Epstein-Glaser Renormalization and Dimensional Regularization

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(based on joint work with Romeo Brunetti, Michael Dütsch and Kai Keller)

Introduction

Quantum field theory is an extremely ambitious and at the same time incredibly successful theory.

Early successes: Renormalization theory of Tomonaga, Schwinger, Feynman and Dyson made high precision predictions for QED (g-2, Lamb shift etc.)

But: Treatment of divergences was inconsistent (overlapping divergences).

After 2 decades of hard work by Stückelberg, Bogoliubov and others Hepp produced the first complete proof of renormalizability.

Zimmermann's Forest Formula solved the combinatorial problems and provided a closed expression for the renormalized scattering amplitudes.

Incorporation of the locality principle by Epstein-Glaser and by Steinmann.

Practical problems with gauge theory circumvented by dimensional regularization ('t Hooft-Veltman, Bollini-Giambiagi).

Forest formula can be adapted to dimensional regularization (Breitenlohner-Maison).

Nowadays: Good, partially excellent understanding of elementary particle physics.

But the perturbative framework continues to to be quite involved.

Recent developments:

- Combinatorial analysis of the Forest formula (Connes-Kreimer)
- New methods for large order calculations, partially inspired from String theory
- New attempts for the strong coupling regime of QCD based on AdS-CFT correspondence
- Local construction of perturbative QFT, in particular for curved spacetimes (Brunetti,Dütsch,F,Hollands,Wald)

Epstein-Glaser Renormalization

Basic idea: Time ordered product of interaction Lagrangeans is determined by operator product up to coinciding points.

$$T\mathcal{L}(x)\mathcal{L}(y) = \begin{cases} \mathcal{L}(x)\mathcal{L}(y) & , \quad \text{if } x \text{ is not in the past of } y \\ \mathcal{L}(y)\mathcal{L}(x) & , \quad \text{if } x \text{ is not in the future of } y \end{cases}$$

Singularities treated by the theory of distributions.

Main feature: Extension to coinciding points is always possible, but unique only up to the addition of a δ-function or its derivatives.

This corresponds directly to the freedom in the choice of renormalization conditions.

Equivalent: Additional finite counterterms can be added to the Lagrangean in every order of perturbation theory (Main Theorem of Renormalization (Stora-Popineau,Pinter,Dütsch-F)).

$$\hat{S} = S \circ Z$$

$\hat{S}, S$ generating functionals of time ordered products, $Z \in \mathrm{Ren}$

General feature of EG-Renormalization: Rigorous , conceptually clear, but difficult for practical applications.

Experience: For every concrete calculation a new method has to be found.

Dimensional Regularization

Observation: In the calculation of Feynman integrals the dimension d of spacetime appears as a parameter.

Formally, the dimension can assume any complex value.

For $\mathrm{Re}\, d$ sufficiently small the Feynman integrals converge to an analytic function of d.

This function can be analytically continued to a meromorphic function on the complex plane.

Subtraction of the principal part of the Laurent series around the physical dimension delivers a well defined expression.

Remaining problem: One has to prove that the subtractions are compatible with the requirement of locality, i.e. the principal part must be a polynomial in the external momenta.

Lower orders: by inspection

General case: Subtractions have to be done according to the Forest Formula.

Connes-Kreimer: Description in terms of a Hopf algebra

Interesting connections to pure mathematics (Number theory, Noncommutative geometry etc.)(Brouder, Frabetti, Marcolli . . .)

Dimensional regularization versus EG renormalization

Question: Is it possible to combine the nice conceptual features of the EG-method with the effective computational tools stemming from Dimensional Regularization, combined with the Forest Formula?

Difficulties: Different concepts

- Position space vs momentum space (no curved spacetime analog)
- No divergences in EG vs poles of meromorphic functions
- Inductive construction without counter terms vs closed formula with counter terms

Single graph:

EG: Feynman graph t with n vertices, renormalized up to order $n-1$ is a distribution on $n-1$ independent difference variables which is well defined outside of the origin.

Scaling degree (Steinmann):

$$\mathrm{sd} t = \inf\{\delta \in \mathbb{R}, \lambda^{\delta} t(\lambda x) \to 0 \text{ for } \lambda \to 0\}$$

Consequence: t is uniquely determined on test functions

$$f \in \mathcal{D}_{\mathrm{div} t}(\mathbb{R}^{4(n-1)})$$

which vanish up to order $\mathrm{div}(t) := \mathrm{sd} t - 4(n-1)$ at the origin.

Extension to all test functions:
Choose a projection

$$W : \mathcal{D}(\mathbb{R}^{4(n-1)}) \to \mathcal{D}_{\mathrm{div}\, t}(\mathbb{R}^{4(n-1)})$$

and set

$$\bar{t} := t \circ W$$

All extensions are of this form for a suitable projection W, and W can be explicitely given by

$$Wf = f - \sum_{|\alpha| \leq \mathrm{sd}\, t - 4(n-1)} w_\alpha \partial^\alpha f(0)$$

with test functions w_α which satisfy

$$\partial^\alpha w_\beta(0) = \delta_{\alpha\beta} \ .$$

DimReg:

Feynman graph t , dimensionally regularized and renormalized at lower orders ("prepared"), is a distribution valued meromorphic function of the dimension d with a principal part at $d = 4$ consisting of δ-functions and derivatives.

