UNIVERSAL KZB EQUATIONS
BRAIDS ON THE TORUS
AND CHEREDNIK ALGEBRAS

- joint work with B. Enriquez and P. Etingof

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I Flat connections, configuration spaces, formality

- Drinfeld's universal KZ connection
- universal KZB connection
- formality of pure braids on the torus

II Realizations and representations

- why "universal" (realizations)
- an isomorphism between the DAHA and the rational Cherednik algebra
- representation theory

III and then...

modular invariance, elliptic polylog, level structures, higher genus...

Drinfeld's universal KZB connection

→ Let t_n be the **Lie algebra** with generators t_{ij} $(1 \leq i \neq j \leq n)$ and relations: $t_{ij} = t_{ji}$, $[t_{ij}, t_{kl}] = 0$ $(\# \{i,j,k,l\} = 4)$,

$$[t_{ij}, t_{ik} + t_{jk}] = 0 \quad (\# \{i,j,k\} = 3).$$

→ we consider the principal $\exp(\hat{t}_n)$-bundle $\mathcal{P} = X_n \times \exp(\hat{t}_n) \downarrow X_n$ on the configuration space $X_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \forall i \neq j,\ z_i \neq z_j\}$ of n points in the plane.

→ the **Knizhnik-Zamolodchikov** connection is

$$\nabla_{KZ} = d - \sum_{i=1}^{n} \Big(\sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} \Big) dz_i$$

<u>Proposition</u>: ∇_{KZ} is a **flat** connection: $\nabla_{KZ}^2 = 0$.

This induces a monodromy representation of the group of pure braids with n strands: $\rho : PB_n := \pi_1(X_n, \cdot) \longrightarrow \exp(\hat{t}_n)$

Passing to **Malcev Lie algebras**: $\mathrm{Lie}(PB_n) \xrightarrow{\mathrm{Lie}(\rho)} \hat{t}_n$.

<u>Proposition</u>: $\mathrm{Lie}(\rho)$ is an isomorphism.

Universal KZB connection

Main goal: replace $\mathbb{C}$ by an elliptic curve $E_\tau = \mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$

→ We define the Lie algebra $t_{1,n}$ generated by $x_1,\dots,x_n, y_1,\dots,y_n$

with relations: $[x_i, x_j] = 0 = [y_i, y_j]$ $(\forall i,j)$, $[x_i, y_j] = [x_j, y_i]$ $(i\neq j)$,

$$\Big[\sum_{j=1}^{n} x_j, y_i\Big] = 0 = \Big[\sum_{j=1}^{n} y_j, x_i\Big] \quad (\forall i),$$

$$\big[x_i, [x_j, y_k]\big] = 0 = \big[y_i, [y_j, x_k]\big] \quad (\#\{i,j,k\}=3).$$

(variation: $\bar{t}_{1,n} := t_{1,n}/_{\langle \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i \rangle}$)

→ We consider a principal $\exp(\hat{t}_{1,n})$-bundle on the configuration space

$X_{\tau,n} := \{(u_1,\dots,u_n)\in E_\tau^n \,/\, \forall i\neq j,\ u_i\neq u_j\}$, that we characterize by

its sections: holomorphic functions $f : \mathbb{C}^n \supset U \longrightarrow \exp(\hat{t}_{1,n})$ such that

$$f(z_1,\dots,z_j+1,\dots,z_n) = f(z_1,\dots,z_n) \quad\text{and}\quad f(z_1,\dots,z_j+\tau,\dots,z_n) = e^{-2\pi i x_j}\cdot f(z_1,\dots,z_n)$$

(variation: $\overline{X}_{\tau,n} := X_{\tau,n}/_{(E_\tau)^{\mathrm{diag}}}$, $\exp(\hat{\bar{t}}_{1,n})$-bundle)

→ One defines $k(z,x) := \dfrac{\theta(z+x)}{\theta(z)\,\theta(x)} - \dfrac{1}{x} \in \mathrm{Hol}\big(\mathbb{C}-(\mathbb{Z}+\tau\mathbb{Z})\big)[\![x]\!]$

(normalization: $\theta'(0)=1$), and $K_{ij}(z) := k(z, \mathrm{ad}(x_i))\big([x_i, y_j]\big)$

Finally, $K_i(z_1,\dots,z_n) := -y_i + \sum_{j\neq i} K_{ij}(z_i - z_j)$

→ The Knizhnik–Zamolodchikov–Bernard connection is given by

$$\nabla_{KZB} := d - \sum_{i=1}^{n} K_i \, dz_i$$

<u>Theorem</u> [C-Enriquez-Etingof]: ∇_{KZB} is flat.

