Renormalization of rough paths

A physical and algebraic approach of stochastic calculus

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Plan

1. Algebraic properties of iterated integrals
2. Rough path construction by Fourier normal ordering
3. Examples of regularizations
4. From constructive field theory to fractional stochastic calculus

Signature of a regular path

$X := (X_t(1), \ldots, X_t(d)) : \mathbb{R} \to \mathbb{R}^d$ smooth path with d components

Signature of X:

$$\mathbf{X}^{ts}(i_1, \ldots, i_n) := \int_s^t dX_{x_1}(i_1) \int_s^{x_1} dX_{x_2}(i_2) \ldots \int_s^{x_{n-1}} dX_{x_n}(i_n).$$

Solution of differential equations

$$dY_t = \sum_{j=1}^{d} V_j(Y_t) dX_t(j)$$

Formal solution:

$$Y_t = Y_s + \sum_{j=1}^{\infty} \sum_{1 \le i_1, \ldots, i_j \le d} [V_{i_1} \cdots V_{i_j} \cdot \mathrm{Id}](Y_s) \ \cdot \ \mathbf{X}^{ts}(i_1, \ldots, i_j).$$

Euler scheme of rank N:

Replace with truncated series ($j \le N$) $\rightsquigarrow$

$$Y_t = \Phi(\mathbf{X}^{ts}(i_1), \ldots, \mathbf{X}^{ts}(i_1, \ldots, i_N); Y_s)$$

Compose $\rightsquigarrow$ $Y_t \simeq \Phi(\mathbf{X}^{t, \frac{n-1}{n}t}; \cdots \Phi(\mathbf{X}^{\frac{2t}{n}, \frac{t}{n}}; \Phi(\mathbf{X}^{\frac{t}{n}, 0}; Y_0) \cdots).$

Shuffle property

$$\mathbf{X}^{ts}(i_1,\ldots,i_{n_1})\mathbf{X}^{ts}(j_1,\ldots,j_{n_2}) = \sum_{\boldsymbol{k}\in\mathrm{Sh}(\boldsymbol{i},\boldsymbol{j})} \mathbf{X}^{ts}(k_1,\ldots,k_{n_1+n_2})$$

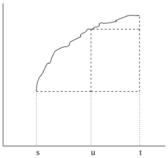
Sh=shuffles
Ex. $\mathbf{X}^{ts}(i_1,i_2).\mathbf{X}^{ts}(j_1) = \mathbf{X}^{ts}(i_1,i_2,j_1) + \mathbf{X}^{ts}(i_1,j_1,i_2) + \mathbf{X}^{ts}(j_1,i_1,i_2)$.

Hopf algebraic interpretation.

$\mathbf{Sh}^d := \{$words with letters in $1,\ldots,d\} \simeq \{$decorated trunk trees$\}$
Product=shuffle product

$\mathbf{X}^{ts}$ has shuffle property $\Longleftrightarrow$ $\mathbf{X}^{ts}$ character of $\mathbf{Sh}^d$.

Chen property



$$\mathbf{X}^{ts}(i_1,\ldots,i_n) = \mathbf{X}^{tu}(i_1,\ldots,i_n) + \mathbf{X}^{us}(i_1,\ldots,i_n) + \sum_k \mathbf{X}^{tu}(i_1,\ldots,i_k)\mathbf{X}^{us}(i_{k+1},\ldots,i_n).$$

Hopf algebraic interpretation.

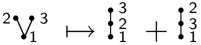
Coproduct of $\mathbf{Sh}^d$: $\Delta((i_1,\ldots,i_n)) = \sum_k (i_1,\ldots,i_k) \otimes (i_{k+1},\ldots,i_n)$.

$\mathbf{X}^{ts}$ has Chen property $\Longleftrightarrow \mathbf{X}^{ts} = \mathbf{X}^{tu} * \mathbf{X}^{us}$.

Tree extension

$\mathbf{H}^d$ Hopf algebra of rooted trees with decoration in $\{1, \ldots, d\}$.

Definition. $\Pi^d : \mathbf{H}^d \to \mathbf{Sh}^d$ Hopf algebra projection



Write $v \twoheadrightarrow w$ if v is above w ($\rightsquigarrow$ tree partial ordering)

$\Pi^d(\mathbb{T}) = \sum$ trunk trees with total ordering compatible with transfered tree partial ordering

Definition (tree iterated integrals).

