Equilateral triangles and the Fano plane

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Abstract. We formulate a definition of equilateral triangles in the complex line that makes sense over the field with seven elements. Adjacency of these abstract triangles gives rise to the Heawood graph, which is a way to encode the Fano plane. Through some reformulation, this gives a geometric construction of the Steiner systems $S(2,3,7)$ and $S(3,4,8)$. As a consequence, we embed the Heawood graph in a torus, and we derive the exceptional isomorphism $\mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{GL}_3(\mathbb{F}_2)$.

1. EQUILATERAL TRIANGLES. Three points b, c, and d in $\mathbb{C}$ form an equilateral triangle if and only if the ratio $(d-b)/(c-b)$ is $-j$ or $-j^2$, where j is a primitive cubic root of unity. With a (projective) view to extend the notion to other fields, this can be written as $[\infty, b, c, d] \in \{-j, -j^2\}$, where the bracket denotes the *cross-ratio* of four distinct elements $a, b, c, d \in \mathbb{P}^1(\mathbb{C})$:

$$[a,b,c,d] = \frac{c-a}{d-a} \times \frac{d-b}{c-b},$$

with the usual conventions about infinity: if $\alpha \in \mathbb{C}^*$, then $\alpha/0 = \infty$, and if $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are such that $\alpha\delta - \beta\gamma \neq 0$, then $(\alpha\infty + \beta)/(\gamma\infty + \delta) = \alpha/\gamma$.

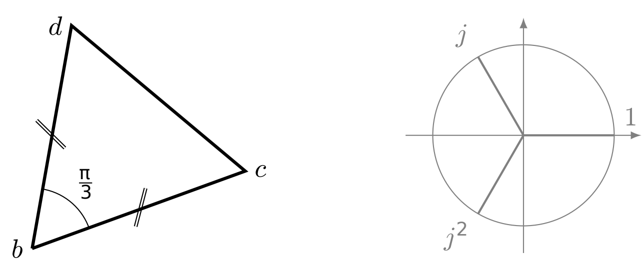

Figure 1. Equilateral triangle in the complex line

Definition. Let $\mathbb{K}$ be a field containing a primitive cubic root of unity j. As in $\mathbb{C}$, there are two of them, the roots of $X^2 + X + 1$. An unordered triple $\{b, c, d\}$ (in short: bcd) of distinct points in $\mathbb{K}$ is said to be an *equilateral triangle* if $[\infty, b, c, d] \in \{-j, -j^2\}$.

More generally, an *equianharmonic quadrangle* is a quadruple of distinct points $\{a, b, c, d\}$ in $\mathbb{P}^1(\mathbb{K})$ such as $[a, b, c, d] \in \{-j, -j^2\}$.

Remark. Harmonic quadrangles are a projective substitute for the notion of middle in affine geometry in the following sense: a quadruple $\{a, b, c, d\}$ is harmonic, *i.e.* $[a, b, c, d] \in \{-1, 2, 1/2\}$, if and only if, when one point is mapped to infinity by a homography, one of the other three is mapped to the middle of the last two ones. Similarly, a quadrangle is equianharmonic if an only if, when one point is mapped to infinity, the other three are mapped to an equilateral triangle.

These two sets of values are remarkable in the fact that, by permuting the four variables in the cross-ratio $\lambda = [a, b, c, d]$, one obtains six different values, namely λ, $(\lambda - 1)/\lambda$, $1/(1 - \lambda)$, $1/\lambda$, $1 - \lambda$, and $\lambda/(\lambda - 1)$, unless the quadrangle is equianharmonic (resp. harmonic), in which case there are only two (resp. three) different values. In particular, in the definition, the order of points and the choice of a cubic root j versus the other one j^2 are irrelevant.

Let us recall a few facts about the action of $\mathrm{PGL}_2(\mathbb{K})$ on $\mathbb{P}^1(\mathbb{K})$ and cross-ratio.

Lemma 1. *The group* $\mathrm{PGL}_2(\mathbb{K})$ *acts simply transitively on ordered triples of distinct points points of* $\mathbb{P}^1(\mathbb{K})$. *Cross-ratio is invariant under* $\mathrm{PGL}_2(\mathbb{K})$.

