Developments in the theory of universality

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- Universality allow testable predictions even if we do not known the details of the microscopic model.
- Connections between universality and renormalization. Deep connections bewteen QFT and statistical physics (statistical field theory).

The Ising model

The paradigmatic model for statistical mechanics is the 2D Ising model

$$H = J \sum_{j=0,1} \sum_{\mathbf{x} \in \Lambda} \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{e}_j} \equiv \sum_{\mathbf{x} \in \Lambda} h_{\mathbf{x}}$$

$\sigma_{\mathbf{x}} = \pm$, Λ is a square lattice, $\mathbf{x} \in \Lambda$, $\mathbf{e}_0 = (0, 1), \mathbf{e}_1 = (1, 0)$.

The Ising model

- The *partition function* is $Z = \sum_\sigma e^{-\beta H(\sigma)}$ and phase transitions appear as non-analyticity points of $f_\beta = -\beta^{-1} \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \log Z$.

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- (Onsager (1944)) The critical temperature is $\tanh \beta_c J = \sqrt{2} - 1$ and the specific heat (second derivative) and the correlations

$$C_v(\beta) \sim -C_1 \log|\beta - \beta_c| + C_2 \qquad < h_{\mathbf{x}} h_{\mathbf{y}} >_{\beta_c} \sim \frac{C}{|\mathbf{x} - \mathbf{y}|^2}$$

while for $\beta \neq \beta_c$ $< h_{\mathbf{x}} h_{\mathbf{y}} >_\beta$ decays faster than any power of $\xi^{-1}|\mathbf{x} - \mathbf{y}|$, with $\xi^{-1} \sim C|T - T_c|$. The critical indices are *pure number* i.e. independent from J.

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- A mathematical proof of universality is achieved in $D \geq 4$ (Aizenamnn (1982), Frohelich (1982)) where is a consequence of a strengthened version of the central limit theorem. In lower dimension is more difficult.

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- There are however systems in which the indices are not pure numbers but depend on the microscopic structure. This happens in planar magnetic materials, carbon nanotubes or spin chains like $KCuF_3$ (Ishiii et al. Nature 2003)

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- There are however systems in which the indices are not pure numbers but depend on the microscopic structure. This happens in planar magnetic materials, carbon nanotubes or spin chains like $KCuF_3$ (Ishiii et al. Nature 2003)
- What is universality in these cases?

Coupled Ising models

- The simplest example with model-dependent exponents is obtained coupling two 2D Ising models

$$H(\sigma,\sigma') = H_J(\sigma) + H_{J'}(\sigma') - \lambda V(\sigma,\sigma')$$

with $H = -J\sum_{j=0,1}\sum_{\mathbf{x}\in\Lambda}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}$ $\sigma_{\mathbf{x}} = \pm$, Λ is a 2D square lattice, $\mathbf{x}\in\Lambda$, $\mathbf{e}_0 = (0,1), \mathbf{e}_1 = (1,0)$.

- V is a short ranged, quartic in the spin and invariant in the spin exchange, like

$$V = \sum_{j=0,1}\sum_{\mathbf{x},\mathbf{y}\in\Lambda} v(\mathbf{x}-\mathbf{y})\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{y}}\sigma'_{\mathbf{y}+\mathbf{e}_j}$$

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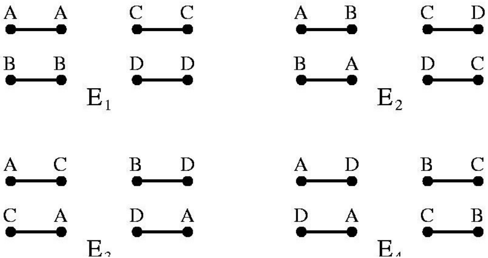
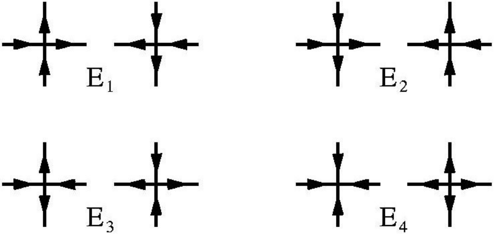
- It is well known that several models in statistical mechanics can be rewritten as coupled Ising models.

The Ashkin-Teller model



In the Ashkin-Teller model the spin has four values A, B, C, D, and two neighbour spins is associated an energy E_0 for AA, BB, CC, DD, E_1 for AB, CD, E_2 for AC, BD, E_3 for AD, BC.

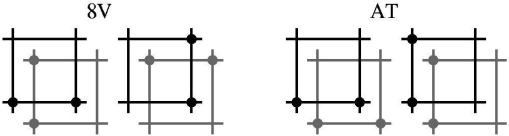
The Eight vertex model



The 8V model is a generalization of the Ice model for the hydrogen bounding in which at each point is associated one among eight vertices.