$\bar{\mathbf{X}}^{ts} := \mathbf{X}^{ts} \circ \Pi^d$, character of $\mathbf{H}^d$; $\bar{\mathbf{X}}^{ts} = \bar{\mathbf{X}}^{tu} * \bar{\mathbf{X}}^{us}$.

Rough paths

$X : \mathbb{R} \to \mathbb{R}^d$ α-Hölder: $|X(t) - X(s)| \leq C|t-s|^\alpha$ $(\alpha \in (0,1))$.

Definition. $((J_X^{ts}(i_1)), \dots, (J_X^{ts}(i_1, \dots, i_N)))$, $N = \lfloor 1/\alpha \rfloor$ rough path over X if:
(i) $J_X^{ts}(i_1) = X_t(i_1) - X_s(i_1)$;
(ii) J_X^{ts} enjoys Chen and shuffle properties;
(iii) Hölder regularity: $|J_X^{ts}(i_1, \dots, i_k)| \leq C|t-s|^{k\alpha}$.

An example: fractional Brownian motion

Fix $\alpha \in (0,1)$.

Definition. Centered Gaussian process $(B_t)_{t\in\mathbb{R}}$ with covariance

$$\mathbb{E} B_s B_t = \frac{1}{2}\left(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}\right).$$

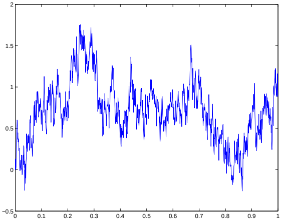


Figure: Path of fractional Brownian motion, $\alpha = 0.3$.

- $\alpha = \frac{1}{2}$: usual Brownian motion
- $\mathbb{E}(B_t - B_s)^2 = O(|t-s|^{2\alpha})$ $\rightsquigarrow$ (Kolmogorov) α^--Hölder paths.

A rough path by approximation

- $d \geq 2 \rightsquigarrow B = (B_t(1), \ldots, B_t(d))$ with d independent components

Replace B_t by ultra-violet regularization B_t^ε, $\varepsilon > 0, \varepsilon \to 0$

$\rightsquigarrow \mathbf{B}^{ts}(i,i)$ always converges when $\varepsilon \to 0$

$\rightsquigarrow$ if $i_1 \neq i_2$, then $\mathbf{B}^{ts}(i_1,i_2)$ converges $(\alpha > 1/4)$, diverges like $\varepsilon^{-\frac{1}{2}(1-4\alpha)}$ $(\alpha < 1/4)$.

Towards Fourier expressions

$$\begin{aligned}\mathbf{B}^{ts}(1,2) =: Area_{ts} &:= \int_s^t B'_{u_1}(1)du_1 \int_s^{u_1} B'_{u_2}(2)du_2 \\ &= Area_{ts}(\partial) + \delta G_{ts}\end{aligned}$$

$=$ Boundary term $+$ Increment.

Two fundamental notions:

Formal integration: $\int^t e^{\mathrm{i}\xi x}dx := \frac{e^{\mathrm{i}\xi t}}{\mathrm{i}\xi} \rightsquigarrow$ skeleton integrals

Fourier projections, $\mathcal{P}^+_{1,2}$ et $\mathcal{P}^-_{1,2}$:

$$\mathcal{P}^{\pm}_{1,2}(f_1 \otimes f_2)(x_1, x_2) = \int\int_{|\xi_1| \lessgtr |\xi_2|} d\xi_1 \ d\xi_2 \hat{f}_1(\xi_1)\hat{f}_2(\xi_2) e^{\mathrm{i}(x_1\xi_1 + x_2\xi_2)}$$

Regularized area using Fubini

Clue: Give two different boundary/increment decompositions for $\mathcal{P}^+_{1,2} Area_{ts}$ and $\mathcal{P}^-_{1,2} Area_{ts}$ using Fubini:

$$\mathcal{P}^+_{1,2} Area_{ts} = \mathcal{P}^+_{1,2} \int_s^t B'_{u_1}(1) du_1 \int_s^{u_1} B'_{u_2}(2) du_2,$$

$$\mathcal{P}^-_{1,2} Area_{ts} = \mathcal{P}^-_{1,2} \int_s^t B'_{u_2}(2) du_2 \int_{u_2}^t B'_{u_1}(1) du_1.$$

Fourier normal ordering: Innermost (rightmost) integrals bear highest Fourier components