If -1 *is not a square in* $\mathbb{K}^*$, *the group* $\mathrm{PSL}_2(\mathbb{K})$ *acts transitively on unordered triples. It does not act transitively on ordered triples.*

Proof. Let (a, b, c) be a triple of distinct points. The homography h defined by

$$\forall z \in \mathbb{P}^1(\mathbb{K}), \quad h(z) = \frac{c-a}{c-b} \times \frac{z-b}{z-a}$$

is the unique element in $\mathrm{PGL}_2(\mathbb{K})$ that maps (a, b, c) to $(\infty, 0, 1)$ (if $\infty \in \{a, b, c\}$, simply erase the corresponding factors). Invariance of cross-ratio is easy to check.

Now, assume -1 is not a square. Then, the homography s defined by $s(z) = 1 - z$ does not belong to $\mathrm{PSL}_2(\mathbb{K})$, so that either h or sh does. Since s maps $(\infty, 0, 1)$ to $(\infty, 1, 0)$, the unicity claimed above shows that $\mathrm{PSL}_2(\mathbb{K})$ is not transitive on ordered triples. On the other hand, both h and sh map $\{a, b, c\}$ to $\{\infty, 0, 1\}$ and one of them lies in $\mathrm{PSL}_2(\mathbb{K})$. Thus, $\mathrm{PSL}_2(\mathbb{K})$ is transitive on unordered triples. ∎

2. EQUIANHARMONIC QUADRANGLES OVER $\mathbb{F}_7$. Let $\mathbb{K} = \mathbb{F}_7$ be the field with 7 elements. Since $7 - 1$ is a multiple of 3, there are two primitive cubic roots of unity in $\mathbb{K}$, namely $j = 2$ and $j^2 = 4$. For example, note that the quadrangles $Q_3 = \{\infty, 0, 1, 3\}$ and $Q_5 = \{\infty, 0, 1, 5\}$ are equianharmonic.

Lemma 2. *There are* 28 *equianharmonic quadrangles in* $\mathbb{P}^1(\mathbb{F}_7)$.

Proof. Observe that the cross-ratio is a homography with respect to every variable. Hence, given three distinct points a, b, c in $\mathbb{P}^1(\mathbb{F}_7)$ and λ in $\mathbb{P}^1(\mathbb{F}_7)$, there is a unique d such that $[a, b, c, d] = \lambda$. If $\lambda \notin \{\infty, 0, 1\}$, the point d is automatically distinct from a, b, and c. By multiplying the number of triples by the number of admissible values for the cross-ratio, one obtains $8 \times 7 \times 6 \times 2$ ordered quadruples and $8 \times 7 \times 6 \times 2/4! = 28$ equianharmonic quadrangles. ∎

Lemma 3. *Equianharmonic quadrangles form a single orbit under* $\mathrm{PGL}_2(\mathbb{F}_7)$, *and two orbits under* $\mathrm{PSL}_2(\mathbb{F}_7)$.

Proof. Let $Q = \{a, b, c, d\}$ be an equianharmonic quadrangle. Let h be the homography that maps (a, b, c) to $(\infty, 0, 1)$. Then, $h(d) = [\infty, 0, 1, h(d)] = [a, b, c, d]$. Since Q is equianharmonic, h maps Q to $\{\infty, 0, 1, 3\}$ or $\{\infty, 0, 1, 5\}$. Since $s : z \mapsto 1 - z$ exchanges these two quadrangles, h or sh maps Q to $\{\infty, 0, 1, 3\}$, which proves the first claim.

Recall that the cardinality of the orbit of $Q_3 = \{\infty, 0, 1, 3\}$ under a group G acting on the set of quadrangles is $|G|/|G_{Q_3}|$, where G_{Q_3} is the stabilizer of Q_3. Since $\mathrm{PGL}_2(\mathbb{F}_7)$ acts transitively on the 28 quadrangles, the stabilizer $\mathfrak{A}$ of Q_3 in $\mathrm{PGL}_2(\mathbb{F}_7)$ has cardinality $336/28 = 12$. The group $\mathfrak{A}$ acts faithfully on Q_3: by lemma 1, if a homography fixes three points, it is the identity. Since its order is 12, it is isomorphic to the alternating group $\mathfrak{A}_4$ (the unique subgroup of index 2 in the symmetric group). In fact, using the proof of lemma 1, let us look for homographies that act on Q_3 like double transpositions: one finds that $z \mapsto 3/z$ acts on Q_3 as $(\infty 0)(13)$; that $z \mapsto (z-3)/(z-1)$ acts as $(\infty 1)(03)$; and $z \mapsto (3z-3)/(z-3)$ acts as $(\infty 3)(01)$. These are involutions and they commute on Q_3, so that they commute on $\mathbb{P}^1(\mathbb{F}_7)$.