Ising Mapping

- Both models can be rewritten, with a suitable choice of the parameters, as coupled Ising models; in the case of the AT for instance $V = \sum_{j=0,1} \sum_{\mathbf{x}\in\Lambda} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}\sigma'_{\mathbf{x}}\sigma'_{\mathbf{x}+\mathbf{e}_j}$.



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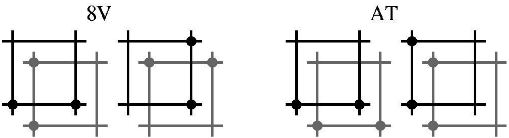

- Despite their similarity, an exact solution (Baxter (1971))exists only in the case of the 8V model and some of the exponents can be computed. They depend from λ, that is it is not in the Ising universality

- The Heisenberg spin chain (physically realized in several compounds like $KCuF_3$) is a quantum generalization of the Ising model; $H =$

$$-\sum_{x=1}^{L-1}[J_1 S_x^1 S_{x+1}^1 + J_2 S_x^2 S_{x+1}^2 - hS_x^3] + \lambda \sum_{1\le x,y\le L} v(x-y) S_x^3 S_y^3$$

where $S_x^\alpha = \sigma_x^\alpha/2$ for $i = 1, 2, \dots, L$ and $\alpha = 1, 2, 3$, σ_x^α being the Pauli matrices and $|v(x-y)| \le Ce^{-\kappa|x-y|}$

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- Despite looks very different, it is related to the previous models: if $v(x-y) = \delta_{|x-y|,1}/2$ and $h = 0$ the hamiltonian of the *XYZ* model commutes with the transfer matrix of the 8V model.

1D INTERACTING FERMIONS

The spin chain can be equivalently written as a model of non relativistic interacting fermions through the Jordan-Wigner transformation $H =$

$$-\frac{1}{2}\sum_{x=1}^{L-1}[a^+_x a^-_{x+1} + a^+_{x+1}a^-_x] - u\sum_{x=1}^{L-1}[a^+_x a^+_{x+1} + a^-_{x+1}a^-_x]$$

$$+h\sum_{x=1}^{L}(\rho_x - \frac{1}{2}) + \lambda\sum_{1\leq x,y\leq L} v(x-y)(\rho_x - \frac{1}{2})(\rho_y - \frac{1}{2})$$

where $a^\pm_x$ are the fermion creation or annihilation operators and $\rho_x = a^+_x a^-_x$, $J_1 = J_2 = 1$, $u = (J_1 - J_2)/2$. This hamiltonian describes non relativistic fermions on a lattice (1D metals).

Conjectures

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- For instance, in the coupled Ising model, if $X_\pm$ are the exponents of the energy or crossover correlations, ν is the exponents of the correlation length, α the exponent of the specific heat

$$X_- X_+ = 1 \quad \nu = \frac{1}{2 - X_+} \quad 2\nu = 2 - \alpha$$

Kadanoff (1977), Kadanoff and Wegner (1971).
In the spin chains or 1D fermions, the same relations hold with a different identification (Luther and Poeschel 1974).

Conjectures

- Even the knowledge of a single exponent can be lacking; in the case of spin chains or 1D fermions, Haldane (1980) conjectured other relations allowing the determination of the exponents in terms of two quantities . (*Luttinger liquid conjecture*)
- In particular if v_s is the Fermi velocity and κ is the susceptibility, $v_N = (\pi\kappa)^{-1}$

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- Even if the critical exponents depend on the extraordinarily complex microscopic details, the universal relations allow concrete and testable predictions in terms of a few measurable parameters.

Check in the solvabe cases

- Its validity can be checked in the XYZ case; the index ν is, if $\cos \bar{\mu} = \lambda$

$$\nu = \frac{\pi}{2\bar{\mu}} = 1 + \frac{2\lambda}{\pi} + O(\lambda^2)$$

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$$v_s = \frac{\pi}{\bar{\mu}} \sin \bar{\mu} \qquad \kappa = [2\pi(\pi/\bar{\mu} - 1) \sin \bar{\mu}]^{-1}$$

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- How can we prove such relations when a solution is lacking?

If $\eta_\alpha, \eta_\alpha^+$, $\alpha = 1, .., N$ are anticommuting Grassman variables $\eta_\alpha \eta_\beta = -\eta_\beta \eta_\alpha$ and the Grassman integration

$$\int d\eta = 0 \quad \int d\eta\eta = 1$$

and extended by linearity; moreover

$$\int \mathcal{D}\eta \mathcal{D}\eta^+ e^{\sum_{\alpha,\beta} \eta_\alpha A_{\alpha,\beta} \eta_\beta^+} = \det A$$

.