This induces a monodromy representation of the group of pure braids on the torus: $\rho_\tau : PB_{1,n} := \pi_1(X_{\tau,n}, \cdot) \longrightarrow \exp(\hat{t}_{1,n})$.

Passing to Malcev Lie algebras: $\mathrm{Lie}(PB_{1,n}) \xrightarrow{\mathrm{Lie}(\rho_\tau)} \hat{t}_{1,n}$.

<u>Theorem</u> [C-Enriquez-Etingof]: $\mathrm{Lie}(\rho_\tau)$ is an isomorphism.

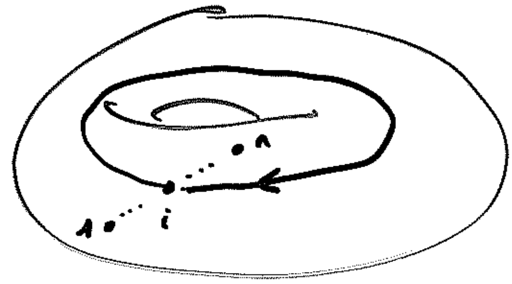
<u>Sketch of the proof</u>: one shows that the associated graded is an isomorphism. For this we construct a surjective morphism:

$$t_{1,n} \longrightarrow\!\!\!\!\rightarrow \mathrm{gr}(\mathrm{Lie}(PB_{1,n})) \xrightarrow{\mathrm{gr}(\mathrm{Lie}(\rho_\tau))} \hat{t}_{1,n}$$