The point is that these involutions belong to $\mathrm{PSL}_2(\mathbb{F}_7)$. Hence, $\mathfrak{A} \cap \mathrm{PSL}_2(\mathbb{F}_7)$ contains a subgroup $\mathfrak{K} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ of order 4; besides, it also contains the order-3 element $z \mapsto 1/(1-z)$, that permutes $\{\infty, 0, 1\}$ cyclically and fixes 3. Hence, the group $\mathfrak{A}$ is included in $\mathrm{PSL}_2(\mathbb{F}_7)$, and $\mathfrak{A}$ is the stabilizer of Q_3 in $\mathrm{PSL}_2(\mathbb{F}_7)$. Therefore, the $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of Q_3 has cardinality $168/12 = 14$. Since $Q_5 = \{\infty, 0, 1, 5\}$ is in the same $\mathrm{PGL}_2(\mathbb{F}_7)$-orbit as Q_3, its stabilizer is conjugated to $\mathfrak{A}$, and the orbit of Q_5 has cardinality 14 too. ∎

Lemma 4. *The complement of an equianharmonic quadrangle in* $\mathbb{P}^1(\mathbb{F}_7)$ *is equianharmonic. Moreover, both are in the same orbit under* $\mathrm{PSL}_2(\mathbb{F}_7)$.

Proof. We start with an example: $[\infty, 0, 1, 3] = 3 = [4, 2, 5, 6]$. The homography defined by $h(z) = 5(z-2)/(z-4) = (6z+2)/(4z+5)$ maps $(4, 2, 5, 6)$ to $(\infty, 0, 1, 3)$, and h belongs to $\mathrm{PSL}_2(\mathbb{F}_7)$. This proves the claim for $\{\infty, 0, 1, 3\}$.

Now, let $Q = \{a, b, c, d\}$ be an equianharmonic quadrangle. By lemma 3, there is a homography $g \in \mathrm{PGL}_2(\mathbb{F}_7)$ such that $g(Q) = \{\infty, 0, 1, 3\}$. Then, the bijection g maps the complement of Q to $\{2, 4, 5, 6\}$, which is equianharmonic. Moreover, $g^{-1}hg$ maps the complement of Q to Q. This proves the claim for a general Q. ∎

3. EQUILATERAL TRIANGLES OVER $\mathbb{F}_7$.

Corollary 5. *There are* 14 *equilateral triangles over* $\mathbb{F}_7$. *They are the following ones:*

$$\begin{array}{ccccccccccccc}
013 & \xrightarrow{z+1} & 124 & \xrightarrow{z+1} & 235 & \xrightarrow{z+1} & 346 & \xrightarrow{z+1} & 450 & \xrightarrow{z+1} & 561 & \xrightarrow{z+1} & 602 \\
{\scriptstyle \frac{1}{z}}\downarrow & & & & & & & & & & & & \\
015 & \xrightarrow[z+1]{} & 126 & \xrightarrow[z+1]{} & 230 & \xrightarrow[z+1]{} & 341 & \xrightarrow[z+1]{} & 452 & \xrightarrow[z+1]{} & 563 & \xrightarrow[z+1]{} & 605.
\end{array}$$

Proof. By lemma 4, one can arrange the 28 equianharmonic quadrangles in 14 complementary pairs. In a given pair, exactly one quadrangle contains ∞. Adding or withdrawing ∞ gives a one-to-one correspondence between equilateral triangles and equianharmonic quadrangles containing ∞, hence between equilateral triangles and pairs of complementary equianharmonic quadrangles. (*E.g.*: 013 corresponds to the pair $\{\{\infty, 0, 1, 3\}, \{2, 4, 5, 6\}\}$.) Hence, the first assertion holds.

To write a list, one starts with 013 and 015. Using invariance of ∞ under affine transformations, one builds 7 new triangles out of the first two with the translation $z \mapsto z + 1$. These are the rows of the list in the corollary. ∎

Remark. The vertical arrow in the statement of corollary 5 has the following meaning. The triangle 013 corresponds to the pair $\{\{\infty, 0, 1, 3\}, \{2, 4, 5, 6\}\}$. The homography $z \mapsto 1/z$ maps this pair to $\{\{\infty, 0, 1, 5\}, \{2, 3, 4, 6\}\}$, which corresponds to 015. This extends to an action of $\mathrm{PGL}_2(\mathbb{F}_7)$ on triangles.