Fermionization for the Ising model

- The *Ising model partition function* (with p.b.c.) (Hurst ,Lieb Schultz Mattis, Kasteleyn, McCoy) can be written as sum of Grassman integrals (the square root)

$$\int \prod_{\omega=\pm,\mathbf{k}} d\psi^+_{\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} e^{-\frac{Z}{L^2}\sum_{\mathbf{k}} \psi^+_{\mathbf{k},\omega} A_{\mathbf{k}} \psi^-_{\mathbf{k},\omega}} = \mathcal{N} \int P_{Z,\mu}(d\psi)$$

where $\psi^{\pm}_{\mathbf{k},\omega}$, $\omega = \pm 1$, $\mathbf{k} = (k_0, k)$ are a finite set of Grassman variables and

$$A_{\mathbf{k}} = \begin{pmatrix} (-i\sin k_0 + \sin k + \mu_{11}) & -\mu + \mu_{12} \\ -\mu + \mu_{21} & -i\sin k_0 - \sin k_1 + \mu_{22} \end{pmatrix}$$

with $\mu = O(|\beta - \beta_c|)$, $\tanh \beta_c J = \sqrt{2} - 1$, $Z = O(1)$, $\mu_{ij} = O(\mathbf{k}^2)$.

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  with $\mu = O(|\beta - \beta_c|)$, $\tanh \beta_c J = \sqrt{2} - 1$, $Z = O(1)$, $\mu_{ij} = O(\mathbf{k}^2)$.
- $P_{Z,\mu}(d\psi)$ is the Gaussian Grassman integration of a Dirac field in $d = 1 + 1$ on a lattice (no fermion doubling).

Fermionization for the coupled Ising model

- The partition function of the coupled Ising model with Hamiltonian $H(\sigma,\sigma') = H_J(\sigma) + H_J(\sigma') - \lambda V(\sigma,\sigma')$ can be exactly written as sum of non quadratic Grassman integrals

$$\int P_{Z,\mu}(d\psi) e^{V(\psi)}$$

where $\lambda_0 = O(\lambda)$

$$V = \lambda_0 \sum_{\mathbf{x}} \psi^+_{\mathbf{x},+} \psi^-_{\mathbf{x},+} \psi^+_{\mathbf{x},-} \psi^-_{\mathbf{x},-} + \dots$$

- $\int P_{Z,\mu}(d\psi) e^V$ is the partition function of a interacting Dirac field with a lattice regularizationr in $d = 1+1$ and mass μ (criticality correspond to massless fermions); if $J \neq J'$ two masses are present.

We are interested in the specific heat C_v and the energy $\varepsilon = +$ and cross-over ($\varepsilon = -$) correlations, defined as

$$G_\beta^\varepsilon(\mathbf{x}-\mathbf{y}) = \lim_{\Lambda\to\infty} \left\langle O_\mathbf{x}^\varepsilon O_\mathbf{y}^\varepsilon \right\rangle_\Lambda - \left\langle O_\mathbf{x}^\varepsilon \right\rangle_\Lambda \left\langle O_\mathbf{y}^\varepsilon \right\rangle_\Lambda \quad , \quad \varepsilon = \pm$$

where $\langle .. \rangle_\Lambda$ is the average over all the spins configurations with weight $e^{-\beta H}$ and

$$O_\mathbf{x}^\epsilon = \sum_{j=0,1} \sigma_\mathbf{x}\sigma_{\mathbf{x}+\mathbf{e}_j} + \varepsilon \sum_{j=0,1} \sigma'_\mathbf{x}\sigma'_{\mathbf{x}+\mathbf{e}_j}$$

They can be also written as Grassman integrals with source $\psi_+^+\psi_-^-$ and $\psi_+^+\psi_-^+$ respectively.

Theorem

(Mastropietro JSP (2003),CMP(2004)) In the coupled Ising model with $J = J'$ and λ small enough

- *The specific heat*

$$C_v \sim -\frac{1}{\alpha}[1 - |\beta - \beta_c|^{-\alpha}] + O(1)$$

  with $\alpha = O(\lambda)$, $\tanh \beta_c J = \sqrt{2} - 1 + O(\lambda)$.

- *If* $\beta \neq \beta_c$ *the energy and crossover correlation* $G^{\varepsilon}_{\beta}(\mathbf{x} - \mathbf{y})$, $\varepsilon = \pm$ *decays faster than any power of* $\xi^{-1}|\mathbf{x} - \mathbf{y}|$, *with* $\xi^{-1} \sim C\,|\beta - \beta_c|^{\nu}$ *with* $\nu = 1 + O(\lambda)$.

- $$G^{\varepsilon}_{\beta_c}(\mathbf{x} - \mathbf{y}) \sim \frac{C_\varepsilon}{|\mathbf{x} - \mathbf{y}|^{2X_\varepsilon}}\ , \textit{as } |\mathbf{x} - \mathbf{y}| \to \infty\ ,$$

  with $X_\pm = 1 + O(\lambda)$.

Remarks

- The series for X_+, X_-, ν, X_T are convergent for small λ; by explicit computation of the lowest order the above result gives the first proof of the fact that the critical exponents are non trivial function of the interaction (in particular for the AT case)

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- The series for X_+, X_-, ν, X_T are convergent for small λ; by explicit computation of the lowest order the above result gives the first proof of the fact that the critical exponents are non trivial function of the interaction (in particular for the AT case)
- In the case of a single perturbed Ising model, it was proved by Pinson and Spencer (2000) that the indices $\nu = 1, X_\pm = 1$, that is are the same as the Ising ones.