$$x_i \longmapsto \sigma(A_i)$$
$$y_i \longmapsto \sigma(B_i)$$

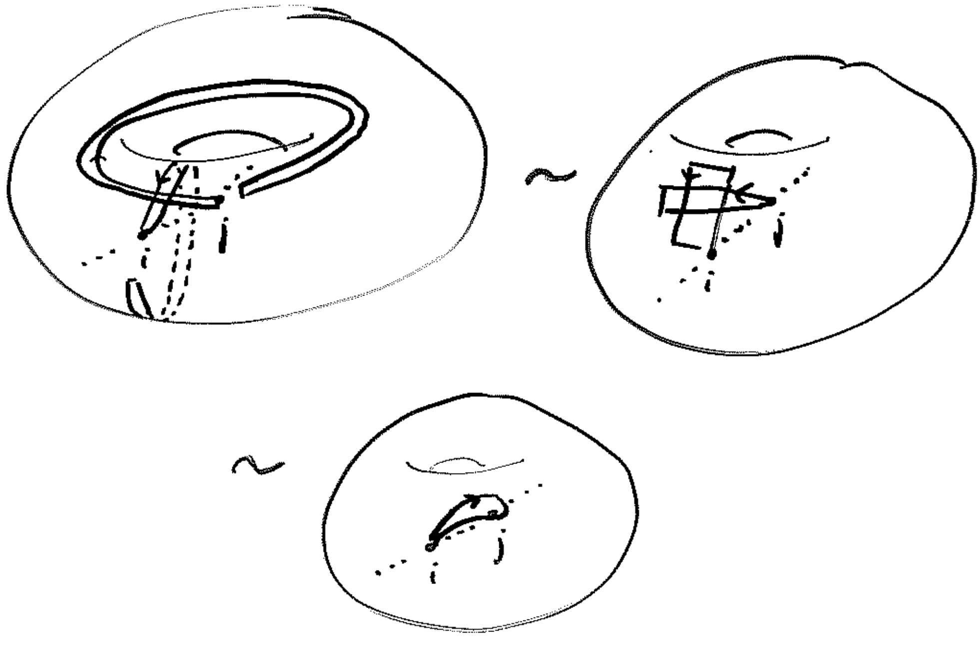
where σ is the **symbol map** and

$A_i =$ 

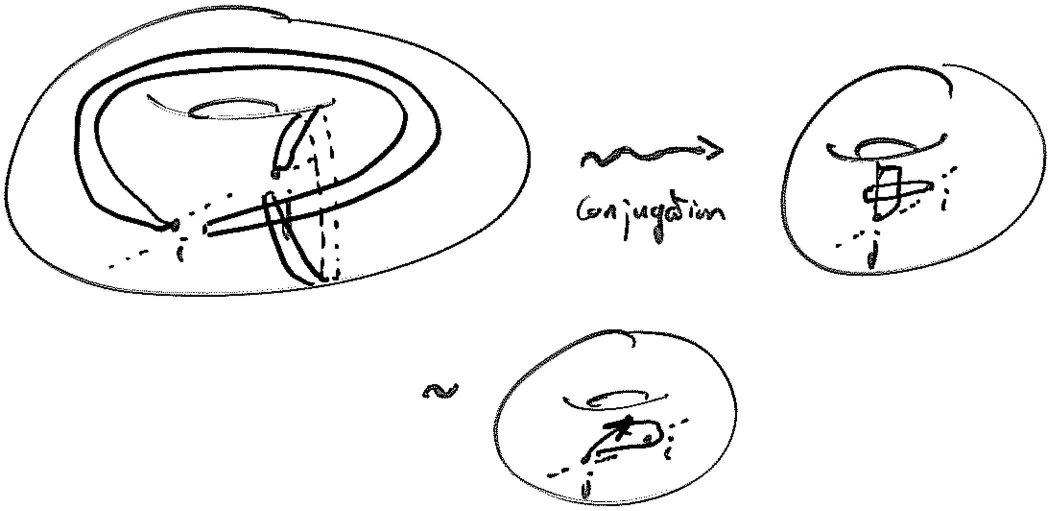
$B_i =$ 

One has to prove that the relations in $t_{1,n}$ are preserved by $x_i \longmapsto \sigma(A_i)$ and $y_i \longmapsto \sigma(B_i)$

$(A_i, B_j) =$



$(A_j, B_i) =$



[Variation: the isomorphism $\mathrm{Lie}(\rho_\tau)$ descends to an isomorphism $\mathrm{Lie}(\overline{PB}_{1,n}) \xrightarrow{\sim} \hat{f}_{1,n}$, where $\overline{PB}_{1,n} := \pi_1(\overline{X}_{\tau,n}, \cdot)$.)

Realizations

- genus zero case: $\mathfrak{g}$ Lie algebra, $t_{\mathfrak{g}} \in S^2(\mathfrak{g})^{\mathfrak{g}} \rightsquigarrow (a,b) \mapsto \langle a,b \rangle$ invariant bilinear form.

  $t_n \longrightarrow U(\mathfrak{g})^{\otimes n}$ is a Lie algebra morphism.

- genus one (elliptic case)

  → $D(\mathfrak{g})$ algebra of polynomial diff. operators on $\mathfrak{g}$.

  generators: x_a, ∂_a $(a \in \mathfrak{g})$

  relations: linearity, $[x_a, x_b] = 0 = [\partial_a, \partial_b]$ and $[\partial_a, x_b] = \langle a, b \rangle$.

  → Notation: $t_{\mathfrak{g}} = \sum_{\alpha} e_\alpha \otimes e_\alpha$.

  We have a quantum momentum map

  $$\mu : \mathfrak{g} \longrightarrow D(\mathfrak{g})$$
  $$a \longmapsto \sum_{\alpha} x_{[a, e_\alpha]} \partial_{e_\alpha}$$

  and another one

  $$\mathfrak{g} \longrightarrow A_n := D(\mathfrak{g}) \otimes U(\mathfrak{g})^{\otimes n}$$
  $$a \longmapsto \mu(a) \otimes 1 + 1 \otimes \sum_{i=1}^{n} a^{(i)}.$$

  → quantum reduction: we have a **Hecke algebra** $H_n(\mathfrak{g}) := A_n /\!/ \mathfrak{g}$.

  $$\left(A_n /\!/ \mathfrak{g} := N(A_n \cdot \mathfrak{g}) / A_n \cdot \mathfrak{g} \right)$$

<u>Proposition</u> [C-Enriquez-Etingof]: $x_i \longmapsto \sum_{\alpha} x_{e_\alpha} \otimes e_\alpha^{(i)}$; $y_i \longmapsto -\sum_{\alpha} \partial_{e_\alpha} \otimes e_\alpha^{(i)}$

defines a Lie algebra morphism $t_{1,n} \longrightarrow H_n(\mathfrak{g})$.