Regularized area : Boundary term

$$\mathcal{P}^+_{1,2} Area_{ts}(\partial) := -\mathcal{P}^+_{1,2} \int_s^t B'_{u_1}(1) du_1 \int^s B'_{u_2}(2) du_2$$
$$= -\delta \left[u \mapsto \int dW_{\xi_1}(1) |\xi_1|^{-\frac{1}{2}-\alpha} e^{iu\xi_1} \int_{|\xi_2| \geq |\xi_1|} dW_{\xi_2}(2) |\xi_2|^{-\frac{1}{2}-\alpha} e^{is\xi_2} \right]_{ts}$$

Lemma. *Let* $F(u) = \int_{\mathbb{R}} dW_\xi e^{iu\xi} a(\xi)$ *with* $|a(\xi)|^2 \lesssim |\xi|^{-1-2\beta}$*: then*

$$\mathbb{E}|F(u_1) - F(u_2)|^2 \lesssim |u_1 - u_2|^{2\beta}.$$

Corollary (Kolmogorov-Centsov) F β^--Hölder.

One finds: $\mathrm{Var}\ a(\xi_1) \lesssim |\xi_1|^{-1-4\alpha} \longrightarrow \mathcal{P}^+_{1,2} Area(\partial)$ $2\alpha^-$-Hölder.

Increment term G (skeleton integral)

$$\mathcal{P}^+_{1,2} G_t = \int dW_{\xi_1}(1) \frac{|\xi_1|^{\frac{1}{2}-\alpha}}{\xi_1+\xi_2} e^{it\xi_1} \, . \, \int_{|\xi_2|\geq|\xi_1|} dW_{\xi_2}(2) |\xi_2|^{-\frac{1}{2}-\alpha} e^{it\xi_2}$$

$$\sim \int dW_{\zeta_1}(1) e^{it\zeta_1} a(\zeta_1), \quad a(\zeta_1) = \frac{1}{\zeta_1} \int_{|\zeta_1-\zeta_2|\leq|\zeta_2|} \frac{dW_{\zeta_2}(2)}{\zeta_2} |\zeta_1-\zeta_2|^{\frac{1}{2}-\alpha} |\zeta_2|^{\frac{1}{2}-\alpha}$$

$$\mathrm{Var}\, a(\zeta_1) \lesssim \frac{1}{\zeta_1^2} \int |\zeta_1-\zeta_2|^{1-2\alpha} |\zeta_2|^{-1-2\alpha} d\zeta_2 \lesssim |\zeta_1|^{-1-4\alpha} \; (\alpha > 1/4), \infty \text{ else!}$$

1. Domain regularization :

$\mathbb{R}^2_+ := \{(\xi_1,\xi_2) : |\xi_1| \leq |\xi_2|\} \rightsquigarrow \mathbb{R}^2_{reg} := \{(\xi_1,\xi_2) \in \mathbb{R}^2_+ : |\xi_1+\xi_2| > C_{reg}|\xi_2|\}$.

2. Regularization by counterterm: $a(\zeta_1) \mapsto \mathcal{R}a(\zeta_1) := a(\zeta_1) - a(0)$.

Theorem: $\mathcal{R}\mathcal{P}^+_{1,2} G_t$ is 2α-Hölder FOR ALL α.

Three fundamental theorems.

1. Existence theorem (Lyons-Victoir 2007). Let X α-Hölder. Then there exists a rough path J_X over X. Non-constructive proof: non-canonical Hölder lifts of sections of principal bundles.
2. Approximation theorem (Lyons-Friz-Victoir). Let X α-Hölder and J_X rough path over X. Then there exists an approximation of X by smooth $X(\varepsilon)$ such that iterated integrals of $X(\varepsilon)$ converge to J_X. Non-constructive proof: compacity type arguments to determine existence of horizontal sub-Riemannian Carnot-Carathéodory geodesics over universal nilpotent group.
3. "Black box" (Lyons-Friz-Victoir-Lejay-Tindel...) Let X α-Hölder and J_X rough path over X. Then one knows how to define integrals along/solve differential equations driven by X – or rather J_X. Explicit numerical analysis constructions.

Plan

Characters of $\mathbf{H}^d$ and $\mathbf{Sh}^d$

Main question. Recall $\Pi^d : \mathbf{H}^d \to \mathbf{Sh}^d$ canonical projection

χ character of $\mathbf{Sh}^d \rightsquigarrow \chi \circ \Pi^d$ character of $\mathbf{H}^d$.

How to go in the reverse direction ?

