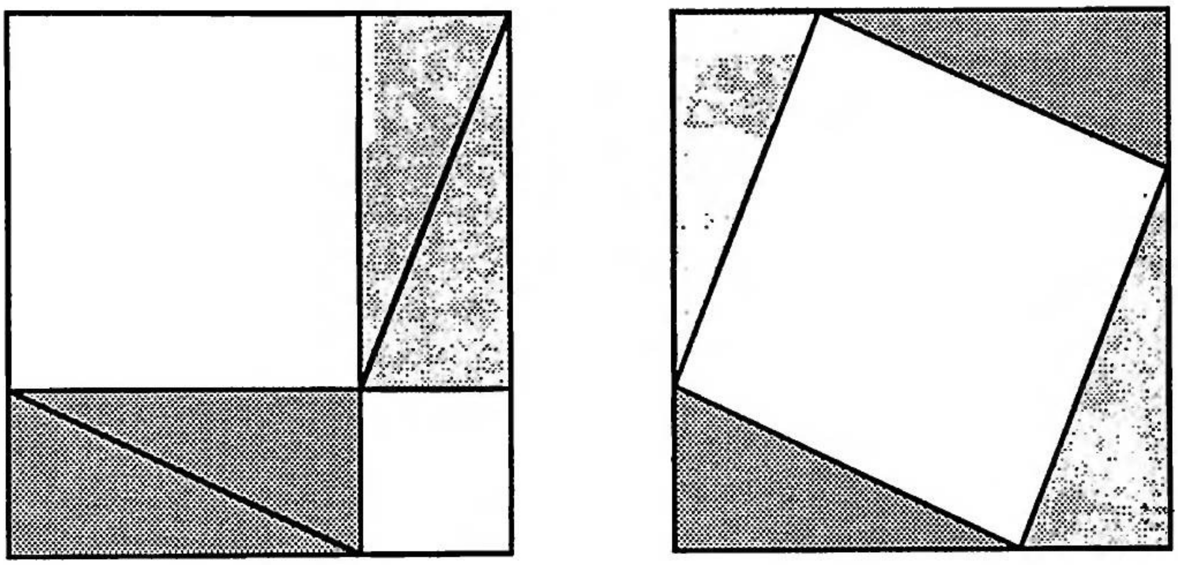
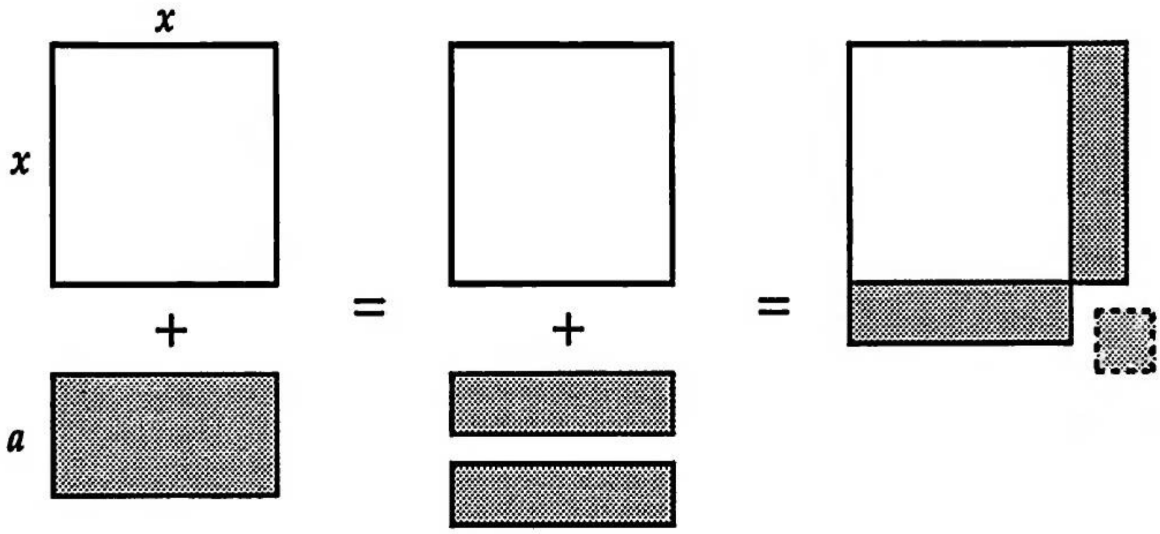
The Pythagorean Theorem I



—adapted from the *Chou pei suan ching*
(author unknown, circa B.C. 200?)

Completing the Square

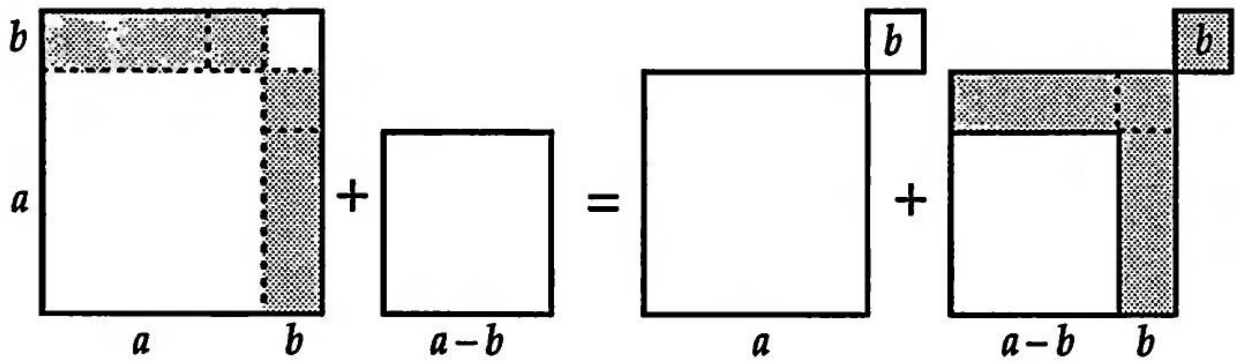
$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



