphi(y_n)} \; .$$

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$\Rightarrow \star_p$ is associative

$\star$-product quantization

$p = H =$ Hadamard function, satisfies

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$F \star_H G$ is a $\star$-product, i.e. it is $\hbar$-dependent deformation of $F \cdot G$,

$$\lim_{\hbar \to 0} F \star_H G = F \cdot G \ ,$$

with

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}(F \star_H G - G \star_H F) = \{F, G\} \ .$$

Time ordered product

The time ordered product corresponding to $\star$-product $\star_H$ must satisfy

$$T(\varphi(x)\varphi(y)) := \begin{cases} \varphi(x) \star_H \varphi(y) & \text{if} \quad x^0 > y^0 \\ \varphi(y) \star_H \varphi(x) & \text{if} \quad y^0 > x^0 \end{cases}$$
$$= \varphi(x) \star_{H_F} \varphi(y)$$

where $H_F(z) := \Theta(z^0)H(z) + \Theta(-z^0)H(-z)(= H_F(-z))$.

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$F_1 \star_{H_F} \dots \star_{H_F} F_n$ exists **only** for **non-local** $F_1, \dots, F_n \in \mathcal{F}_0$, since $\nexists (H_F(z))^n$.

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Time ordered product of local functionals

Assumption: All observables $\in \mathcal{F}_{\mathrm{loc}}, \mathcal{F}$ are **polynomial** in φ.

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can be defined by **renormalization** as follows:

T_n's are linear and totally symmetric maps defined in terms of S-**matrix** (= generating functional)

$$S : \mathcal{F}_{\mathrm{loc}} \to \mathcal{F}$$
$$T_n(V^{\otimes n}) = S^{(n)}(0)(V^{\otimes n}) \equiv \frac{d^n}{d\lambda^n} S(\lambda V)|_{\lambda=0} \ .$$

Defining axioms for S-matrix (causal perturbation theory)

Causality $S(A+B) = S(A) \star_H S(B)$ if $\mathrm{supp}\, A$ is later than $\mathrm{supp}\, B$.

Starting element $S(0) = 1$, $S^{(1)}(0) = \mathrm{id}$

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(Unitarity) $\overline{S(-V)} \star_H S(\overline{V}) = 1$ (complex conjugation)

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Proof: construction of the time ordered products T_n by induction on n (Epstein-Glaser).

Stückelberg - Petermann RG

Definition: Stückelberg - Petermann RG $\mathcal{R}$ is the **group** of analytic bijections $Z : \mathcal{F}_{\rm loc} \to \mathcal{F}_{\rm loc}$ with

Starting element

$$Z(0) = 0\,, \quad Z^{(1)}(0) = \mathrm{id}\,, \quad Z = \mathrm{id} + O(\hbar)$$

Locality Z is local:

$$Z(A+B+C) = Z(A+B) - Z(B) + Z(B+C)$$

if $\mathrm{supp}\, A \cap \mathrm{supp}\, C = \emptyset$

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(Poincaré invariance)

(Unitarity)

(almost homogeneous Scaling)

Change of renormalization prescription

Main Theorem (Uniqueness): (i) Given two renormalization prescriptions S and $\hat{S}$ there exists a unique $Z \in \mathcal{R}$ with $\hat{S} = S \circ Z$.

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Sketch of Proof: (ii) direct verification

(i) inductive construction of $Z^{(n)} \equiv Z^{(n)}(0)$, $n \in \mathbb{N}$:

example $\hat{S}(V) = S(Z(V))$ to 3rd order in V

$$\hat{S}^{(3)}(V^{\otimes 3}) = S^{(3)}(V^{\otimes 3}) + c\, S^{(2)}(V \otimes Z^{(2)}(V^{\otimes 2})) + Z^{(3)}(V^{\otimes 3})$$

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$$S^{(3)}(V^{\otimes 3}) + c\, S^{(2)}(V \otimes Z^{(2)}(V^{\otimes 2})) = (S \circ Z_2)^{(3)}(V^{\otimes 3})$$