Anisotropic AT model

Theorem

(Giuliani,Mastropietro CMP,PRL(2005)) In the case of the anisotropic AT model ($J \neq J'$) there are two critical temperatures, T_c^+ and T_c^- such that

$$|T_c^+ - T_c^-| \sim |J - J'|^{X_T}$$

with $X_T = 1 + O(\lambda)$ and

$$C_v \sim -\Delta^\alpha \log \frac{|T - T_c^-| \cdot |T - T_c^+|}{\Delta^2}$$

where $2\Delta^2 = (T - T_c^-)^2 + (T - T_c^+)$.

- The analysis is based on Wilsonian Renormalization Group and multiscale analysis. The Grassmann variables are written as $\psi_{\mathbf{k}} = \sum_{h=-\infty}^{1} \psi_{\mathbf{k}}^{(h)}$ with $\psi_{\mathbf{k}}^{(h)}$ living at momentum scales $O(\mathbf{k}) = O(2^h)$. After the integration of the fields $\psi^{(0)}, .., \psi^{(h)}$ we get

$$\int P_{Z,\mu}(d\psi) e^{V} = e^{L^2 N_h} \int P_{Z_h,\mu_h}(d\psi^{(\leq h)}) e^{V^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})}$$

where Z_h is the wave function renormalization, μ_h the effective mass; V^h is sum of monomials of any degree and λ_h the effective coupling.

- The physical observables are expressed as renormalized series in λ_k; contrary to the original series in λ, there are no divergences (Gallavotti trees: no overlapping divergences). Analyticity in λ_k follows from the compensations between Feynman graphs coming from the minus signs due to anticommutativity (Caianiello 1973); technically via Gram bounds in the Battle-Brydges-Federbush formula for truncated expectations (Gawedzki and Kupiainen (1985)).

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- The effective coupling λ_h converges, as $h \to -\infty$, to a function $\lambda_{-\infty}$, analytic function of λ (*line of fixed points*), as a consequence of asymptotic Ward Identities (Benfatto, Mastropietro (2004)).

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- The exponents are convergent power series in $\lambda_{-\infty}$. Do they verify the universal relations?

An universality result for the coupled Ising model

Theorem

(Benfatto,Falco,Mastropietro CMP(2009)) If the coupling of the coupled Ising model is small enough

$$X_+(\lambda) = \frac{1}{X_-(\lambda)} \quad \nu = \frac{1}{2 - X_+(\lambda)} \quad \alpha = \frac{2 - 2X_+(\lambda)}{2 - X_+(\lambda)}$$

and in the case of the anisotropic AT model the transition index verify

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The last relation was never proposed; the others were proposed by Kadanoff (1977), Kadanoff and Wegner (1971) and imply the hyperscaling relation $2\nu = 2 - \alpha$.

- We introduce the QFT model, if $j_\mu = \bar\psi_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}$

$$\int P(d\psi^{(\le N)}) e^{\tilde\lambda_\infty \int d\mathbf{x} v(\mathbf{x}-\mathbf{y}) j_{\mu,\mathbf{x}} j_{\mu,\mathbf{y}}}$$

where $P(d\psi^{(\le N)})$ have propagator $\frac{\chi_N(\mathbf{k})}{\not{\mathbf{k}}}$ with a smooth cut-off function vanishing for $|\mathbf{k}| \ge 2^N$ and $v(\mathbf{x}-\mathbf{y})$ a short range symmetric interaction.

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- A multiscale integration is now necessary also in the ultraviolet region (superrenormalizable) to perform the limit $N \to \infty$; in the infrared is similar to the previous one, with effective coupling called $\tilde{\lambda}_h$.

Equivalence with a QFT model

- While the short distance (large momenta) behavior of the two models are completely different, the large distance behavior is expressed by critical indices η; they are analytic in $\tilde{\lambda}_{-\infty}$, $\eta \equiv \eta(\tilde{\lambda}_{-\infty}) = a_1 \tilde{\lambda}_{-\infty} + a_2 \tilde{\lambda}^2_{-\infty} + \ldots$ where the coefficients a_i are equal to the ones in the spin model.

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- The crucial point is that one can make a *fine tuning* of the bare coupling $\tilde{\lambda}_{\infty}$ so that $\lambda_{-\infty} = \tilde{\lambda}_{-\infty}$, so that with this choice the indices are identical: of course $\tilde{\lambda}_{\infty}(\lambda)$ is an analytic function of λ depending on all the details of the spin model.
- We have found that, for a suitable value of the bare coupling $\tilde{\lambda}_{\infty}$, the indices of the large distance behavior of the spin or QFT model are the same. And so what?