—Charles D. Gallant

Algebraic Areas I

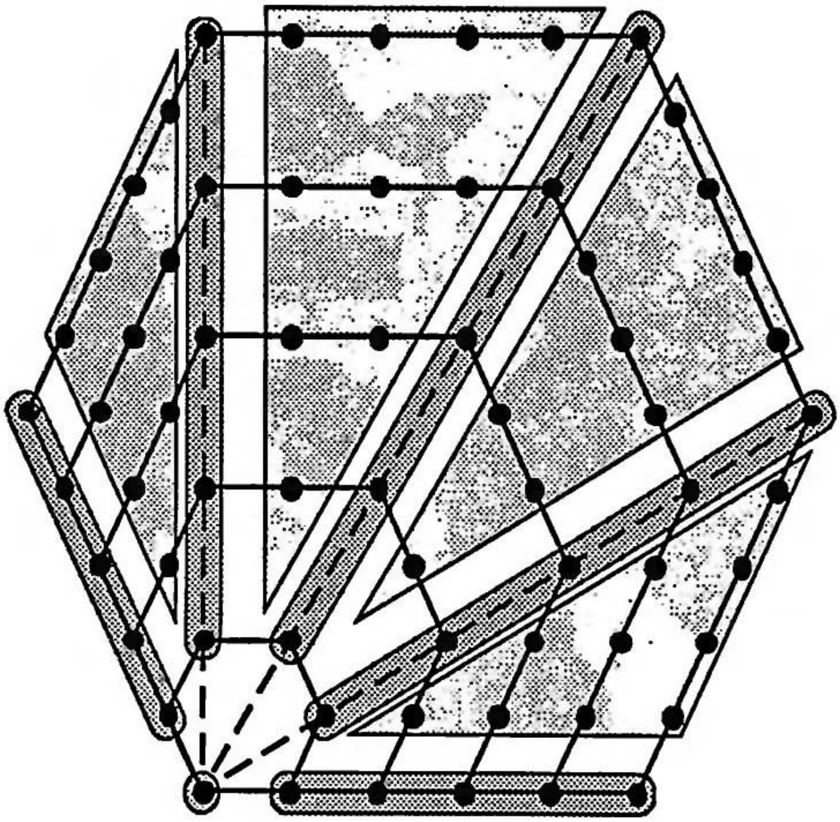
$$(a+b)^2+(a-b)^2 = 2(a^2+b^2)$$



—Shirley Wakin

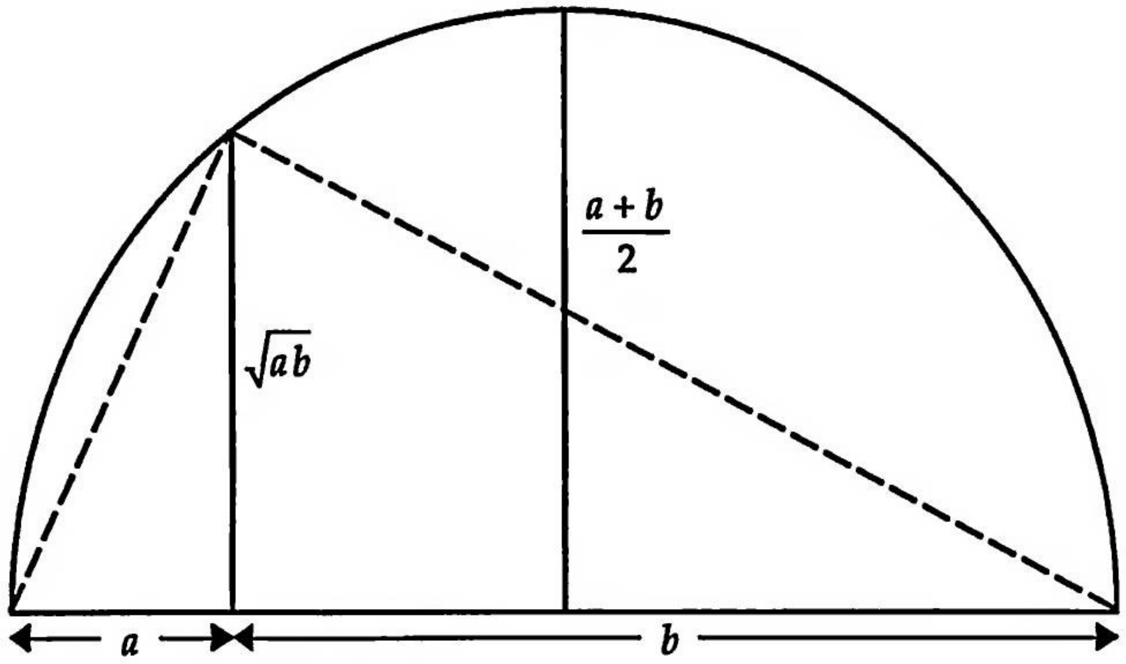
The k^{th} n-gonal Number is

$$1 + (k-1)(n-1) + \frac{1}{2}(k-2)(k-1)(n-2)$$



—Dave Logothetti

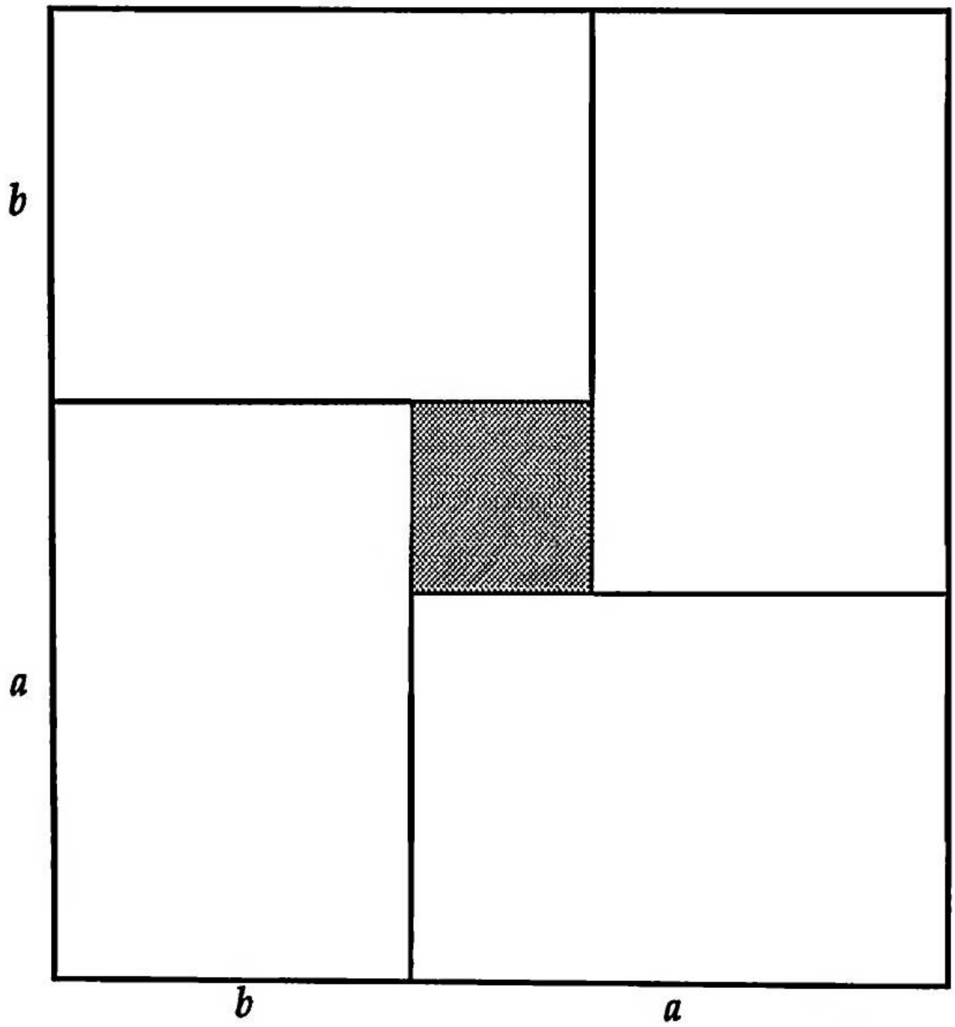
The Arithmetic Mean—Geometric Mean Inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