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Part (ii) $\Rightarrow S \circ Z_2$ is admissible S-matrix which coincides with $\hat{S}$ in orders $k \leq 2$. Setting

$$Z^{(3)} := \hat{S}^{(3)} - (S \circ Z_2)^{(3)}\ , \quad Z_3(V) := Z_2(V) + \frac{Z^{(3)}(V^{\otimes 3})}{3!}$$

it follows $Z_3 \in \mathcal{R}$. □

Outline

Regularized Feynman propagator

Propagator of time-ord. product is $H_F(z)(= H_F(-z))$.
As a cutoff we approximate H_F by a family of symmetric testfunctions (or sufficiently regular distributions) $(p_\Lambda)_{\Lambda>0}$:

$$\lim_{\Lambda\to\infty} p_\Lambda = H_F \quad \text{in appropriate topology}$$

and for $\Lambda = 0$ it is required that $p_0 = 0$.

Regularized time-ordered product

Def: regularized time-ordered product

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Def: regularized S-matrix (corresponding generating functional)

$$S_\Lambda \,:\, \mathcal{F} \to \mathcal{F}\,;\, S_\Lambda(F) = \sum_n \frac{1}{n!}\, T_\Lambda(F^{\otimes n}) = e^F_{\star_{p_\Lambda}} \;.$$

Proposition:

S_Λ is invertible (in contrast to S).

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Proof: write $\star_{p_\Lambda}$ alternatively as

$$F \star_{p_\Lambda} G = \tau_\Lambda \left(\tau_\Lambda^{-1} F \cdot \tau_\Lambda^{-1} G\right) ,$$

where

$$\tau_\Lambda F \doteq \exp(i\hbar \Gamma_\Lambda) F$$

with

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$$\Gamma_\Lambda \doteq \frac{1}{2} \int dx\, dy \;\; p_\Lambda(x-y) \, \frac{\delta^2}{\delta\varphi(x)\delta\varphi(y)} .$$

With that

$$S_\Lambda = \tau_\Lambda \circ \exp \circ \tau_\Lambda^{-1}$$

and hence

$$\exists \quad S_\Lambda^{-1} = \tau_\Lambda \circ \log \circ \tau_\Lambda^{-1} . \quad \square$$

Example: Euklidean theory with mass $m > 0$

(following Salmhofer). Let $K(x)$ be a smooth approximation of $\theta(1-x)$. We set (in momentum space)

$$\hat{p}_\Lambda(k) := \frac{1}{(2\pi)^2\,(k^2+m^2)}\,K(\frac{k^2}{\Lambda^2}) \quad (k^2 \equiv k_0^2 + \vec{k}^2)$$

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$$p_{\Lambda,\Lambda_0} := p_{\Lambda_0} - p_\Lambda \quad (0 < \Lambda \le \Lambda_0 < \infty)$$

contribution only for $\Lambda^2 < k^2 < \Lambda_0^2$ (UV- and IR-cutoff).

Example: ϵ-regularized relativistic theory with $m > 0$

(following Keller, Kopper and Schophaus). Let $\epsilon > 0$ and

$$\hat{p}_\Lambda(k) := \frac{i\, e^{-\Lambda^{-1}(k\eta_\epsilon k + (\epsilon + i)m^2)}}{(2\pi)^2\,(k\eta_\epsilon k + (\epsilon + i)m^2)}\ ,$$

where

$$k\eta_\epsilon k := k_0^2\,(\epsilon - i) + \vec{k}^2\,(\epsilon + i)\ ,$$

hence $\quad \mathrm{Re}(k\eta_\epsilon k + (\epsilon + i)m^2) > 0 \quad \forall k\ .$

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Obviously (pointwise)

$$\lim_{\epsilon\downarrow 0}\lim_{\Lambda\to\infty}\hat{p}_\Lambda(k) = \frac{1}{(2\pi)^2\,(m^2 - k^2 - i0)}\ , \quad \lim_{\Lambda\to 0}\hat{p}_\Lambda = 0\ .$$