Variations on H

H Hopf algebra of (non-decorated) rooted trees

$\mathbf{H}_{ho}$ Hopf algebra of heap-ordered rooted trees:

$\mathcal{F}_{ho} = \uplus_{n\geq 0}\mathcal{F}_{ho}(n)$ heap-ordered forests with n vertices

If $\mathbb{F} \in \mathcal{F}_{ho}(n)$ then $(i \twoheadrightarrow j) \Rightarrow (n \geq i > j \geq 1)$

Ex. $\bullet_1 . \mathfrak{I}^{2}_{1} = \bullet_1 \mathfrak{I}^{3}_{2}$, $\mathfrak{I}^{2}_{1} . \bullet_1 = \mathfrak{I}^{2}_{1} \bullet_3$

$$\Delta({}^{2}\mathbb{V}^{3}_{1}) = {}^{2}\mathbb{V}^{3}_{1} \otimes 1 + 1 \otimes {}^{2}\mathbb{V}^{3}_{1} + 2\,\mathfrak{I}^{2}_{1} \otimes \bullet_1 + \bullet_1 \otimes \bullet_1 \bullet_2 .$$

Equivalent notations.

I^{ts} : (path $X : \mathbb{R} \to \mathbb{R}^d$) $\times$ (decorated forest) $\to \mathbb{R}$, or
((signed) measure in $Meas(\mathbb{R}^n)$) $\times$ (heap-ordered forest
$\mathbb{F} \in \mathcal{F}_{ho}(n)$), $I^{ts}_X((\mathbb{F},\ell)) = I^{ts}_{\mu_{(X,\ell)}}(\mathbb{F})$
where $\mu_{(X,\ell)}(dx_1,\ldots,dx_n) = \otimes_{i=1}^n dX_{x_i}(\ell(i))$.
The second definition extends to arbitrary measures, $\rightsquigarrow I^{ts}_{\mu}(\mathbb{F})$

Tree-order-preserving symmetries

$\mathbb{F} \in \mathcal{F}_{ho}(n)$ with n vertices

Definition.

$$\begin{aligned} S_{\mathbb{F}} &:= \{\sigma \in \Sigma_n \mid (i \twoheadrightarrow j) \Rightarrow (\sigma^{-1}(i) > \sigma^{-1}(j))\} \\ &= \{\sigma \in \Sigma_n \mid \sigma^{-1}(\mathbb{F}) \in \mathcal{F}_{ho}(n)\}. \end{aligned}$$

Example. $\mathbb{F} = \overset{\bullet}{\bullet}{}^{2}_{1} \ \bullet_{3}$

$Sh((1,2),(3)) = \{\sigma = (1\ 2\ 3), (1\ 3\ 2), (3\ 1\ 2)\}$

$\rightsquigarrow \sigma^{-1} = (1\ 2\ 3), (1\ 3\ 2), (2\ 3\ 1)$

$\rightsquigarrow \sigma^{-1}(\mathbb{F}) = \overset{\bullet}{\bullet}{}^{2}_{1} \ \bullet_{3},\ \overset{\bullet}{\bullet}{}^{3}_{1} \ \bullet_{2},\ \overset{\bullet}{\bullet}{}^{3}_{2} \ \bullet_{1}.$

I^{ts} depends only on the topology of $\mathbb{F}$

$$\rightsquigarrow I^{ts}_{\mu_{(X,\ell)}}(\mathbb{F}) = I^{ts}_{\mu_{(X,\ell)}\circ\sigma}(\sigma^{-1}(\mathbb{F})). \tag{2.1}$$

Fourier normal ordering

$\mu \in Meas(\mathbb{R}^n)$, standard example:
$\mu_{(X,\ell)}(dx_1, \ldots, dx_n) = \otimes_{i=1}^n dX_{x_i}(\ell(i))$.
Fourier projections.

$$\mathcal{P}^\sigma \mu = \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi_{\sigma(1)}| \le \ldots \le |\xi_{\sigma(n)}|} \mathcal{F}\mu(\xi_1, \ldots, \xi_n)\right).$$

$\mu^\sigma := (\mathcal{P}^\sigma \mu) \circ \sigma = \mathcal{P}^{\mathrm{Id}}(\mu \circ \sigma)$ is Fourier normal-ordered