—Charles D. Gallant

The Arithmetic Mean—Geometric Mean Inequality II

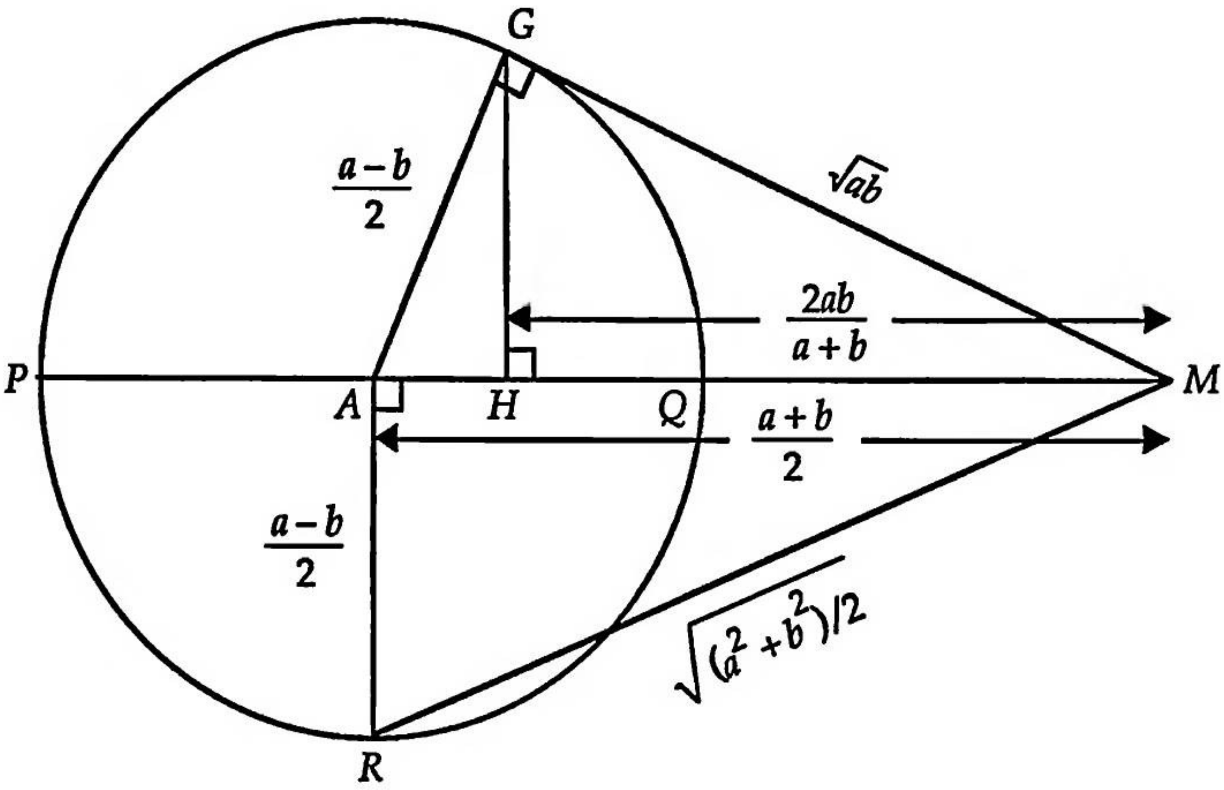


$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

—Doris Schattschneider

The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality I



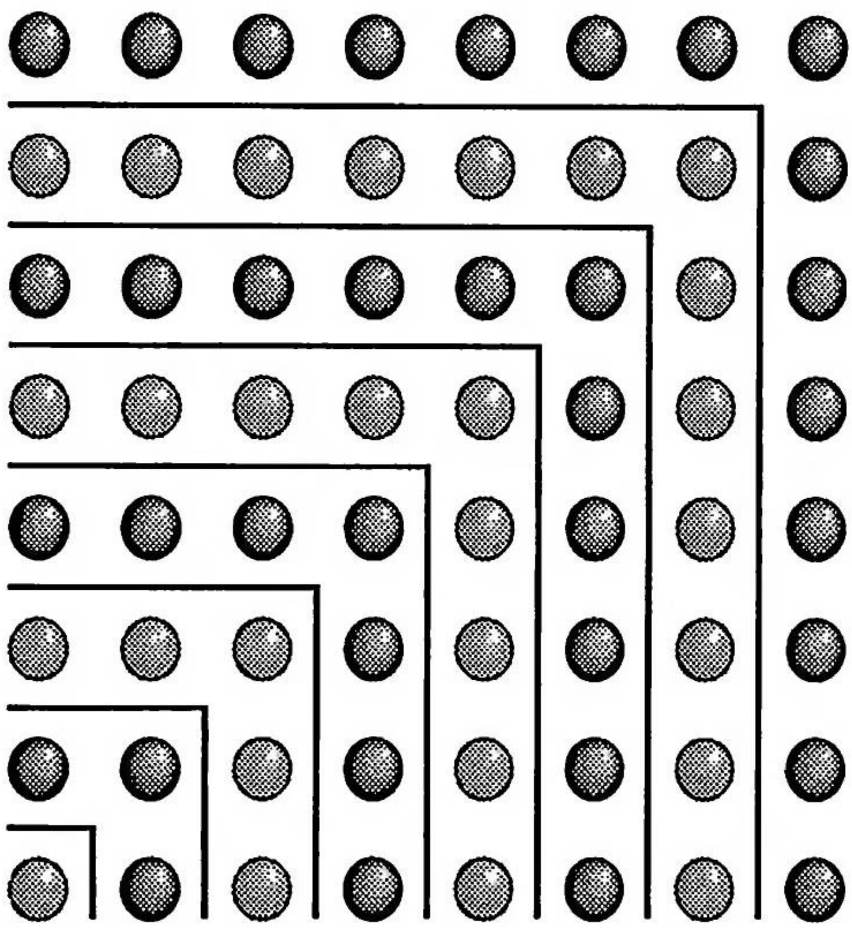
$$PM = a,\ QM = b,\ a > b > 0$$

$$HM < GM < AM < RM$$

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{(a^2+b^2)/2}$$

—RBN

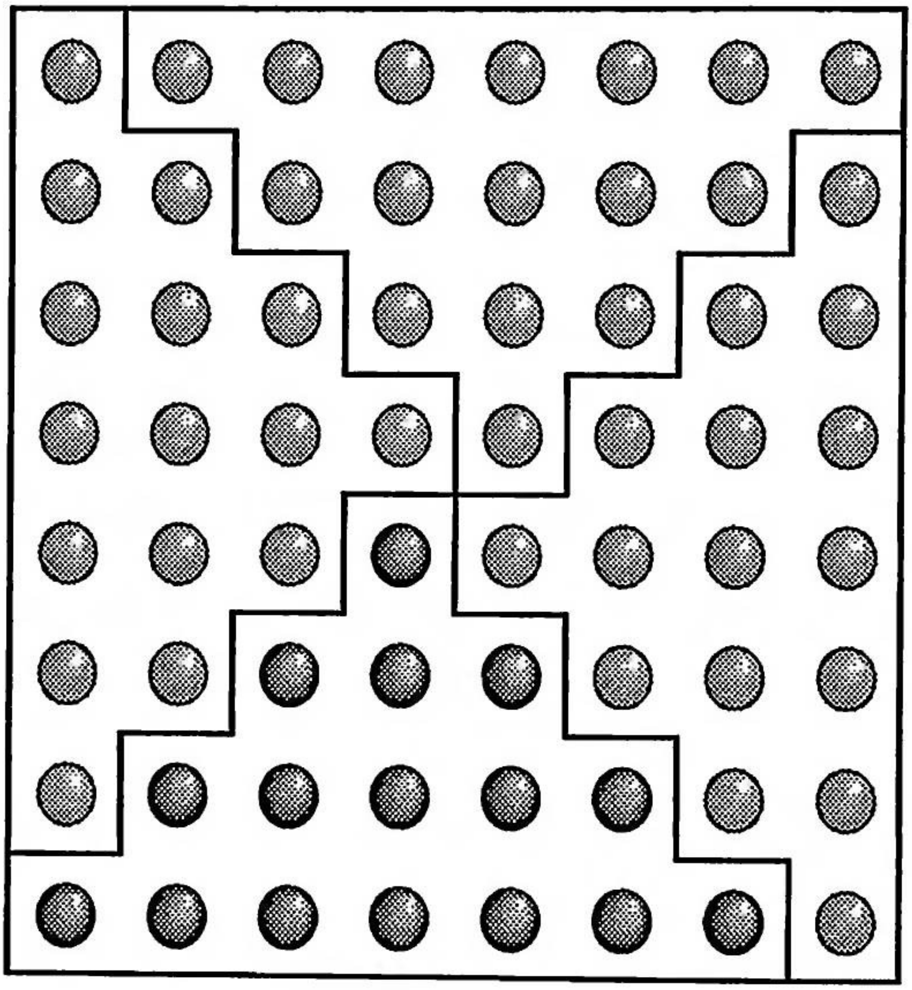
Sums of Odd Integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

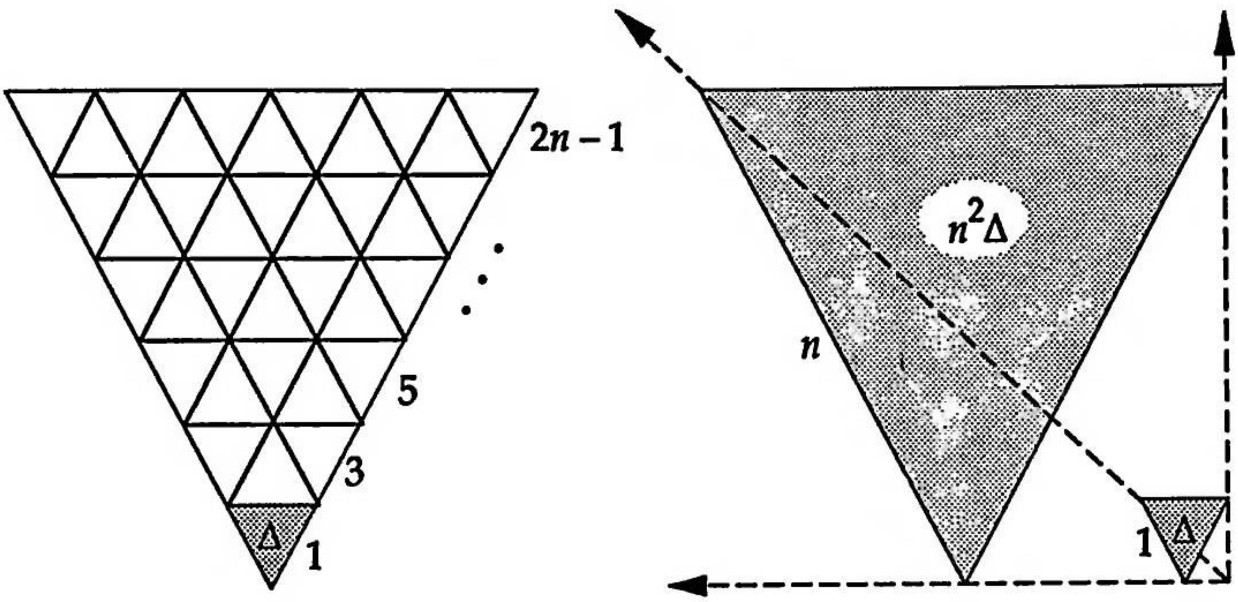
—Nicomachus of Gerasa (circa A. D. 100)

Sums of Odd Integers II



$$1+3+\cdots+(2n-1)=\frac{1}{4}(2n)^2=n^2$$

Sums of Odd Integers III



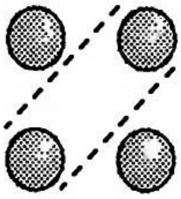
$$\Delta + 3\cdot\Delta + \cdots + (2n-1)\cdot\Delta \;=\; A \;=\; n^2\cdot\Delta$$

$$\sum_{i=1}^{n} (2i-1) \;=\; n^2$$

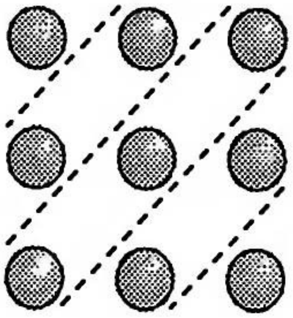
—Jenő Lehel

Squares and Sums of Integers

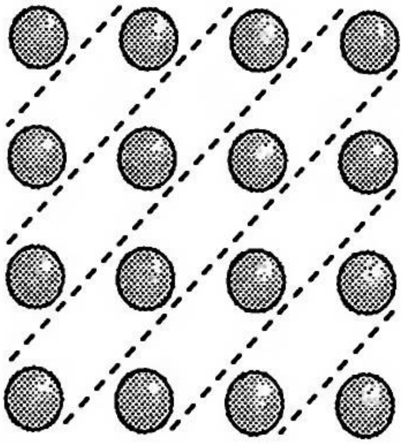
I.



$$1+2+1=2^2$$



$$1+2+3+2+1 = 3^2$$

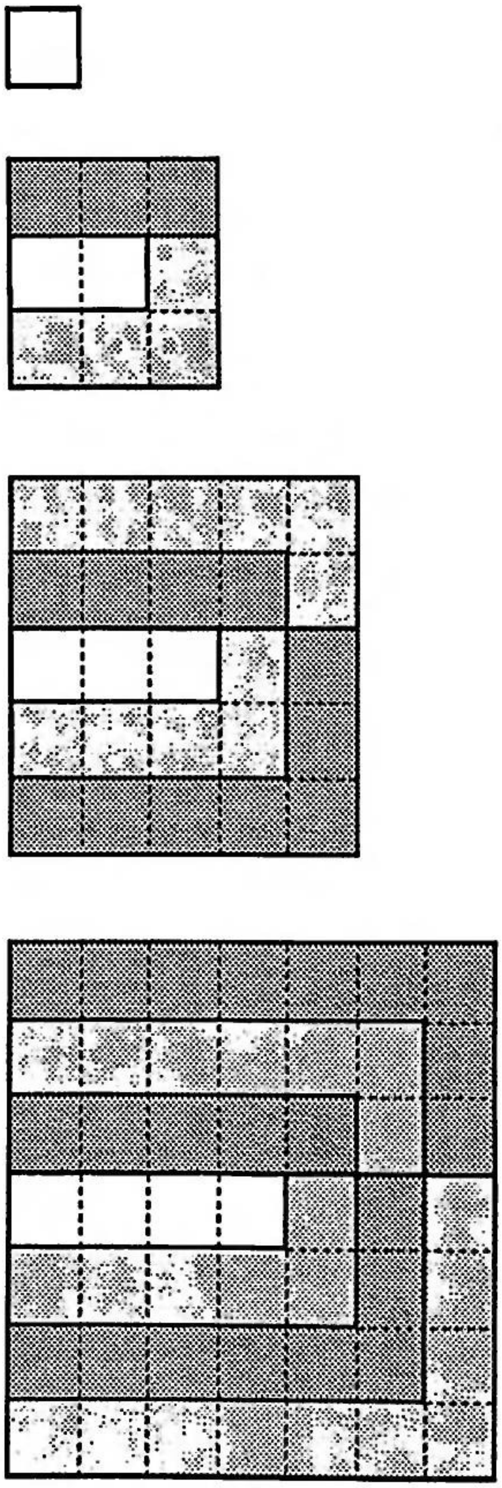


$$1+2+3+4+3+2+1 = 4^2$$

$$1+2+\cdots+(n-1)+n+(n-1)+\cdots+2+1 = n^2$$

—"The ancient Greeks"
(as cited by Martin Gardner)