Construction of S (renormalization) by adding counterterms and removing cutoff

S_Λ diverges for $\Lambda \to \infty$,

Construction of S (renormalization) by adding counterterms and removing cutoff

S_Λ diverges for $\Lambda \to \infty$,
but for a renormalizable model there exists $\forall\Lambda$ a $Z_\Lambda \in \mathcal{R}$ with

$$\lim_{\Lambda\to\infty} S_\Lambda \circ Z_\Lambda = S \ .$$

Z_Λ adds the **local** counter terms which are needed that limit exists.

Outline

Definition of effective potential in terms of (unknown) S

Know that S exists (e.g. from Epstein - Glaser).
Define V_Λ ("effective potential at scale Λ") by
(Exact theory with interaction V)=(cutoff theory with V_Λ)

$$S(V) = S_\Lambda(V_\Lambda) \ , \quad \text{i.e.} \quad V_\Lambda := S_\Lambda^{-1} \circ S\,(V)$$

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In general V_Λ is not an element of $\mathcal{F}_{\mathrm{loc}}$.

For $\Lambda = 0$ we have $S_0(V) = e^V$, hence

$$V_0 = \log \circ S(V)\ ,$$

but $\lim_{\Lambda\to\infty} V_\Lambda$ does not exist in general.

Theorem (Flow equation)

$$\frac{d}{d\Lambda} V_\Lambda = -\frac{1}{2}\frac{d}{d\lambda}|_{\lambda=\Lambda}(V_\Lambda \star_{p_\lambda} V_\Lambda)$$
$$= -\frac{\hbar}{2}\int dx\, dy\, \frac{d\, p_\Lambda(x-y)}{d\Lambda}\, \frac{\delta V_\Lambda}{\delta\varphi(x)} \star_{p_\Lambda} \frac{\delta V_\Lambda}{\delta\varphi(y)}$$

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Sketch of Proof

$$0 = \frac{d}{d\Lambda}S_\Lambda(V_\Lambda) = \frac{d}{d\lambda}|_{\lambda=\Lambda}S_\lambda(V_\Lambda) + \frac{d\,V_\Lambda}{d\Lambda} \star_{p_\Lambda} S_\Lambda(V_\Lambda)$$
$$\Rightarrow \frac{d\,V_\Lambda}{d\Lambda} = -\frac{d}{d\lambda}|_{\lambda=\Lambda}S_\lambda(V_\Lambda) \star_{p_\Lambda} S_\Lambda(V_\Lambda)^{-1}$$

Using $S_\lambda(F) = e^F_{\star_{p_\lambda}}$ it results the assertion.

Construction of S by solving flow equation

Flow equation can be integrated in perturbation theory (expansion in V: $V_\Lambda = V + \mathcal{O}(V^2)$)

$$\frac{d}{d\Lambda} V_\Lambda^{(n)} = \sum_{k=1}^{n-1} -\frac{1}{2} \frac{d}{d\lambda}|_{\lambda=\Lambda} (V_\Lambda^{(k)} \star_{p_\lambda} V_\Lambda^{(n-k)}) \ .$$

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V_Λ diverges in general for $\Lambda \to \infty$. But

$$\lim_{\Lambda\to\infty} S_\Lambda(V_\Lambda)$$

exists and gives the wanted $S(V)$.

Outline

Usual procedure (Euklidean, following Salmhofer)

Def. of effective action G_{Λ,Λ_0}

$$e^{G_{\Lambda,\Lambda_0}(\psi)} := \int d\mu_{p_{\Lambda,\Lambda_0}}(\phi)\, e^{-\lambda V(\phi+\psi)} \;,$$

where $p_{\Lambda,\Lambda_0} := p_{\Lambda_0} - p_\Lambda$ and $V =$ unrenormalized interaction.
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Flow equation

Computing $\frac{\partial}{\partial\Lambda}$ of this functional integral one derives the **flow equation**. Perturbation theory: flow eq. expresses $\frac{\partial G^{(r)}_{\Lambda,\Lambda_0}}{\partial\Lambda}$ (= term of order r in coupling constant λ) in terms of $G^{(k)}_{\Lambda,\Lambda_0}$, $k < r$ (inductively known).