Measure-splitting decomposition:

$$\mu = \sum_{\sigma \in \Sigma_n} \mu^\sigma \circ \sigma.$$

First definition of permutation graph $\mathbb{T}^\sigma$

$\mathfrak{t}_n \in \mathcal{F}_{ho}(n)$: trunk heap-ordered tree with n vertices
$\sigma \in \Sigma_n \rightsquigarrow \mathbb{T}^\sigma \in \mathbf{H}_{ho}(n)$ defined by

$$I_\mu^{ts}(\mathfrak{t}_n) = I_{\mu\circ\sigma}^{ts}(\mathbb{T}^\sigma).$$

Example. $\sigma = (231)$:

$$\int_s^t dx_{\ell(1)} \int_s^{x_{\ell(1)}} dx_{\ell(2)} \int_s^{x_{\ell(2)}} dx_{\ell(3)} \, (\ldots) = \int_s^t dx_{\ell(2)} \int_s^{x_{\ell(2)}} dx_{\ell(3)} \int_{x_{\ell(2)}}^t dx_{\ell(1)} (\ldots)$$

$$= - \int_s^t dx_1 \int_s^{x_1} dx_2 \int_s^{x_1} dx_3 (\ldots) + \int_s^t dx_1 \int_s^{x_1} dx_2 \, . \int_s^t dx_3 (\ldots)$$

$\longrightarrow$ (after permuting indices): $\mathbb{T}^\sigma = -\,{}^{3}\!\bigvee_{1}^{2} + \mathfrak{l}_{1}^{2} \; \bullet_{3}$

Second definition of permutation graph $\mathbb{T}^\sigma$

Definition. FQSym Hopf algebra of free quasi-symmetric functions

FQSym = formal sums of permutations

Preliminary remark. Fix $k \leq n$ and $\sigma \in \Sigma_n$, then σ writes uniquely as $\zeta^{-1} \circ (\sigma_1 \otimes \sigma_2)$ or $(\sigma_1 \otimes \sigma_2) \circ \varepsilon$, with $\varepsilon, \zeta \in \mathrm{Sh}(k, n-k)$.

Product. $\sigma_1 \in \Sigma_k, \sigma_2 \in \Sigma_{n-k} \rightsquigarrow$

$$\sigma_1.\sigma_2 = \sum_{\varepsilon \in \mathrm{Sh}(k,n-k)} (\sigma_1 \otimes \sigma_2) \circ \varepsilon.$$

Example.

$$(123)(21) = (12354) + (12534) + (15234) + (51234) + (12543)$$
$$+(15243) + (51243) + (15423) + (51423) + (54123).$$

Coproduct. $\Delta(\sigma) = \sum_{k=0}^n \sigma_1^{(k)} \otimes \sigma_2^{(k)}$.

Example.

$$\Delta((231)) = 1 \otimes (231) + Std(2) \otimes Std(31) + Std(23) \otimes Std(1) + (231) \otimes 1$$
$$= 1 \otimes (231) + (1) \otimes (21) + (12) \otimes (1) + (231) \otimes 1.$$

An explicit Hopf algebra isomorphism

Theorem (L. Foissy+J.U.)

1. Let $\Theta : \mathbf{H}_{ho} \to \mathrm{FQSym}, \mathbb{F} \in \mathcal{F}_{ho} \mapsto \sum_{\sigma \in S_{\mathbb{F}}} \sigma$. Then Θ is a Hopf algebra isomorphism.
2. $\mathbb{T}^{\sigma} = \Theta^{-1}(\sigma^{-1})$.

Corollary.

1. $$\varepsilon^{-1}(\mathbb{T}^{\sigma_1}.\mathbb{T}^{\sigma_2}) = \sum_{\zeta \in \mathrm{Sh}(k,n-k)} \mathbb{T}^{\zeta^{-1} \circ (\sigma_1 \otimes \sigma_2) \circ \varepsilon};$$
2. $$\Delta(\mathbb{T}^{\sigma}) = \sum_k \sum_{\sigma = (\sigma_1 \otimes \sigma_2) \circ \varepsilon} \mathbb{T}^{\sigma_1} \otimes \mathbb{T}^{\sigma_2}.$$

General rough path construction by Fourier normal ordering

Theorem. Let $\phi^t_{\mathbb{F}} : \mathcal{P}^{\mathbb{F}} Meas(\mathbb{R}^n) \to \mathbb{R}$, $\mu \mapsto \phi^t_\mu(\mathbb{F})$ $(t \in \mathbb{R}, \mathbb{F} \in \mathcal{F}_{ho}(n))$ linear and invariant under tree-order-preserving symmetries, i.e.