Lemma 6. *There is a canonical action of* $\mathrm{PGL}_2(\mathbb{F}_7)$ *on equilateral triangles. All triangles are in the same* $\mathrm{PGL}_2(\mathbb{F}_7)$*-orbit, but there are two* $\mathrm{PSL}_2(\mathbb{F}_7)$*-orbits described by the lines of corollary 5.*

Proof. By invariance of cross-ratio, a homography in $\mathrm{PGL}_2(\mathbb{F}_7)$ maps an equiharmonic quadrangle to another one. Since this action on parts of $\mathbb{P}^1(\mathbb{F}_7)$ commutes with taking the complement, it maps a pair of complementary equianharmonic quadrangles. Hence, the group $\mathrm{PGL}_2(\mathbb{F}_7)$ acts on the set of pairs of complementary equianharmonic quadrangles. But there is a one-to-one correspondence between such pairs and equilateral triangles, one inherits an action of $\mathrm{PGL}_2(\mathbb{F}_7)$ on triangles: if abc is a triangle and $g \in \mathrm{PGL}_2(\mathbb{F}_7)$, one defines $g \cdot abc$ as the triangle corresponding to $g \cdot \big\{\{\infty, a, b, \}, \mathbb{P}^1 \setminus \{\infty, a, b, c\}\big\}$.

Two triangles abc and $a'b'c'$ are in the same $\mathrm{PGL}_2(\mathbb{F}_7)$-orbit by lemma 1: the homography that maps (a, b, c) to (a', b', c') also maps $\big\{\{\infty, a, b, c\}, \mathbb{P}^1 \setminus \{a, b, c\}\big\}$ to $\big\{\{\infty, a', b', c'\}, \mathbb{P}^1 \setminus \{a', b', c'\}\big\}$.

As for $\mathrm{PSL}_2(\mathbb{F}_7)$-orbits, lemma 4 implies that the pairs of the form $\{Q, \mathbb{P}^1 \setminus Q\}$, where Q runs over a $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of quadrangles, are orbits of pairs. Hence, by lemma 3, there are two orbits of pairs, corresponding to two orbits of triangles. Since triangles in the same line in corollary 5 are in the same orbit (simply apply $z \mapsto z + 1$), the two lines are exactly the two orbits. ∎

E.g.: Let $h(z) = 1/z$. Then $h\big(\{\infty, 0, 1, 3\}\big) = \{0, \infty, 1, 5\}$, so h maps the triangle 013 to 015; moreover, $h\big(\{\infty, 1, 2, 4\}\big) = \{0, 1, 2, 4\} = \mathbb{P}^1(\mathbb{F}_7) \setminus \{\infty, 3, 5, 6\}$, so that h maps the triangle 124 to 356.

Equilateral triangles over $\mathbb{F}_7$ as equilateral triangles on a torus. Say that two (equilateral) triangles (over $\mathbb{F}_7$) are *adjacent* if they have two vertices in common. It is easy to check that a triangle in either line of corollary 5 is adjacent to exactly three triangles, and all three lie on the other line. The upshot is that one can arrange triangles over $\mathbb{F}_7$ as equilateral triangles in the real plane in a periodic manner. By glueing the sides of a fundamental parallelogram, one tiles a torus by 14 triangles (fig. 2).

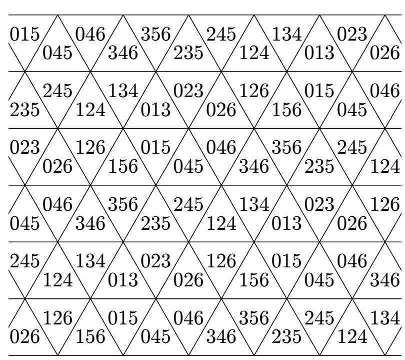

Figure 2. Triangles over $\mathbb{F}_7$ as triangles on a torus

The picture is more appealing when one replaces the adjacency graph, embedded in the torus, by its dual graph: the meeting points of triangles become 7 hexagons that tile the torus (fig. 3). Since each face touches all the others, this tiling shows that the chromatic number of the torus is at least 7. A polyhedral version of this tiling was discovered by L. Szilassi [**6**].