Arithmetic Progressions with Sum Equal to the Square of the Number of Terms



$$\sum_{k=n}^{3n-2} k = (2n-1)^2; \quad n = 1, 2, 3, \cdots.$$

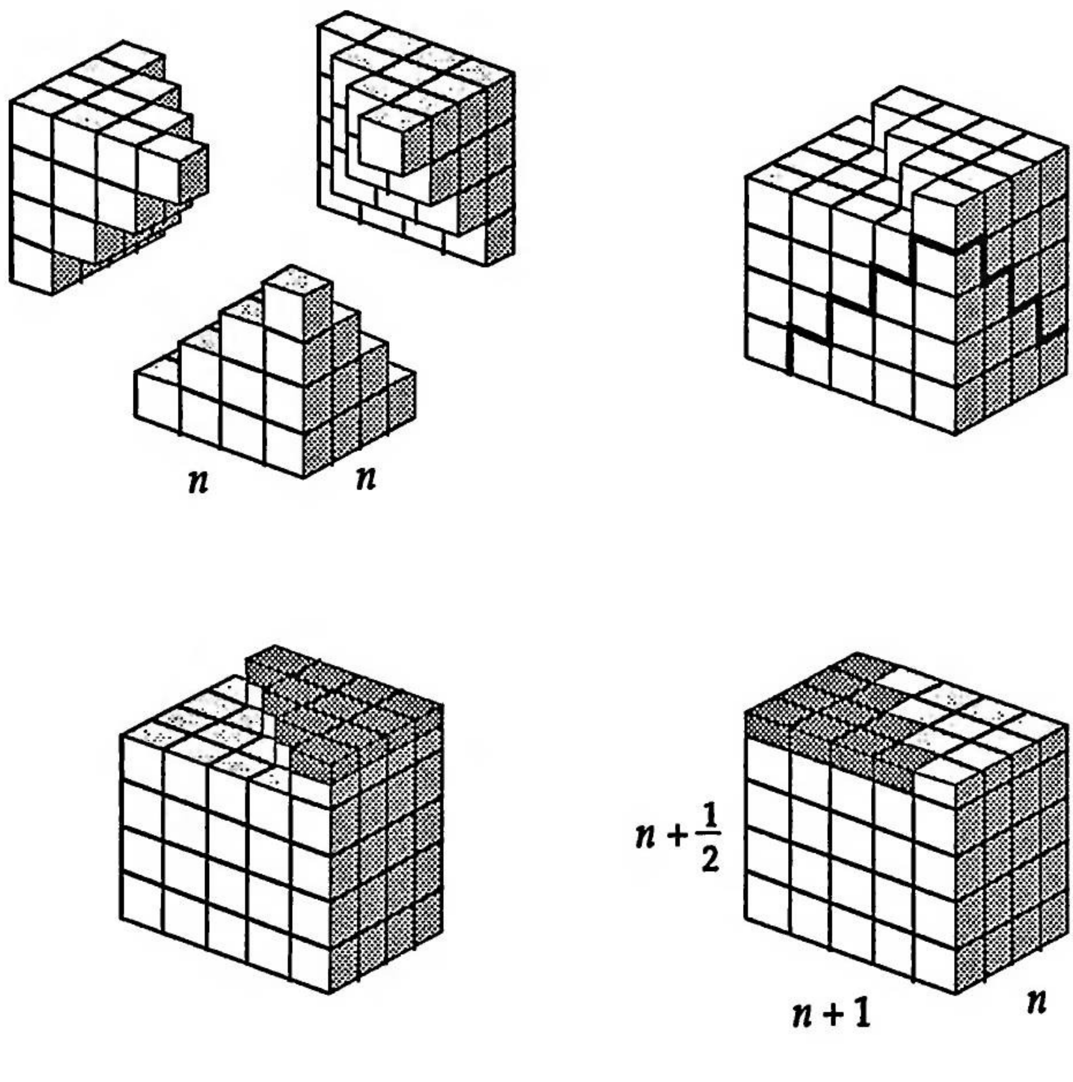
$n = 4$

$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$

—James O. Chilaka

Sums of Squares I

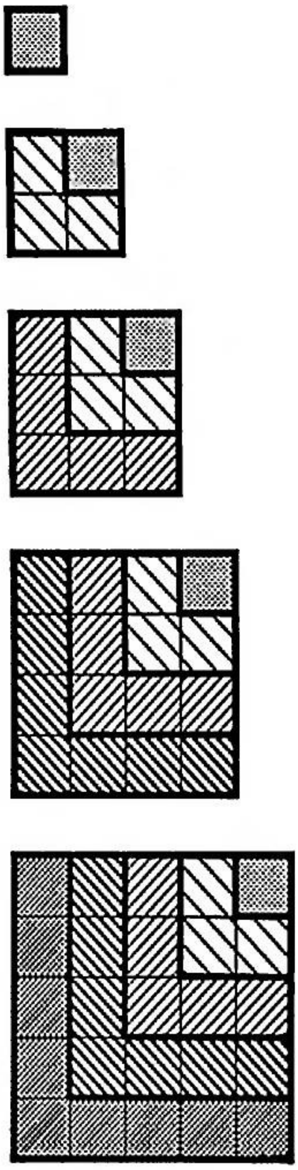
$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$$

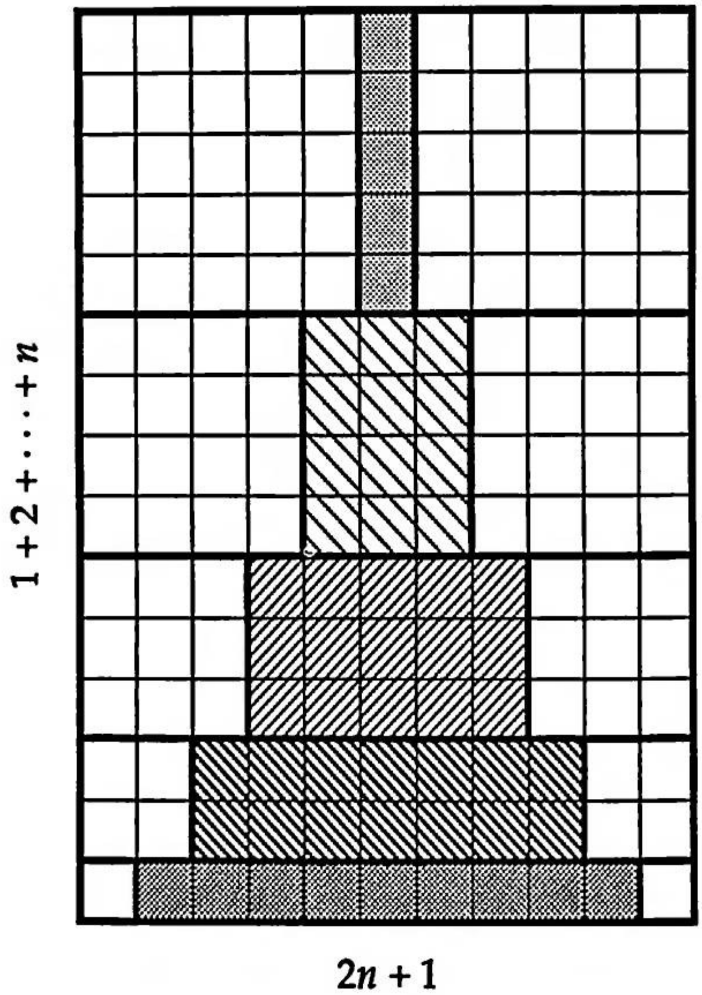


—Man-Keung Siu

Sums of Squares II

$$3(1^2+2^2+\cdots+n^2) = (2n+1)(1+2+\cdots+n)$$

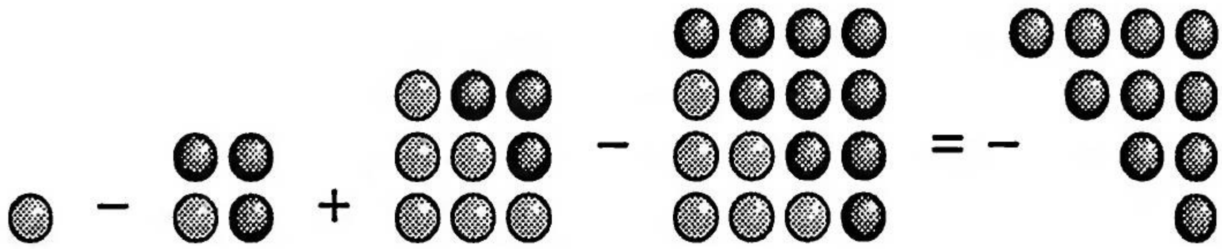




—Martin Gardner and Dan Kalman
(independently)

Alternating Sums of Squares

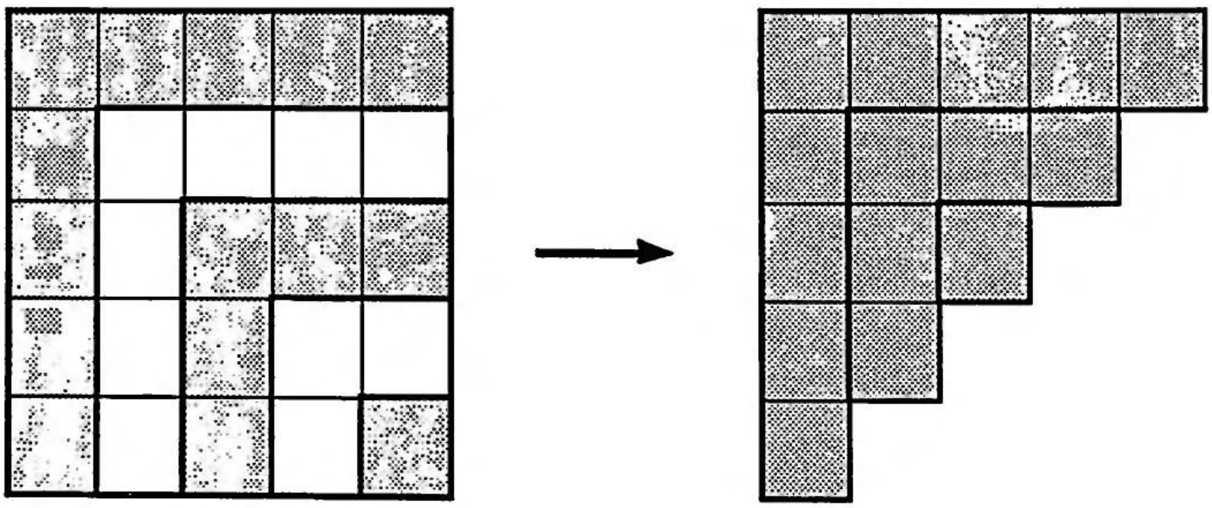
I.