$$\phi^t_\mu(\mathbb{F}) = \phi^t_{\mu\circ\sigma}(\sigma^{-1}(\mathbb{F})), \quad \sigma \in S_{\mathbb{F}},$$

and such that:

- $\phi^t_{dX(i)}(\mathfrak{t}_1) - \phi^s_{dX(i)}(\mathfrak{t}_1) = X_t(i) - X_s(i)$.
- $\phi^t_{\mu_1}(\mathbb{T}_1)\phi^t_{\mu_2}(\mathbb{T}_2) = \phi^t_{\mu_1\otimes\mu_2}(\mathbb{T}_1.\mathbb{T}_2)$.

Then:

1. $\chi^t_X((\mathfrak{t}_n,\ell)) := \sum_{\sigma\in\Sigma_n} \phi^t_{\mu^\sigma_{(X,\ell)}}(\mathbb{T}^\sigma)$ is a character of $\mathbf{Sh}^d$.
2. $J^{ts}_X((\mathfrak{t}_n,\ell)) := \chi^t_X * (\chi^s_X \circ S)(\mathfrak{t}_n,\ell)$ is a rough path over X.

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First elementary example.

Let $\phi_\mu^t(\mathfrak{t}_n) = 0$ for every $n \geq 2$.

Let X be an α-Hölder path (e.g. a path of $(B_t(1), \ldots, B_t(d))$).

Then $J_X^{ts}(i_1, \ldots, i_n)$, $n \leq \lfloor 1/\alpha \rfloor$ is $n\alpha$-Hölder.

Second example. Fourier domain regularization

Definition (skeleton integrals)

$$\mathrm{SkI}^t_\mu(\mathbb{T}) = \int d\xi_1 \ldots d\xi_n (\mathcal{F}\mu)(\xi_1, \ldots, \xi_n) \cdot \int^t \int^{x_2-} \ldots \int^{x_n-} e^{\mathrm{i}\langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} d\boldsymbol{x}$$

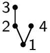
where $\int^t e^{\mathrm{i}x\xi} = \frac{e^{\mathrm{i}t\xi}}{\mathrm{i}\xi}$.
Computation:

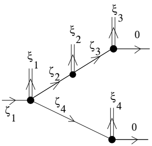
$$\mathrm{SkI}^t_\mu(\mathbb{T}) = \int d\xi_1 \ldots d\xi_n (\mathcal{F}\mu)(\xi_1, \ldots, \xi_n) \cdot \frac{e^{\mathrm{i}t(\xi_1 + \ldots + \xi_n)}}{\prod_i (\xi_i + \sum_{j \twoheadrightarrow i} \xi_j)}.$$

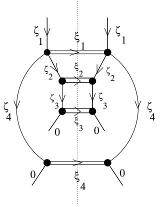
$\rightsquigarrow$ Integrate over subdomain of $\mathbb{R}^n_+ = \{|\xi_1| \leq \ldots \leq |\xi_n|\}$ where denominator large, e.g. $|\xi_i + \sum_{j \twoheadrightarrow i} \xi_j| > C \sup_{j \twoheadrightarrow i} |\xi_j|$.

Third example. BPHZ renormalization for fBm.

Associate to a skeleton integral $\mathrm{SkI}_B(\mathbb{T})$ a Feynman half-diagram and a Feynman diagram.

Example. $\mathbb{T} =$ 

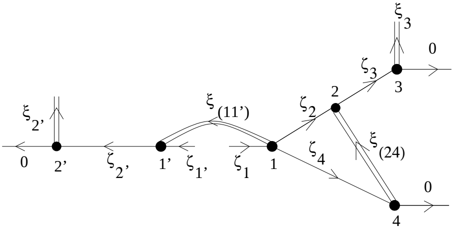


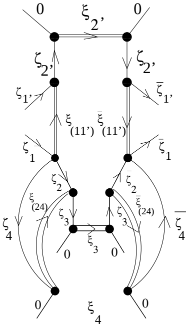


Feynman rules $\rightsquigarrow$ Diagram evaluation $A_{G^{\frac{1}{2}}(\mathbb{T})}(\boldsymbol{\zeta})$ or $A_{G(\mathbb{T})}(\zeta_1)$.