<u>Remark</u>: $H_n(\mathfrak{g})$ acts on $\left(\mathcal{O}_{\mathfrak{g}} \otimes V_1 \otimes \cdots \otimes V_n \right)^{\mathfrak{g}}$, where V_i are $\mathfrak{g}$-modules.

DAHA and Cherednik algebra

→ The rational Cherednik algebra $H_n(k)$, of type A_{n-1} and level k, is the quotient of $\mathbb{C}[S_n] \ltimes \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ by relations:

$$\sum_{i=1}^{n} x_i = 0 = \sum_{i=1}^{n} y_i \; , \; [x_i, x_j] = 0 = [y_i, y_j] \; (\forall i,j),$$

$$[x_i, y_j] = \frac{1}{n} - k\,\sigma_{ij} \quad (i \neq j).$$

Proposition: $\forall a, b \in \mathbb{C}$, there is a morphism of Lie algebras

$$\xi_{a,b} : \bar{t}_{1,n} \longrightarrow H_n(k)$$
$$x_i \longmapsto a x_i$$
$$y_i \longmapsto b y_i$$

→ The double affine Hecke algebra $\mathcal{H}_n(q,t)$, of type A_{n-1}, is the quotient of the group algebra of $\bar{B}_{1,n} = \pi_1^{\text{orbifold}}\left(X_{\mathcal{E},n} / S_n \right)$ by

$$(T - q^{-1}t)(T + q^{-1}t^{-1}) = 0$$

where T is any "small loop" around the divisor (in the counterclockwise sens).

→ For any representation V of $H_n(k)$, and a, b formal parameters, the monodromy representation and the morphism $\xi_{a,b}$ induce an $\mathcal{H}_n(q,t)$-module structure on V for $q = e^{-2\pi i\, ab/n}$ and $t = e^{2\pi i\, kab}$.

Taking $a = b$ and $V = H_n(k)$ one has an isomorphism

$$\widehat{\mathcal{H}_n(q,t)} \xrightarrow{\sim} H_n(k)[\![a]\!][a^{-1}]$$

Representation theory

→ Let $N \mid n$, $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$ and $V_N = \left(S\mathfrak{g} \otimes (\mathbb{C}^N)^{\otimes n}\right)^{\mathfrak{g}}$

Then

$$U(\bar{F}_{1,n}) \rtimes S_n \longrightarrow H_n(\mathfrak{g}) \rtimes S_n \longrightarrow End(V_N)$$

$$U(\bar{F}_{1,n}) \rtimes S_n \searrow H_n(N/n) \nearrow H_n(\mathfrak{g}) \rtimes S_n$$

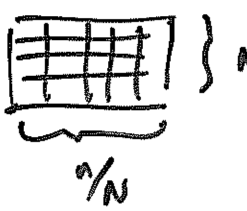
The map $H_n(N/n) \longrightarrow H_n(\mathfrak{g}) \rtimes S_n$ is given by:

$$\sigma \in S_n \longmapsto \sigma \; ; \; x_i \longmapsto \sum_\alpha x_{e_\alpha} \otimes e_\alpha^{(i)} \; ; \; y_i \longmapsto \frac{N}{n} \sum_\alpha \partial_{e_\alpha} \otimes e_\alpha^{(i)}.$$

→ There is a PBW theorem for Cherednik algebras:

$$H_n(k) \simeq \mathbb{C}[x_1, \dots, x_n] \otimes \mathbb{C}[S_n] \otimes \mathbb{C}[y_1, \dots, y_n] \quad \text{as v.s.}$$

→ $\forall \pi \in Rep(S_n)$, one has a notion of **lowest weight module** $L(\pi)$ of weight π.

Theorem [C-Enriquez-Etingof]: $V_N \simeq L\Big($  $\Big)$

One can obtain character formula ...