Equivalence with a QFT model

- While the short distance (large momenta) behavior of the two models are completely different, the large distance behavior is expressed by critical indices η; they are analytic in $\tilde{\lambda}_{-\infty}$, $\eta \equiv \eta(\tilde{\lambda}_{-\infty}) = a_1 \tilde{\lambda}_{-\infty} + a_2 \tilde{\lambda}^2_{-\infty} + \ldots$ where the coefficients a_i are equal to the ones in the spin model.
- The crucial point is that one can make a *fine tuning* of the bare coupling $\tilde{\lambda}_\infty$ so that $\lambda_{-\infty} = \tilde{\lambda}_{-\infty}$, so that with this choice the indices are identical: of course $\tilde{\lambda}_\infty(\lambda)$ is an analytic function of λ depending on all the details of the spin model.
- We have found that, for a suitable value of the bare coupling $\tilde{\lambda}_\infty$, the indices of the large distance behavior of the spin or QFT model are the same. And so what?

Ward Identities for the QFT model

- The advantage of this is that in the QFT model the indices can be explicitly computed as functions of $\tilde{\lambda}_\infty$; this follows from the fact that the QFT model verify extra gauge symmetries

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$$\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar{\psi}_{\mathbf{k}+\mathbf{p}} >=< \psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}} > - < \psi_{\mathbf{k}+\mathbf{p}}\bar{\psi}^{-}_{\mathbf{k}+\mathbf{p}} > +\Delta_N(\mathbf{k},\mathbf{p})$$

where $\Delta_N =< \delta j_{\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}+\mathbf{p}} >$ with $\delta_{J\mathbf{p}} = \int d\mathbf{k}[(\chi_N^{-1}(\mathbf{k}+\mathbf{p})-1)(\not{\mathbf{k}}+\not{\mathbf{p}})-(\chi_N^{-1}(\mathbf{k})-1)\not{\mathbf{k}}]\bar{\psi}_{\mathbf{k}}\psi_{\mathbf{k}+\mathbf{p}}$.

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- $$\frac{\chi_N(\mathbf{k})}{\not{\mathbf{k}}}\not{\mathbf{p}}\frac{\chi_N(\mathbf{k}+\mathbf{p})}{\not{\mathbf{k}}+\not{\mathbf{p}}} =$$
  $$\frac{\chi_N(\mathbf{k})}{\not{\mathbf{k}}} - \frac{\chi_N(\mathbf{k}+\mathbf{p})}{\not{\mathbf{k}}+\not{\mathbf{p}}} + \frac{\chi_N(\mathbf{k})}{\not{\mathbf{k}}}C(\mathbf{k},\mathbf{p})\frac{\chi_N(\mathbf{k}+\mathbf{p})}{\not{\mathbf{k}}+\not{\mathbf{p}}}$$

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$$\lim_{N\to\infty} \Delta_N(\mathbf{k},\mathbf{p}) = \tau \hat{v}(\mathbf{p})\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_{\mathbf{k},\omega}\bar{\psi}_{\mathbf{k}+\mathbf{p},\omega} >$$

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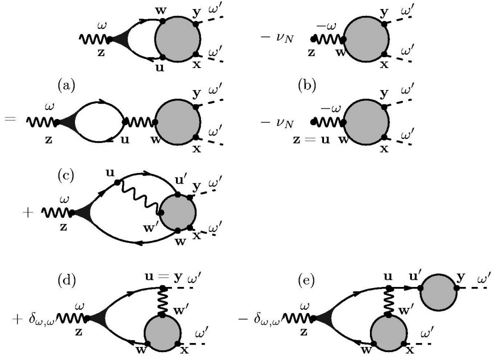
- The coefficient τ is linear in $\tilde{\lambda}_\infty$ (Mastropietro JMP 2007): in the case of the axial WI, this is the non-perturbative analogue of the anomaly non renormalization in QED4.



In a RG analysis $\Delta_N(\mathbf{k}, \mathbf{p})$ the terms $\delta j \psi^+ \psi^-$ are marginal; one subtracts a local term, and one can further decompose them in a sum of terms with have scaling negative dimension (see c,d,e) except *a*, which is compensated by the local term (b), if $\nu_N = \frac{\tilde{\lambda}_\infty}{4\pi}$.

- The WI in the limit $N \to \infty$ have the form, if $j_{5,\mu} = \bar\psi \gamma_\mu \gamma_5 \psi$

$$\gamma_\mu \mathbf{p}_\mu < j_{\mu,\mathbf{p}} \psi_{\mathbf{k},\omega} \bar\psi_{\mathbf{k}+\mathbf{p}} >= A[< \psi_{\mathbf{k}} \bar\psi_{\mathbf{k}} > - < \psi_{\mathbf{k}+\mathbf{p}} \bar\psi^{-}_{\mathbf{k}+\mathbf{p}} >]$$

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  with $A^{-1} = 1 - \tau v(\mathbf{p}) \qquad \bar{A}^{-1} = 1 + \tau v(\mathbf{p})$.
- Similar relation were *postulated* by Johnson (1961) in its solution of the Thriring model $v(\mathbf{x}) = \delta(\mathbf{x})$(their value was fixed by self-consistency); here they are derived by a functional integral (essential to prove that the exponents are the same as the spin models).