$A_{G^{\frac{1}{2}}(\mathbb{T})}$: $1/\zeta$, $|\xi|^{\frac{1}{2}-\alpha}$; $\qquad A_{G(\mathbb{T})}$: $1/\zeta$, $|\xi|^{1-2\alpha}$

$$\mathrm{SkI}^t_B(\mathbb{T}) := \int \prod_{v\in V(\mathbb{T})} dW_{\xi_v}(\ell(v)) \frac{e^{\mathrm{i}t\zeta_1}}{\zeta_1} A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1,\boldsymbol{\xi}) \rightsquigarrow \mathrm{Var}(\ .\ ) = \int \frac{d\zeta_1}{\zeta_1^2} A_{G(\mathbb{T})}(\zeta_1).$$


ξ3
0
ζ3
2
3
ξ(11')
ζ2
ξ2'
ξ(24)
ζ4
0
0
2'
ζ2'
1'
ζ1'
ζ1
1
0
4



BPHZ renormalization

Theorem.
Let

1. $$\mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1) := \sum_{\mathbb{F}\in\mathcal{F}^{div}(G(\mathbb{T}))} \prod_{g\in\mathbb{F}} (-\tau_g) A_{G^{\frac{1}{2}}(\mathbb{T})}(\boldsymbol{\zeta})$$

2. $$\mathcal{R}\mathrm{SkI}_B^t(\mathbb{T}) := \int \prod_{v\in V(\mathbb{T})} dW_{\xi_v}(\ell(v)) \frac{e^{\mathrm{i}t\zeta_1}}{\zeta_1} \mathcal{R}A_{G^{\frac{1}{2}}(\mathbb{T})}(\zeta_1, \boldsymbol{\xi})$$

Then

$$|\mathrm{Var}(\mathcal{R}\mathrm{SkI}_B^t - \mathcal{R}\mathrm{SkI}_B^s)(\mathbb{T})| \lesssim |t-s|^{2|V(\mathbb{T})|\alpha}.$$

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Singular penalizations

Definition (stationary field associated to fBm).

$$\phi_{1,2}(t) = \int \frac{e^{it\xi}}{|\xi|^{\alpha+1/2}} dW_{1,2}(\xi), \quad \mathbb{E}|\mathcal{F}\phi_{1,2}(\xi)|^2 = \frac{1}{|\xi|^{1+2\alpha}}.$$

Associated Gaussian measure: $d\mu(\phi)$
Idea: penalize trajectories with many small area bubbles by replacing $d\mu(\phi)$ with $\frac{1}{Z(\lambda)} e^{-\frac{1}{2}\lambda^2 \int \mathcal{L}_{int}(t)dt}$ where $\lambda \ll 1$ and $\mathcal{L}_{int}$ quadratic in the Lévy area, $\mathcal{A}$.
"Trick": $e^{-\frac{1}{2}\int \lambda^2 \mathcal{A}^2} = \int e^{i\lambda \int \mathcal{A}\sigma} d\mu(\sigma)$
Associated Gaussian measure : $d\mu(\sigma)$, $\mathbb{E}|\mathcal{F}\sigma(\xi)|^2 = \frac{1}{|\xi|^{1-4\alpha}}$

Cultural note : field theory

Classical language in elementary particle physics and in statistical physics.

Multi-scale Fourier analysis $\rightsquigarrow$ integrating w.r. to highest Fourier scales yields an effective theory at low frequency (=at large distances) with renormalized parameters

Examples:

$\lambda \rightsquigarrow \lambda^j$ effective parameter for $2^j \lesssim |\xi| \lesssim 2^{j+1}$;

$\frac{1}{|\xi|^{1-4\alpha}} \rightsquigarrow \frac{1}{|\xi|^{1-4\alpha}+b^j}$, b^j=effective mass of the σ-field

Other examples:

• Weakly self-avoiding paths or ϕ^4 theory (D=4): free theory at large distances ($\lambda^j \to 0$ quand $j \to -\infty$)

• Quantum chromodynamics: free theory at small distances.