Renormalization ("Minimal Subtraction" (MS))

$$t^{\mathrm{MS}} = \lim_{d \to 4} \left(t(d) - \sum_{n=1}^{\infty} \int_C \frac{dz}{2\pi i} (d-4)^{-n} z^{n-1} t(4+z) \right)$$

where C denotes a little circle around $z = 0$.

Connection to EG: Choose W such that $t \circ W = t^{MS}$

Always possible for the prepared Feynman graph (pole terms are local), but W depends on $t(d)$.

Remaining problem: EG induction requires decomposition of the space of configurations into subsets C_I such that the points $x_i, i \in I$ are not in the past of $x_j, j \notin I$.

No counter part exists in the Forest Formula; moreover, this decomposition requires a C^∞-decomposition of the unit and is not constructive.

Way out: use regularization and the fact that

$$\sum_\alpha \langle t(d), w_\alpha \rangle (-1)^{|\alpha|} \partial^\alpha \delta(x)$$

is meromorphic in d with the same principal part as $t(d)$.

Ansatz: Interpret the Main Theorem of Renormalization as a relation between renormalized and regularized S-matrix

$$S_{\mathrm{ren}} = S_{\mathrm{reg}} \circ Z_{\mathrm{counter}}$$

At nth order by the chain rule (Faà di Bruno formula)

$$S_{\mathrm{ren}}^{(n)} = \sum_{P \in \mathrm{Part}(\{1,\ldots,n\})} S_{\mathrm{reg}}^{(|P|)} \left(\bigotimes_{I \in P} Z_{\mathrm{counter}}^{(|I|)} \right)$$

Minimal subtraction: Use $S^{(1)} = \mathrm{id}$, set $Z^{(1)} = \mathrm{id}$ and

$$Z_{\mathrm{counter}}^{(n)} = -\mathrm{pp} \sum_{P \in \mathrm{Part}(\{1,\ldots,n\}), |P|>1} S_{\mathrm{reg}}^{(|P|)} \left(\bigotimes_{I \in P} Z_{\mathrm{counter}}^{(|I|)} \right),$$

for $n > 1$ (pp principal part of a Laurent series).

Explicit form of S_{reg}:

$$S_{\mathrm{reg}} = e^{\frac{1}{2}D} \circ \exp \circ e^{-\frac{1}{2}D}$$

with

$$D = \langle H_F(d), \frac{\delta^2}{\delta\varphi^2} \rangle$$

At nth order (m_n n-fold product)

$$S^{(n)}_{\mathrm{reg}} = m_n \circ \exp(\sum_{i<j} D_{ij})$$

Ansatz for $Z^{(n)}_{counter}$:

$$Z^{(n)}_{counter} = m_n \circ z^{(n)}$$

$z^{(n)}$ functional differential operator for functionals of n independent fields.

Leibniz rule:

$$\frac{\delta}{\delta\varphi} \circ m_n = m_n \circ (\sum_{i=1}^{n} \frac{\delta}{\delta\varphi_i})$$

Insertion into recursion formula for counter terms

$$z^{(n)} = -\mathrm{pp} \sum_{|P|>1} \exp(\sum_{i<_P j} D_{ij}) \prod_{I\in P} z^{(I)}$$

with $i <_P j$ if $1 \leq i < j \leq n$ and $i, j \notin I$, $I \in P$.

Solution of the recursion relation:
Definition of a EG-forest:
An EG-forest $F \in \mathfrak{F}_n$ is a family of subsets $I \subset \{1, \ldots, n\}$, $|I| > 1$ such that for $I, J \in F$ either $I \subset J$, $J \subset I$ or $I \cap J = \emptyset$ holds.
EG-Forest formula

$$S^{(n)}_{\rm ren} = \lim_{d \to 4} \sum_{F \in \mathfrak{F}_n} \prod_{I \in F} R_I S^{(n)}_{\rm reg}(d)$$

Definition of R_I: R_I acts on the dimension variables associated to pairs $i, j \in I$ as $(-\text{principal part})$, for $I < J$, R_I is performed before R_J.

Crucial: "dimensions" can be chosen independently for every pair of indices.

Question: The EG forests are special Zimmermann forests. Why are the other subtractions irrelevant?

Answer (Zimmermann 1975): The other subtractions are spurious.

Example: ("setting sun") 3 lines connecting 2 vertices, no derivatives.

Unrenormalized graph: $t = \Delta_F^3$

$\mathrm{sd}\, t = 6 \implies t$ is defined on test functions vanishing at 0 to 2nd order.

$t^{\text{ren}} = t \circ W$, W projects on space of admissible test functions

Subgraph $t_2 = \Delta_F^2$ ("fish") has scaling degree 4, hence is defined on test functions which vanish at 0.

But the pointwise product $\Delta_F \cdot (Wf)$ vanishes already at 0, since the singularity of Δ_F is $\frac{1}{x^2}$, hence if W_2 is the projection onto test functions vanishing at the origin, we find

$$\langle \Delta_F^2, W_2 \Delta_F \cdot (Wf) \rangle = \langle \Delta_F^2, \Delta_F \cdot (Wf) \rangle = \langle \Delta_F^3, Wf \rangle$$

Conclusions

- Dimensional regularization can be understood as a regularization of the Feynman propagator in 4 dimensional position space. (Scheck et al.)
- Techniques of distribution theory allow a unique determination of Feynman graphs at dimensions near to, but different from 4.
- Arising combinatorics similar, but much simpler than in the BPHZ theory.
- Description in terms of Hopf algebras possible in the case of renormalizable theories (orbit of Ren contained in a finite dimensional affine subspace)(see the talk of Kai Keller).