$$\sum_{k=1}^{n}(-1)^{k+1}k^2 = (-1)^{n+1}T_n = (-1)^{n+1}\frac{n(n+1)}{2}$$

—Dave Logothetti

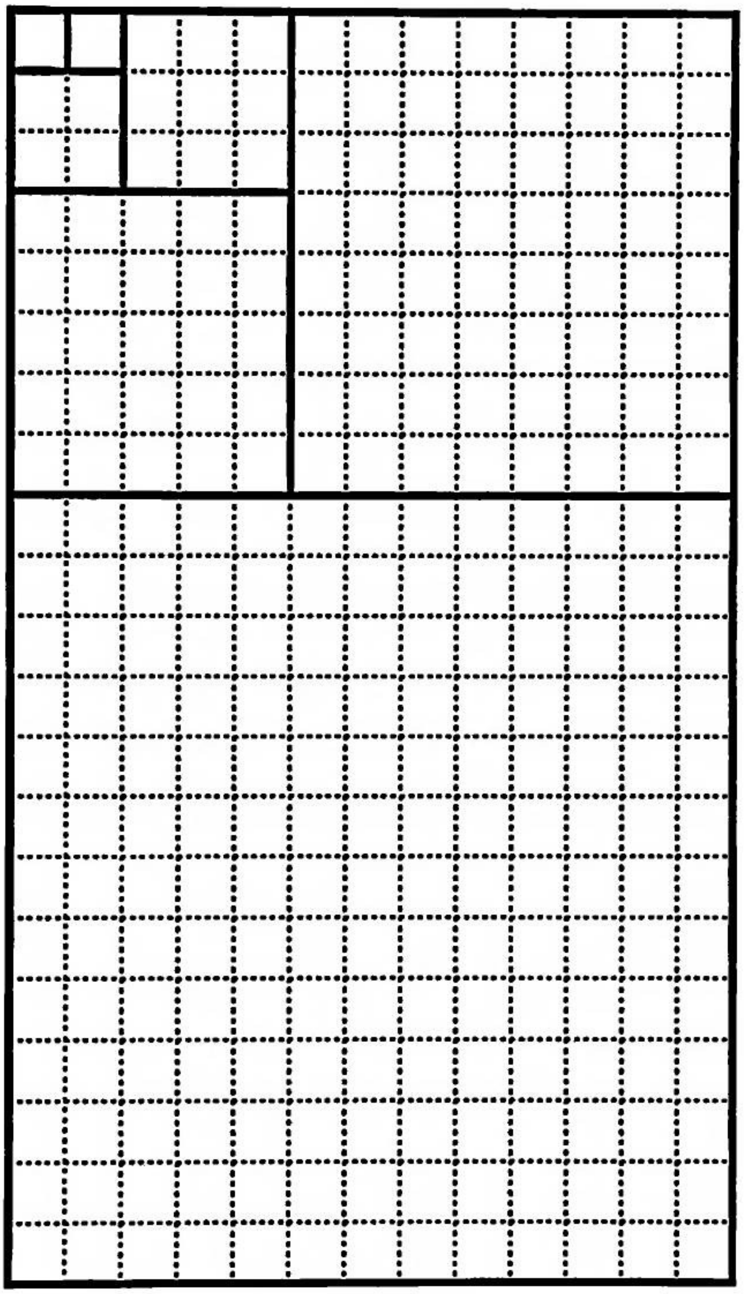
II.



$$n^2-(n-1)^2+\cdots+(-1)^{n-1}(1)^2 = \sum_{k=0}^{n}(-1)^k(n-k)^2 = \frac{n(n+1)}{2}$$

—Steven L. Snover

Sums of Squares of Fibonacci Numbers



$$F_1 = F_2 = 1;\ F_{n+2} = F_{n+1} + F_n \Rightarrow F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$$

—Alfred Brousseau

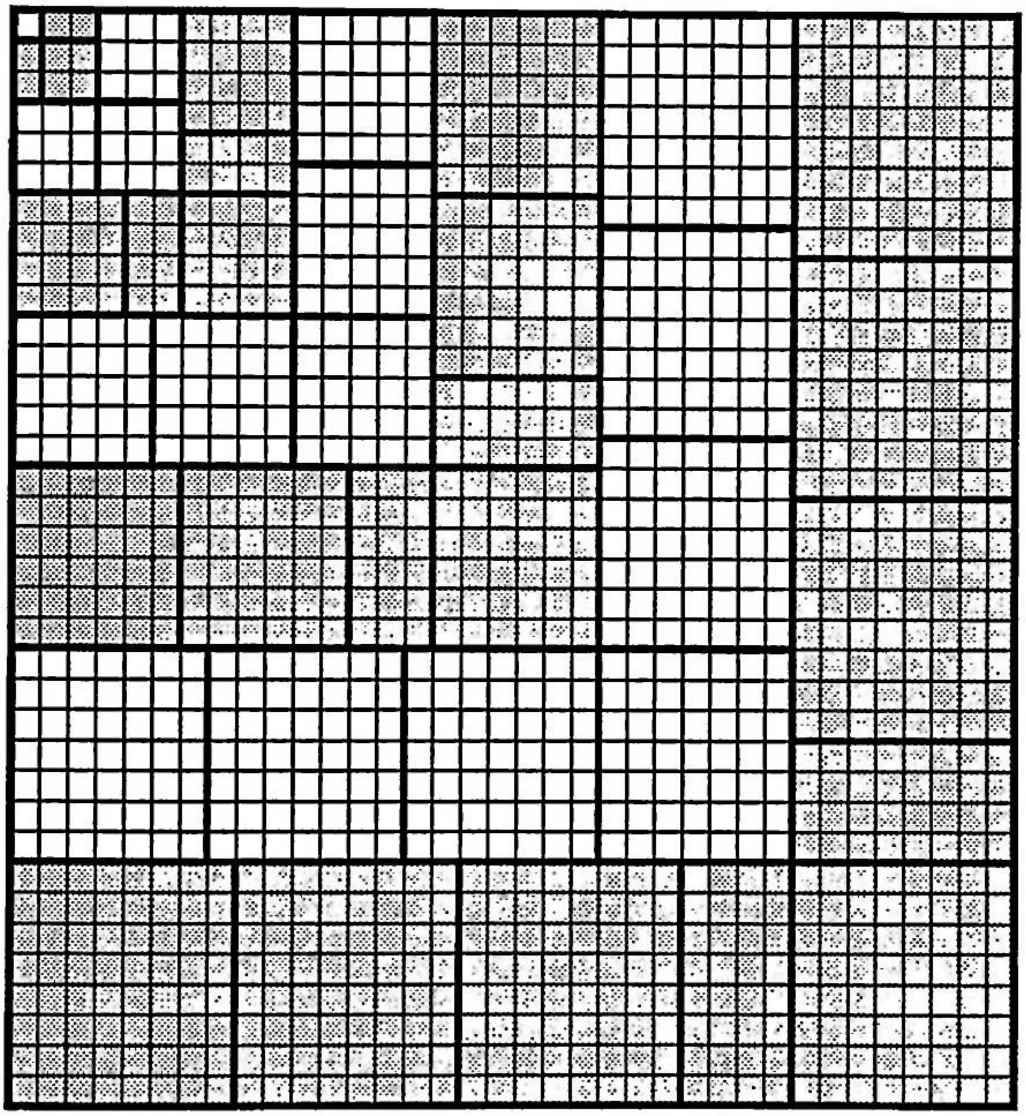
Sums of Cubes I

$$1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$$

—Solomon W. Golomb

Sums of Cubes II

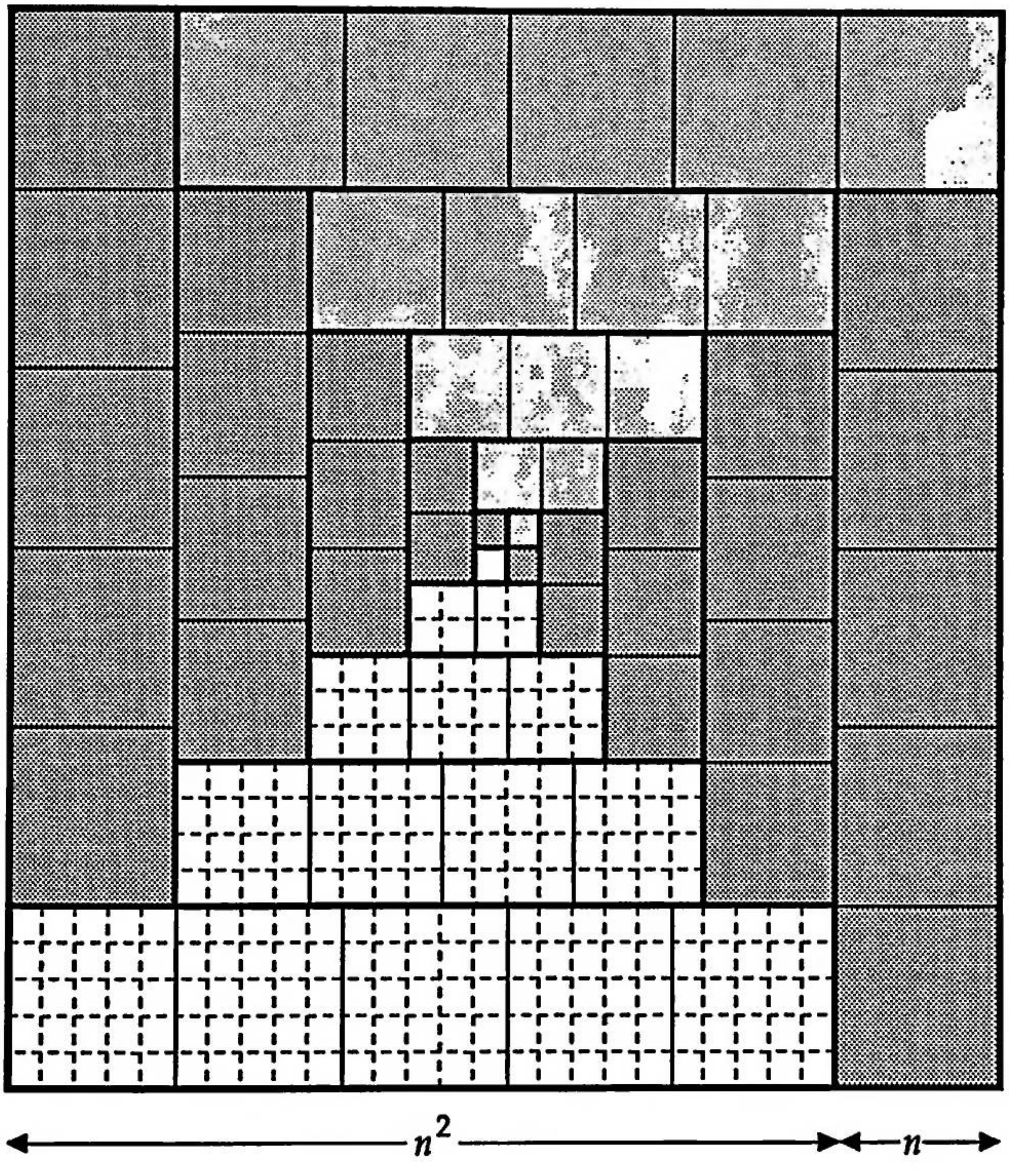
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—J. Barry Love