Closed equations

- One can combine the WI with the Schwinger-Dyson equation and it turns out that the critical indices are written in terms of the anomaly

$$X_+ = 1 - \frac{1}{1+\tau}\frac{\tilde{\lambda}_\infty}{2\pi} \quad X_- = 1 + \frac{1}{1-\tau}\frac{\tilde{\lambda}_\infty}{2\pi}$$

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- The indices have a simple expression in terms of $\tilde{\lambda}_\infty$; all the model dependence is in the function $\tilde{\lambda}_\infty(\lambda) = a\lambda + b\lambda^2 + ...$
- The crucial point are: the exponents are the same as in a QFT, and we can choose its regularization so that the anomaly non renormalization holds.

The fact that τ is linear in the bare coupling λ_∞ is the non-perturbative analogue of a property called in QED anomaly non renormalization, and proved by Adler and Bardeen by an accurate analysis of the Feynman graph expansion. Writing the model as in the equivalent way as a boson-fermion model

$$\gamma_\mu \mathbf{p}_\mu < j_{5,\mu,\mathbf{p}} \psi_{\mathbf{k},\omega} \bar{\psi}_{\mathbf{k}+\mathbf{p}} >=$$
$$[< \psi_{\mathbf{k}} \bar{\psi}_{\mathbf{k}} > - < \psi_{\mathbf{k}+\mathbf{p}} \bar{\psi}^{-}_{\mathbf{k}+\mathbf{p}} >] + \tau \varepsilon_{\mu,\nu} < A_{\nu,\mathbf{P}} \psi_{\mathbf{k},\omega} \bar{\psi}_{\mathbf{k}+\mathbf{p}} >$$

Anomalies

- Note that had we considered a local current-current interaction (Thirring model) with $v(\mathbf{x}) = \delta_M(\mathbf{x})$ with $\lim_{M\to\infty} v_M(\mathbf{x}) = \delta(\mathbf{x})$, , still the ultraviolet fermionic (N) and bosonic M cut-off can be removed (CMP2008)

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- τ has higher orders corrections if the fermionic cut-off is removed after the bosonic one ($N \to \infty, M \to \infty$)
- τ it is not renormalized if they are removed in the opposite way $M \to \infty, N \to \infty$.
- The wave function renormalization must be chosen as $Z_N \sim 2^{\eta N}$ with $\eta > 0$ and $V(\sqrt{Z}\psi)$.

The quantum spin chain in its fermionic representation

$$-\frac{1}{2}\sum_{x=1}^{L-1}[a_x^+ a_{x+1}^- + a_{x+1}^+ a_x^-] - u\sum_{x=1}^{L-1}[a_x^+ a_{x+1}^+ + a_{x+1}^- a_x^-]$$
$$+h\sum_{x=1}^{L}(\rho_x - \frac{1}{2}) + \lambda \sum_{1\leq x,y\leq L} v(x-y)(\rho_x - \frac{1}{2})(\rho_y - \frac{1}{2})$$

where $a_x^\pm$ are the fermion creation or annihilation operators and $\rho_x = a_x^+ a_x^-$, $J_1 = J_2 = 1$, $u = (J_1 - J_2)/2$. This hamiltonian describes non relativistic fermions on a lattice (1D metals). We denote $\mathbf{x} = (x, x_0)$, $O_\mathbf{x} = e^{Hx_0} O_x e^{-Hx_0}$ and, if $A = O_{\mathbf{x}_1} \ldots O_{\mathbf{x}_n}$, $\langle A \rangle = \lim_{L\to\infty} \frac{\mathrm{Tr} e^{-\beta H} \mathbf{T}(A)}{\mathrm{Tr} e^{-\beta H}}$, $\mathbf{T}$ being the time order product and T denoting truncation.

Critical indices

For λ small enough (Benfatto,Mastropietro RMP (2002))
$S_{\mathbf{x}}^{(3)} = a_{\mathbf{x}}^{+} a_{\mathbf{x}}^{-} - \frac{1}{2}$, when $J_1 = J_2$ $\left\langle S_{\mathbf{x}}^{(3)} S_{\mathbf{0}}^{(3)} \right\rangle_T \sim$

$$\cos(2p_F x)\frac{1+O(\lambda)}{2\pi^2[x^2+(v_s x_0)^2]^{X_+}} + \frac{1}{2\pi^2[x^2+(v_s x_0)^2]}(1+O(\lambda))$$

- $p_F = \cos^{-1}(h+\lambda) + O(\lambda)$ (if $h=0$ $p_F = \pi/2$ by symmetry)
- $v_s = \sin p_F + O(\lambda)$, $v_F = \sin p_F$
- κ (the susceptibility) is the limit $p \to 0$ of the 2D FT of $\left\langle S_{\mathbf{x}}^{(3)} S_{\mathbf{0}}^{(3)} \right\rangle_T$ at $p_0 = 0$