$(\phi, \partial\phi, \sigma)$-model

Ultra-violet cut-off: $|\xi| \lesssim 2^\rho$

$$Z(\lambda) = \int d\mu^{\to\rho}(\phi) d\mu^{\to\rho}(\sigma) e^{-\mathrm{i}\lambda \int \mathcal{P}^+(\partial\phi_1(x)\phi_2(x))\sigma(x)dx}$$

Renormalization: $b^j \approx \lambda^2 2^{\rho(1-4\alpha)}$ for every j

The interaction introduces a screening mass $\approx \infty$

$\rightsquigarrow$ by integration by parts:

$$\langle |\mathcal{F}(\partial A^\pm)(\xi)|^2 \rangle_\lambda = \frac{1}{\lambda^2} |\xi|^{1-4\alpha} \left[1 - |\xi|^{1-4\alpha} \langle |(\mathcal{F}\sigma_+)(\xi)|^2 \rangle_\lambda \right]. \quad (4.1)$$

Perturbative "proof"

Feynman diagrams: formal expansion

$e^{-\mathrm{i}\mathcal{L}_{int}} = \sum \frac{(-\mathrm{i}\mathcal{L}_{int})^n}{n!} \rightsquigarrow$ Wick formula

Bubble:

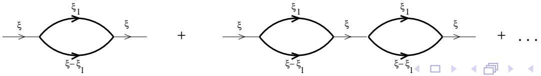
$$-|\xi|^{1-4\alpha} \cdot (-\mathrm{i}\lambda)^2 \int_{|\xi_1|<|\xi-\xi_1|}^{\Lambda} d\xi_1 \left\{ \left(\mathbb{E}[|\mathcal{F}\sigma_+(\xi)|^2]\right)^2 \ \mathbb{E}[|\mathcal{F}(\partial\phi_1)(\xi_1)|^2]\ \mathbb{E}[|\mathcal{F}\phi_2(\xi-\xi_1)|^2] \right\}$$

$$= \lambda^2 |\xi|^{4\alpha-1} \int_{|\xi_1|<|\xi-\xi_1|}^{\Lambda} d\xi_1 |\xi_1|^{1-2\alpha} |\xi-\xi_1|^{-1-2\alpha} \sim_{\Lambda\to\infty} K\lambda^2 (\Lambda/|\xi|)^{1-4\alpha}, \quad (4.2)$$

Bubble series: $\frac{1}{|\xi|^{1-4\alpha}} \rightsquigarrow \frac{1}{|\xi|^{1-4\alpha}+b}$, $b \approx \lambda^2 \Lambda^{1-4\alpha}$

$$\frac{1}{\lambda^2}|\xi|^{1-4\alpha}\left[1 - \frac{1}{1+K'\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}\right] = \frac{1}{\lambda^2}|\xi|^{1-4\alpha} \cdot \frac{K'\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}{1+K'\lambda^2(\Lambda/|\xi|)^{1-4\alpha}}$$

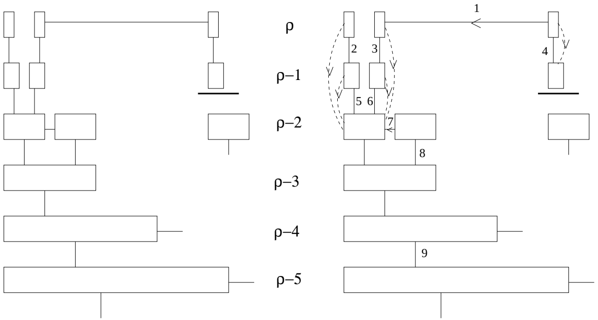
$$\to_{\Lambda\to\infty} \frac{1}{\lambda^2}|\xi|^{1-4\alpha}. \quad (4.3)$$



Constructive proof (I)

Multi-scale vertical slicing $\psi = \sum_j \psi^j$, $supp(\mathcal{F}\psi^j) \subset [2^{j-1}, 2^{j+1}]$
$\rightsquigarrow$ Horizontal slicing : one degree of freedom per dyadic interval Δ^j of length 2^{-j}

Cluster expansion: finite-order expansion within each interval Δ^j, and approximate decoupling of degrees of freedom $\rightsquigarrow$ polymers $\mathbb{P}$.



Constructive proof (II)

$$Z_V^{\to\rho}(\lambda) = \sum_n \frac{1}{n!} \sum_{\mathbb{P}_{1,\ldots,\mathbb{P}_n}\ \mathrm{non-overlapping}} F_{HV}(\mathbb{P}_1)\ldots F_{HV}(\mathbb{P}_n),$$

$\ln Z_V^{\to\rho}(\lambda) = |V| \sum_{j=0}^{\rho} 2^j f_V^{j\to\rho}$, where $f_V^{j\to\rho} \to_{|V|\to\infty} O(\lambda)$

Renormalization: The local parts of diverging diagrams are resummed into an exponential scale after scale $\Leftrightarrow$ covariance renormalized by the mass counterterm b^j

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