Sums of Cubes IV

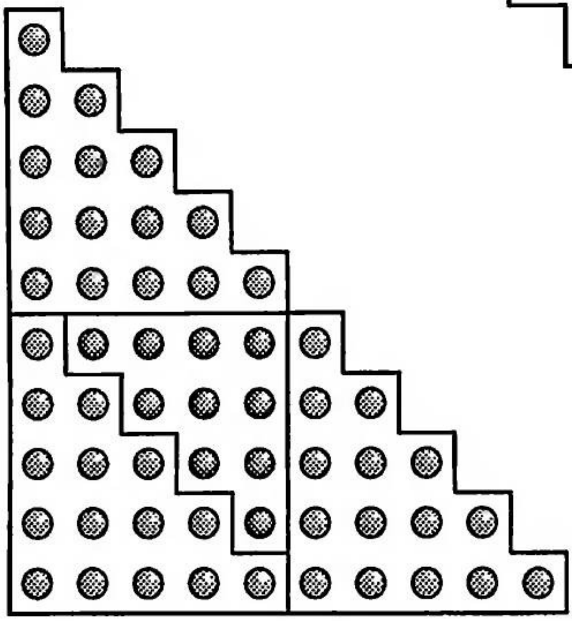
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}[n(n+1)]^2$$



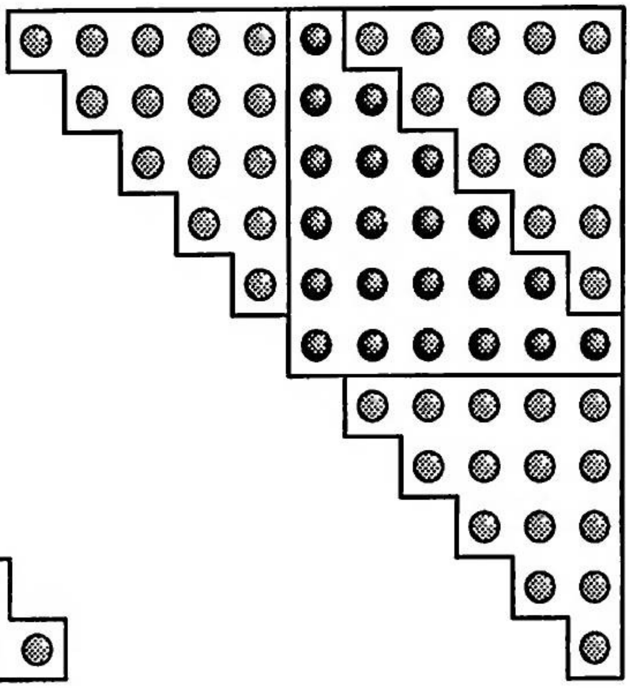
—Antonella Cupillari and Warren Lushbaugh
(independently)

Identities for Triangular Numbers

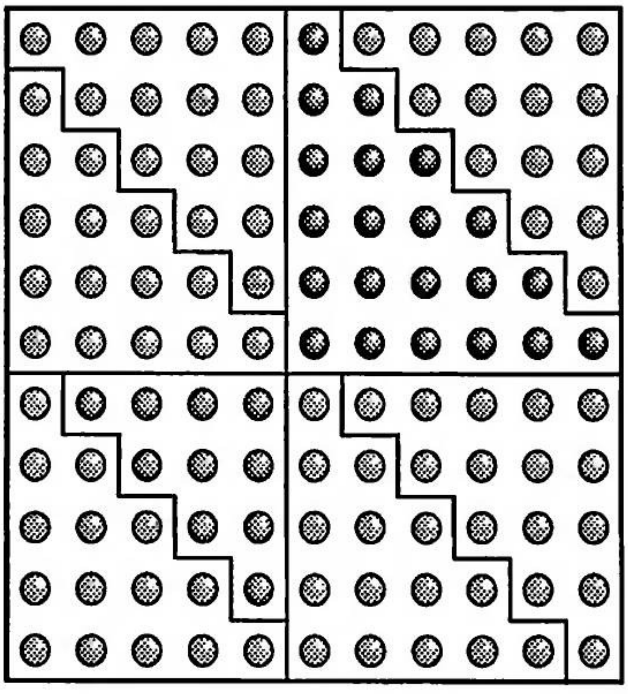
$$T_n = 1 + 2 + \cdots + n \quad \Rightarrow$$