Critical indices

- If $J_1 \neq J_2$ $\left\langle S^{(3)}_{\mathbf{x}} S^{(3)}_{\mathbf{0}} \right\rangle_T$ decays with correlation length $\xi \sim C|J_1 - J_2|^{\bar\nu}$ with $\bar\nu = 1 + O(\lambda)$
- The fermionic 2-point function $\left\langle a^-_{\mathbf{x}} a^+_{\mathbf{0}} \right\rangle_T$ for $J_1 = J_2$ decays at large distance as a power law with index $1 + \eta$, $\eta = O(\lambda^2)$
- The correlations of the Cooper pair operator $\rho^c_{\mathbf{x}} = a^+_{\mathbf{x}} a^+_{\mathbf{x}'} + a^-_{\mathbf{x}} a^-_{\mathbf{x}'}$, $\mathbf{x}' = (x+1, x_0)$ decays at large distance with a power law with index X_-.

Theorem

(Benfatto,Mastropietro 2009) For λ small enough

$$X_+ X_- = 1 \,,$$
$$\bar{\nu} = \frac{1}{2 - X_+^{-1}} \quad , \quad 2\eta = X_+ + X_+^{-1} - 2 \,,$$

Moreover

$$\kappa = \frac{1}{\pi} \frac{X_+}{v_s} \,.$$

$X_+ = 1 - \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)}{\pi \sin p_F} + O(\lambda^2)$ (cfr with $X_+ = 1 - \frac{2\lambda}{\pi} + O(\lambda^2)$ of the exact XYZ solution)

Ideas of the proof

- The density and the current operators are $\rho_x = S^3_x + \frac{1}{2} = a^+_x a^-_x$ and $J_x = \frac{1}{2i}[a^+_{x+1} a^-_x - a^+_x a^-_{x+1}]$

$$\frac{\partial \rho_{\mathbf{x}}}{\partial x_0} = e^{Hx_0}[H, \rho_x]e^{-Hx_0} = -i[J_{x,x_0} - J_{x-1,x_0}]$$

If $(v_F = \sin p_F)$ $J_{\mathbf{x}} = v_F j_{\mathbf{x}}$

$$ip_0 < \hat{\rho}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} > + p v_F < \hat{j}_{\mathbf{p}} \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} > \sim [\langle \hat{a}^+_{\mathbf{k}} \hat{a}^-_{\mathbf{k}} \rangle - \langle \hat{a}^+_{\mathbf{k}+\mathbf{p}} \hat{a}^-_{\mathbf{k}+\mathbf{p}} \rangle]$$

Note also that (H_0 is the quadratic part)

$$[H_0, \hat{J}_p] = \frac{1}{L} \sum_k \sin k(\cos(k+p) - \cos k) \hat{a}^+_{k+p} \hat{a}_k$$

The reference model

The partition function of the reference model (Lorentz invariant and not hamiltonian) is, if $\psi^{\pm}_{\pm,\mathbf{x}}$ are Grassman variables, $\mathbf{x} \in R^2$

$$\int P(d\psi^{(\leq N)}) e^{\tilde{\lambda}_\infty \int d\mathbf{x} v(\mathbf{x}-\mathbf{y}) j_{\mu,\mathbf{x}} j_{\mu,\mathbf{y}}}$$

where $\psi^{\pm} = (\psi^{\pm}_{+}, \psi^{\pm}_{-})$ is a Grassman spinor, $P(d\psi^{(\leq N)})$ have propagator,

$$g_{\pm}(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_N(\mathbf{k})}{-ik_0 \pm ck}$$

where $\chi_N(\mathbf{k})$ is a smooth cut-off function vanishing for $|\mathbf{k}| \geq 2^N$, $v(\mathbf{x}-\mathbf{y})$ a short range symmetric interaction and $j_\mu = \bar{\psi}\gamma_\mu\psi$, $\gamma_0 = \sigma_x$, $\gamma_1 = \sigma_y$.

Relation with the spin chain

It is possible to choose $c = v_s$ and $\tilde{\lambda}_\infty$ as convergent series in λ (depending on all the details of the spin hamiltonian) so that

- The critical exponents of the two models are the same.

Relation with the spin chain

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- The critical exponents of the two models are the same.
- For $\mathbf{k}$ $\mathbf{k}, \mathbf{k}+\mathbf{p}$ small if $\mathbf{p}_F = (0, \omega p_F)$, $\omega = \pm$, $J_\mathbf{x} = v_F j_\mathbf{x}$

$$< \hat{\rho}_\mathbf{p} \hat{a}^+_{\mathbf{k}+\mathbf{p}^\omega_F} \hat{a}^-_{\mathbf{k}+\mathbf{p}+\mathbf{p}^\omega_F} >\sim Z^{(3)} < \hat{j}_{0,\mathbf{p}} \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} >$$

$$< \hat{j}_\mathbf{p} \hat{a}^+_{\mathbf{k}+\mathbf{p}^\omega_F} \hat{a}^-_{\mathbf{k}+\mathbf{p}+\mathbf{p}^\omega_F} >\sim \tilde{Z}^{(3)} < \hat{j}_{1,\mathbf{p}} \hat{\psi}^+_{\mathbf{k},\omega} \hat{\psi}^-_{\mathbf{k}+\mathbf{p},\omega} >$$

$$\frac{\tilde{Z}^{(3)}}{Z^{(3)}} = 1 + a_1 \lambda + O(\lambda^2)$$

  Similar relation hold for the 2-point function with constant Z

- Asymptotic to the relativistic model with different density and current renormalizations. Crucial: The fact that $Z^{(3)} \neq \tilde{Z}^{(3)}$ is the effect of the irrelevant operators breaking the relativistic symmetry.