Solving the flow equation

$$G^{(r)}_{\Lambda,\Lambda_0} = G^{(r)}_{\Lambda_0,\Lambda_0} - \int_{\Lambda}^{\Lambda_0} d\Lambda' \, \frac{\partial G^{(r)}_{\Lambda',\Lambda_0}}{\partial \Lambda'}$$

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Freedom in choosing **boundary value** G_{Λ_0,Λ_0}:
For $G_{\Lambda_0,\Lambda_0} = V$: $\lim_{\Lambda_0\to\infty} G_{\Lambda,\Lambda_0}$ does not exist (usual UV-divergences)!

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such that this limit exists.

The theory is 'perturbatively renormalizable' if this is possible by a *finite* number of counterterms.

In case of ϕ_4^4

$$G_{\Lambda_0,\Lambda_0} = \lambda\phi^4(1 + \sum_{r\geq 2} c_{\Lambda_0}^{(r)}) + \sum_{r\geq 2}(a_{\Lambda_0}^{(r)}\,\phi^2 + b_{\Lambda_0}^{(r)}\,(\partial\phi)^2)\ .$$

Comparison with our formalism

- $$V_\Lambda := S_\Lambda^{-1} \circ S(V) \quad \text{corresponds to} \quad \lim_{\Lambda_0 \to \infty} G_{\Lambda,\Lambda_0} \; .$$

  In particular for $\Lambda = 0$ we have

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  $\to S_\Lambda^{-1} \circ S$ corresponds to "integrating out the degrees of freedom above scale Λ".

  Existence of $\lim_{\Lambda_0 \to \infty} G_{\Lambda,\Lambda_0}$ involves renormalization (addition of suitable counterterms). Also definition of V_Λ presupposes renormalization, since V_Λ is defined in terms of the exact (=renormalized) S-matrix.

▶

$$\text{Def. } V_\Lambda \quad \Rightarrow \quad V_\Lambda = S_\Lambda^{-1} \circ S_{\Lambda_0}(V_{\Lambda_0})$$

i.e. $S_\Lambda^{-1} \circ S_{\Lambda_0}$ is "flow of eff. potential from Λ_0 to Λ".

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  From $\lim_{\Lambda\to\infty} S_\Lambda \circ Z_\Lambda = S = \lim_{\Lambda_0\to\infty} S_{\Lambda_0} \circ Z_{\Lambda_0}$ we (heuristically) obtain

  $$S_\Lambda^{-1} \circ S_{\Lambda_0} \approx Z_\Lambda \circ Z_{\Lambda_0}^{-1} \in \mathcal{R} \quad (\text{for} \quad \Lambda, \Lambda_0 \to \infty)$$

  ($\mathcal{R}$ = Stückelberg-Petermann RG)

Outline

Stückelberg - Petermann group $\mathcal{R}$

a finite renormalization $S \to \hat{S}$ can equivalently be expressed by a transformation $Z \in \mathcal{R}$ of the interaction (by means of $\hat{S}(V) = S(Z(V)))$.
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RG in the sense of Wilson

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RG in the sense of Wilson

Effective potential can be defined and flow equation can simply be proved in the framework of causal perturbation theory.

Flow of effective potential from Λ_0 to Λ is the map

$$S_\Lambda^{-1} \circ S_{\Lambda_0} \approx Z_\Lambda \circ Z_{\Lambda_0}^{-1} \in \mathcal{R}$$

heuristically for $\Lambda, \Lambda_0 \to \infty$, i.e. Wilsons flow can be approximated by a subfamily of the Stückelberg - Petermann group.