$$3T_n + T_{n-1} = T_{2n}$$

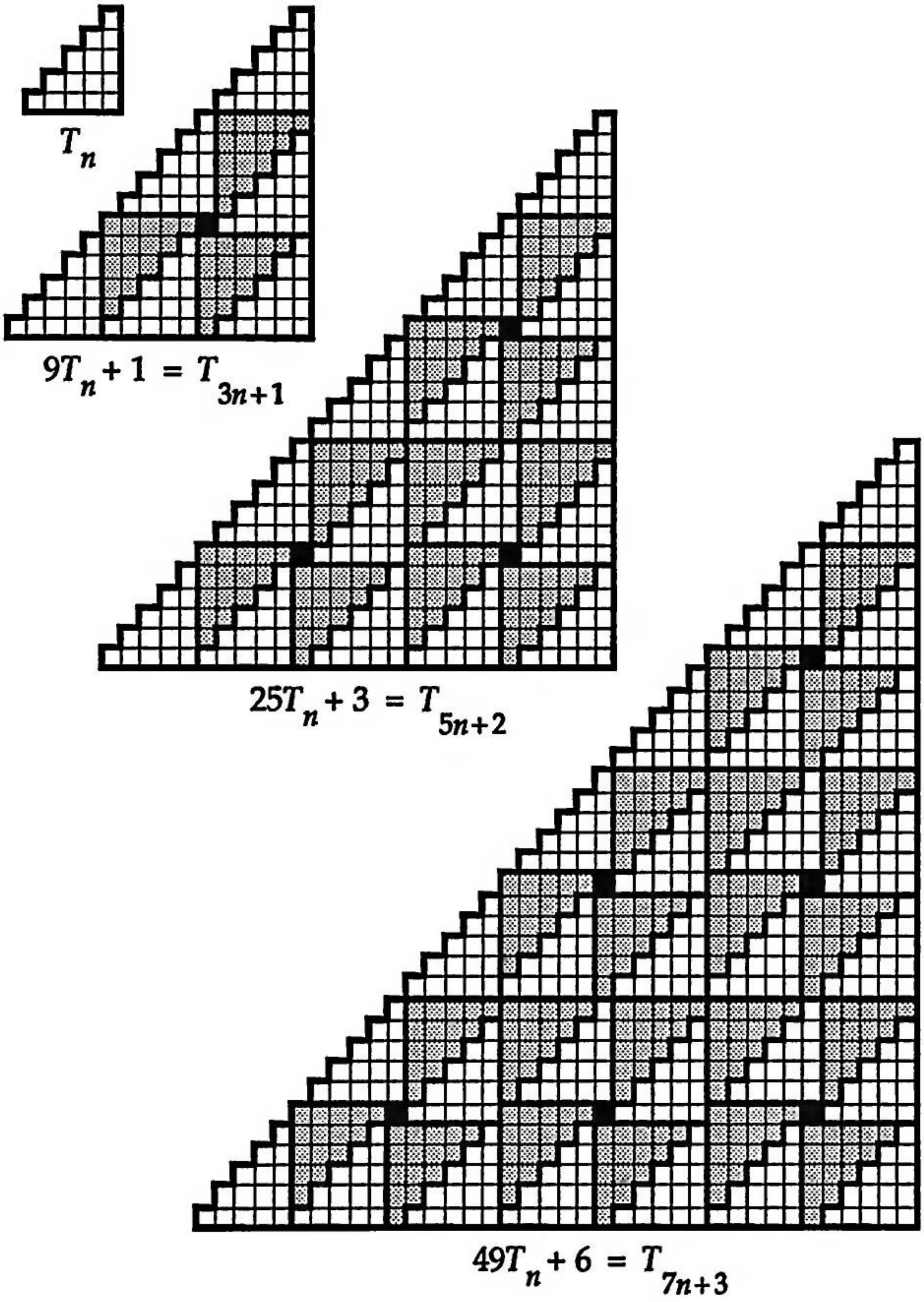


$$3T_n + T_{n+1} = T_{2n+1}$$



$$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$$

A Triangular Identity

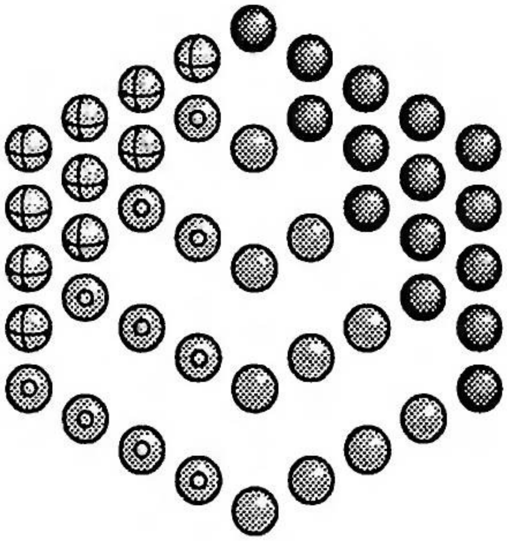


$$T_n = 1 + 2 + \cdots + n \Rightarrow (2k+1)^2 T_n + T_k = T_{(2k+1)n+k}$$

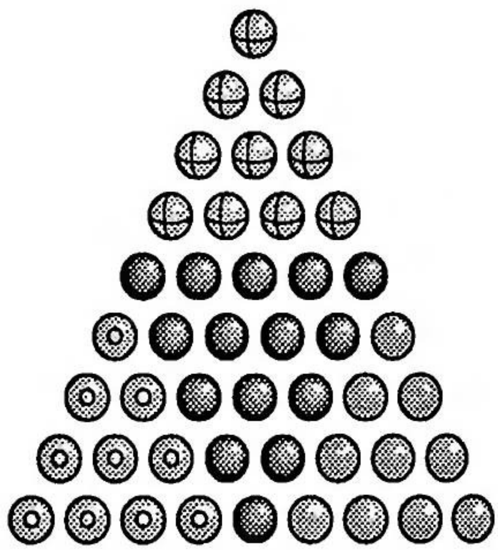
—RBN

Every Hexagonal Number is a Triangular Number

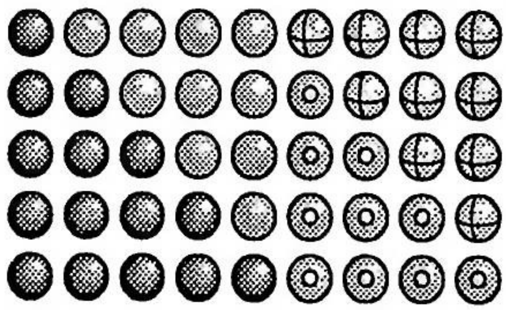
$$\left.\begin{array}{l} H_n = 1+5+\cdots+(4n-3) \\ T_n = 1+2+\cdots+n \end{array}\right\} \Rightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n-1)$$