Ward Identities for the spin chain

- $$ip_\mu < j_{\mu,\mathbf{p}} \psi_{\mathbf{k}} \psi^+_{\mathbf{k}+\mathbf{p}} >= A[< \psi_{\mathbf{k}} \psi^+_{\mathbf{k}} > - < \psi_{\mathbf{k}+\mathbf{p}} \psi^+_{\mathbf{k}+\mathbf{p}} >]$$
  $$ip_\mu < j_{5,\mu,\mathbf{p}} \psi_{\mathbf{k}} \psi^+_{\mathbf{k}+\mathbf{p}} >= \bar{A}[< \psi_{\mathbf{k}} \psi^+_{\mathbf{k}} > - < \psi_{\mathbf{k}+\mathbf{p}} \psi^+_{\mathbf{k}+\mathbf{p}} >]$$

with $A^{-1} = 1 - \tau \qquad A^{-1} = 1 + \tau$.

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- $$i\mathbf{p}_\mu < j_{\mu,\mathbf{p}}\psi_\mathbf{k}\psi^+_{\mathbf{k+p}} >= A[< \psi_\mathbf{k}\psi^+_\mathbf{k} > - < \psi_{\mathbf{k+p}}\psi^+_{\mathbf{k+p}} >]$$
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- $$ip_0 < \hat{\rho}_\mathbf{p}\hat{a}^+_\mathbf{k}\hat{a}^-_{\mathbf{k+p}} > + p\tilde{v}_J < \hat{j}_\mathbf{p}\hat{a}^+_\mathbf{k}\hat{a}^-_{\mathbf{k+p}} >\sim B[\langle \hat{a}^+_\mathbf{k}\hat{a}^-_\mathbf{k}\rangle - \langle \hat{a}^+_{\mathbf{k+p}}\hat{a}^-_{\mathbf{k+p}}\rangle]$$
  $$ip_0 < \hat{\mathrm{j}}_\mathbf{p}\hat{a}^+_\mathbf{k}\hat{a}^-_{\mathbf{k+p}} > + p\tilde{v}_N < \hat{\rho}_\mathbf{p}\hat{a}^+_\mathbf{k}\hat{a}^-_{\mathbf{k+p}} >\sim \bar{B}[\langle \hat{a}^+_\mathbf{k}\hat{a}^-_\mathbf{k}\rangle - \langle \hat{a}^+_{\mathbf{k+p}}\hat{a}^-\rangle]$$
  with $B = \frac{Z^{(3)}}{Z}(1-\tau)^{-1}$, $\bar{B} = \frac{\tilde{Z}^{(3)}}{Z}(1+\tau)^{-1}$, $\tilde{v}_N = v_s\frac{Z^{(3)}}{\tilde{Z}^{(3)}}$, $\tilde{v}_J = v_s\frac{\tilde{Z}^{(3)}}{Z^{(3)}}$.
- One extra WI. Three different velocities

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  $\tilde{v}_J = v_s \frac{\tilde{Z}^{(3)}}{Z^{(3)}}$.
- One extra WI. Three different velocities
- By comparing with lattice WI $B = \frac{Z^{(3)}}{Z}(1-\tau)^{-1} = 1$

- By another WI, again derived by the reference model, if $D_\pm(\mathbf{p}) = -ip_0 \pm cp$

$$< \hat{\rho}_{\mathbf{p}} \hat{\rho}_{\mathbf{p}} >= \frac{1}{4\pi v_s Z^2} \frac{(Z^{(3)})^2}{1-(\tilde{\lambda}_\infty/4\pi v_s)^2} \left[2 - \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} - \frac{D_+(\mathbf{p})}{D_-(\mathbf{p})} \right] ,$$

so that

$$\kappa = \frac{1}{\pi v_s} \frac{1}{Z^2} \frac{(Z^{(3)})^2}{1-(\tilde{\lambda}_\infty/4\pi v_s)^2} = \frac{1}{\pi v_s} \frac{1-(\tilde{\lambda}_\infty/4\pi v_s)}{1+(\tilde{\lambda}_\infty/4\pi v_s)} = \frac{X_+}{\pi v_s}$$

Conclusions

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- Extensions of our methods will allow hopefully to prove universal relations in an even wider class of models and to prove other relations between spin or dynamic exponents.