H_5

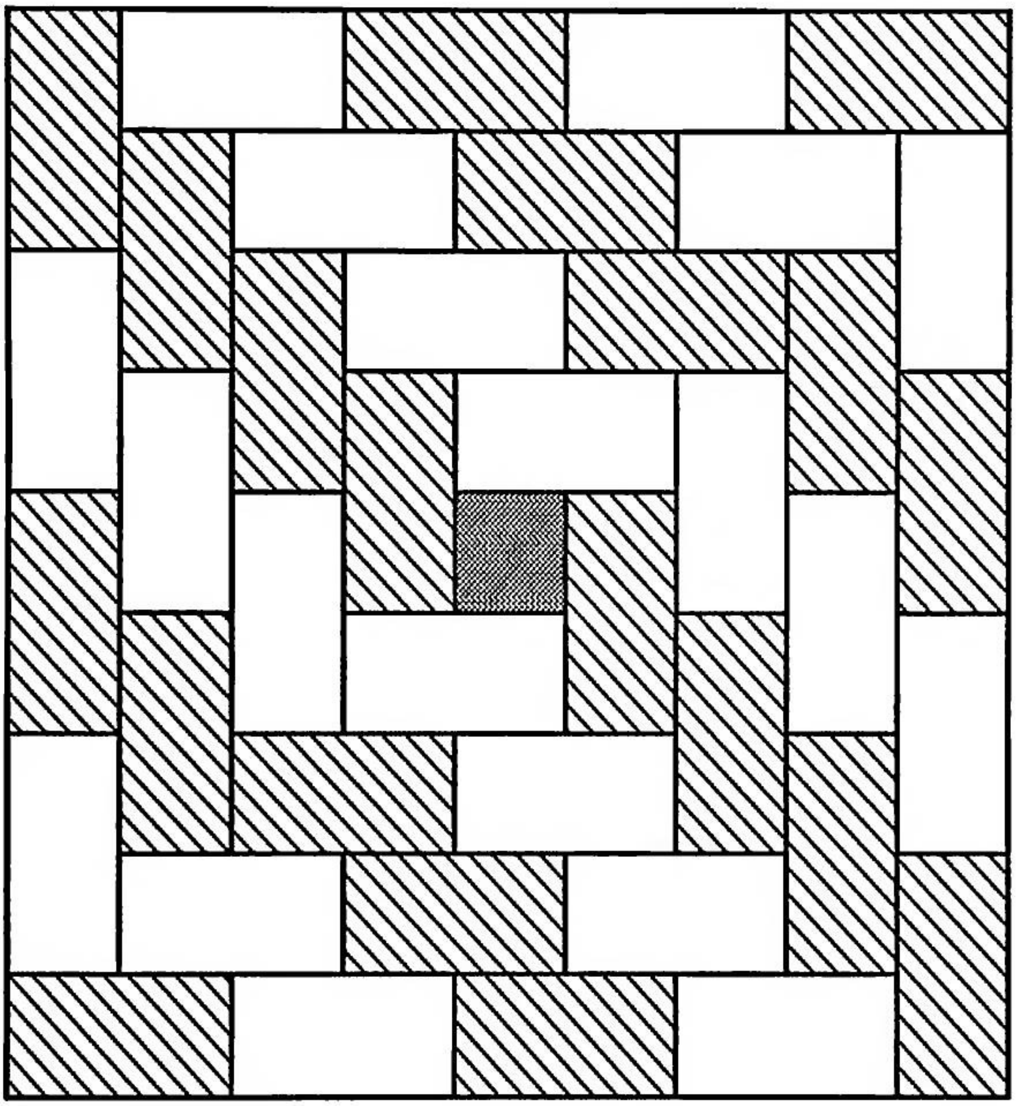


T_9



5·9

One Domino = Two Squares: Concentric Squares

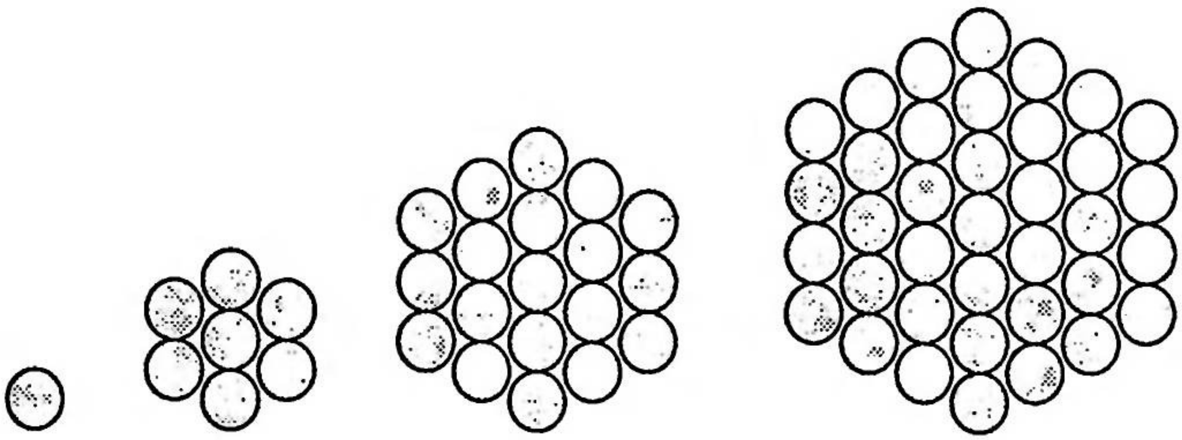


$$1 + 4 \cdot 2 + 8 \cdot 2 + 12 \cdot 2 + 16 \cdot 2 = 9^2$$

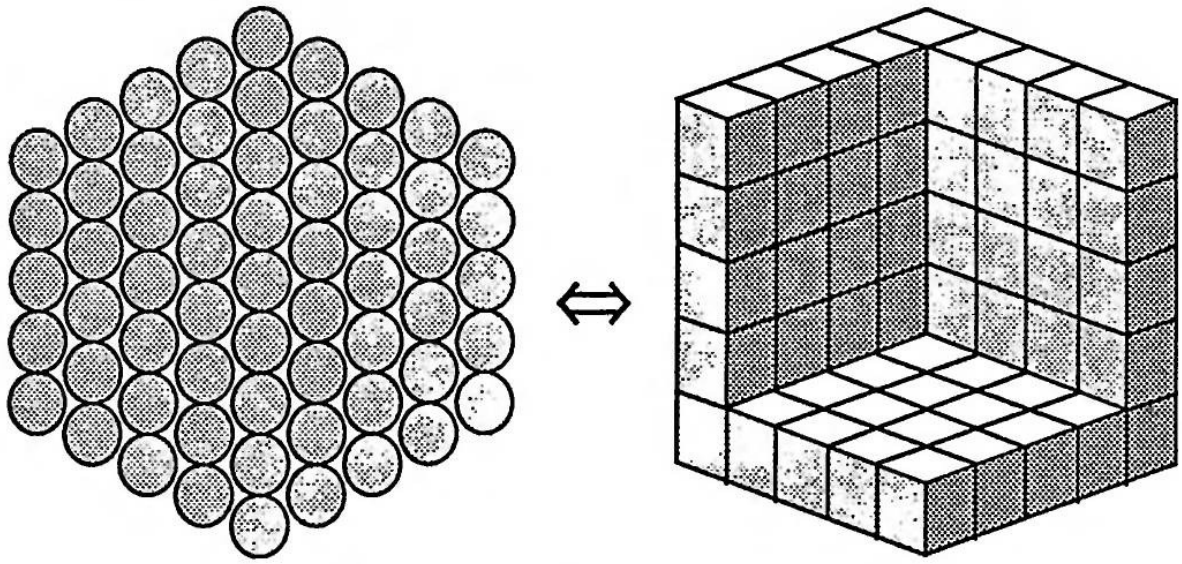
$$1 + 2\sum_{k=1}^{n} 4k = (2n+1)^2$$

—Shirley A. Wakin

Sums of Hex Numbers Are Cubes



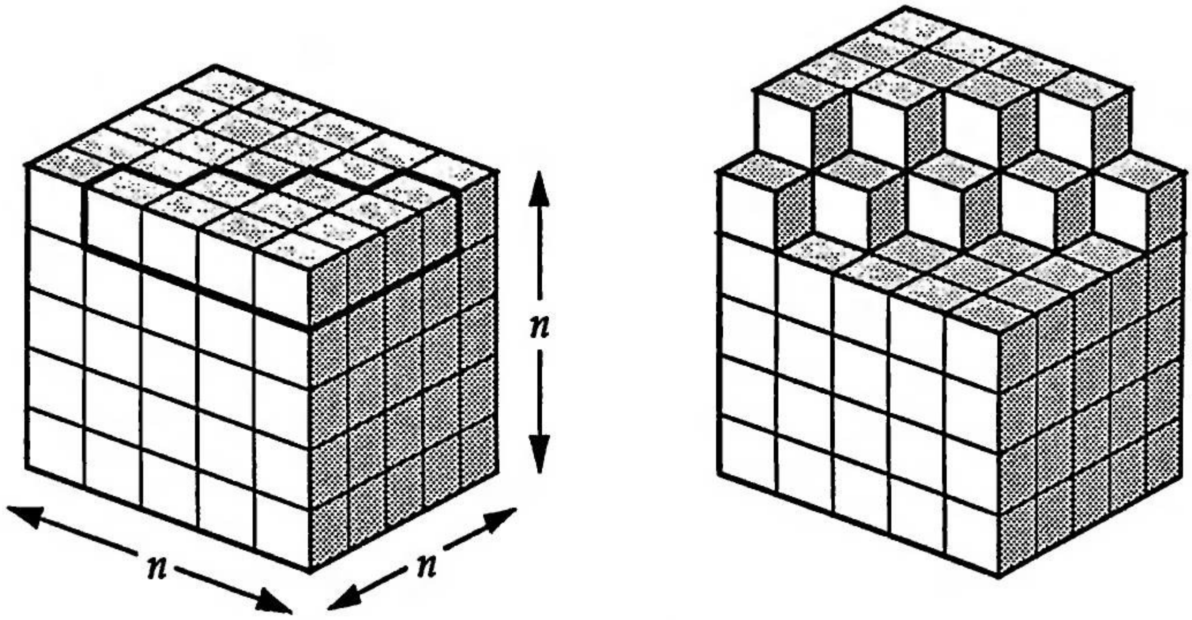
$h_1 = 1$ $h_2 = 7$ $h_3 = 19$ $h_4 = 37$

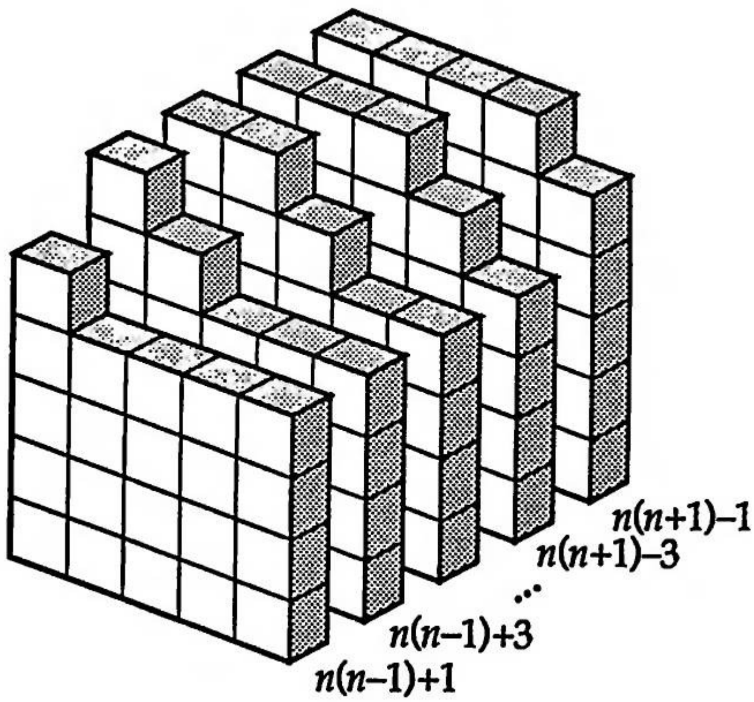


$$h_n = n^3 - (n-1)^3$$

$$\therefore h_1 + h_2 + \cdots + h_n = n^3.$$

Every Cube is the Sum of Consecutive Odd Numbers





$$1^3 = 1$$

$$2^3 = 3 + 5$$

$$3^3 = 7 + 9 + 11$$

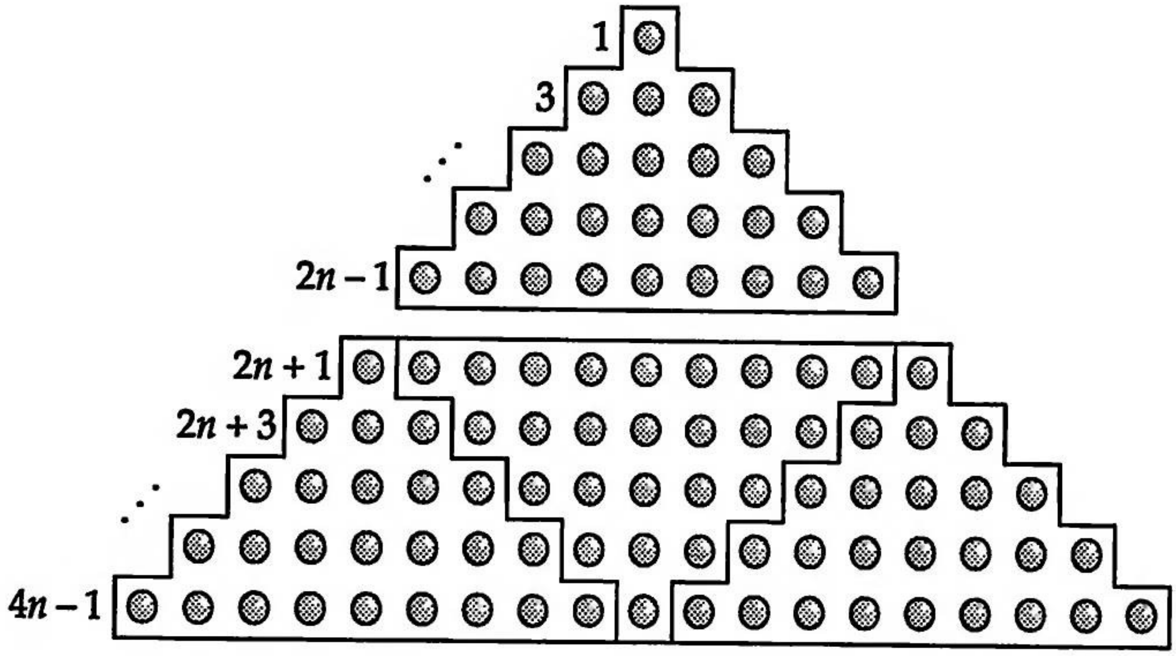
$$\vdots$$

$$n^3 = [n(n-1)+1] + \cdots + [n(n+1)-1]$$

—RBN

On a Property of the Sequence of Odd Integers (Galileo, 1615)

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \cdots .$$



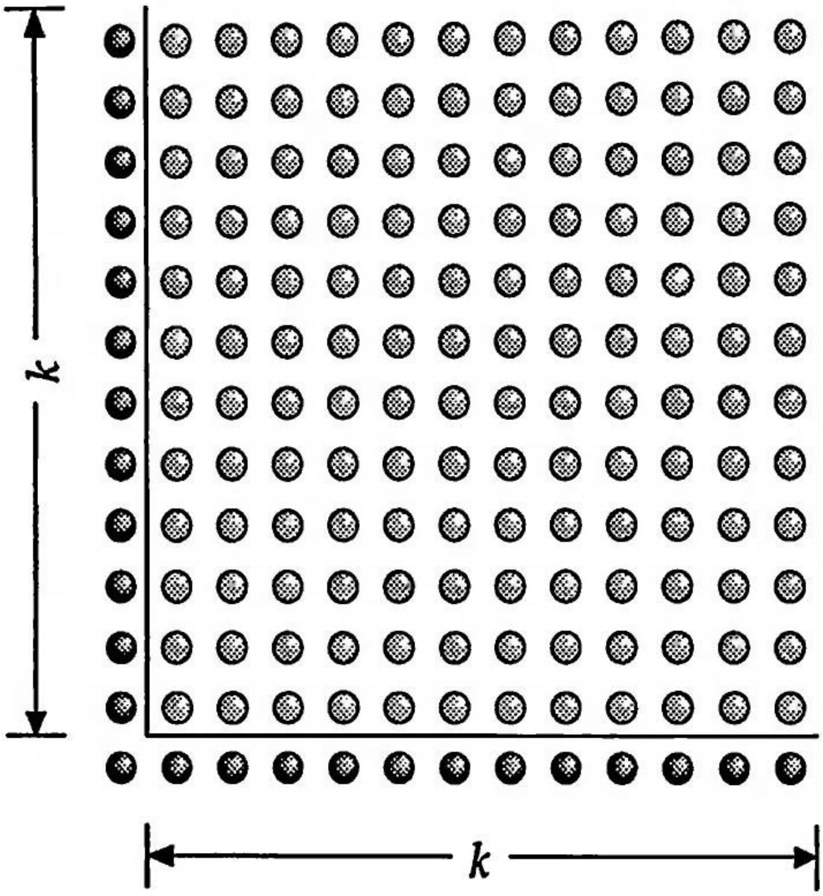
$$\frac{1+3+\ldots+(2n-1)}{(2n+1)+(2n+3)+\ldots+(4n-1)} = \frac{1}{3}$$

REFERENCE

S. Drake, *Galileo Studies,* The University of Michigan Press, Ann Arbor, 1970, pp. 218-219.

—RBN

The Existence of Infinitely Many Primitive Pythagorean Triples



$$n^2 = 2k+1 \;\Rightarrow\; k^2+n^2 = (k+1)^2 \;\;\&\;\; (k, k+1) = 1$$

—Charles Vanden Eynden