Equilateral triangles and the Fano plane. Forgetting the tilings, let us consider the graph with 14 vertices labeled by equilateral triangles over $\mathbb{F}_7$, where two vertices are

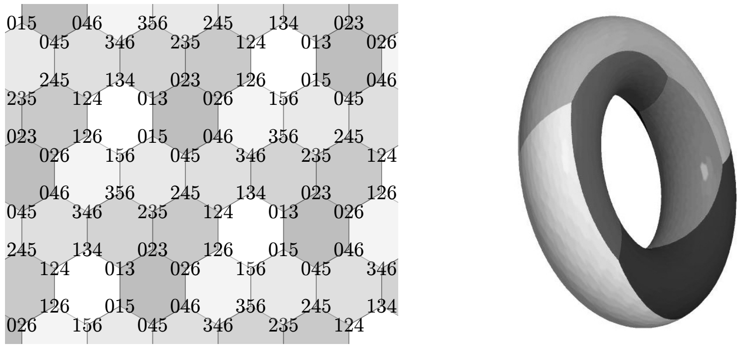

Figure 3. Seven hexagons tiling a torus

connected if the corresponding triangles are adjacent. This gives rise to a graph known as the Heawood graph (see [**4**]). It is bipartite because a triangle on a line of corollary 5 is adjacent to triangles on the other line.

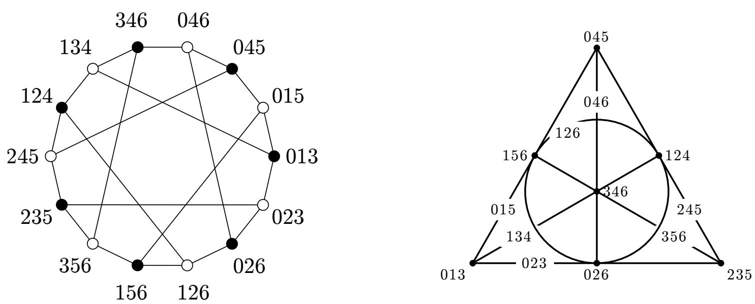

Figure 4. Incidence of triangles: Heawood graph and Fano plane

The Heawood graph is the incidence graph of the Fano plane: one can label vertices and lines of the Fano plane by triangles, so that adjacency of triangles corresponds to incidence (fig. 4).

Application: two exceptional isomorphisms.

Theorem 7. *One has:* $\mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{GL}_3(\mathbb{F}_2)$.

Proof. One can label vertices of the Fano plane by nonzero vectors in $\mathbb{F}_2^3$ so that the third vertex on the line containing v and v' is labelled by $v + v'$ (put for instance the canonical basis vectors on the vertices of the triangle and use this rule to complete the labelling). Hence, a permutation f of vertices and lines of the Fano plane preserves, once completed into a map $\mathbb{F}_2^3 \to \mathbb{F}_2^3$ by setting $f(0) = 0$, is additive. However, on a prime field, additivity is equivalent to linearity, so that automorphisms of the Fano plane are linear automorphisms of $\mathbb{F}_2^3$. This is in fact a special case of the "fundamental theorem of projective geometry" ([**1**, Theorem 2.26]), by which any incidence preserving map is a projective map, *i.e.* an element in $\mathrm{PGL}_3(\mathbb{F}_2)$. ∎

Remark. The action of $\mathrm{PGL}_2(\mathbb{F}_7)$ on $\mathrm{PSL}_2(\mathbb{F}_7)$ by conjugation embeds the former group in $\mathrm{Aut}(\mathrm{PSL}_2(\mathbb{F}_7))$. On the other hand, the automorphism group $\mathrm{Aut}(\mathrm{GL}_3(\mathbb{F}_2))$ is the semidirect product of $\mathrm{GL}_3(\mathbb{F}_2)$ and $\mathbb{Z}/2\mathbb{Z}$ acting by $g \mapsto (g^T)^{-1}$. Hence: $\mathrm{PGL}_2(\mathbb{F}_7) \simeq \mathrm{Aut}\,\mathrm{GL}_3(\mathbb{F}_2)$.

4. STEINER SYSTEMS. Recall a *Steiner system* with parameters (t, k, n), written $S(t, k, n)$, is a set of cardinality n and a collection of k-sets called *blocks* such that every t-set is contained in a unique block.

Proposition 8. *An orbit under* $\mathrm{PSL}_2(\mathbb{F}_7)$ *of equianharmonic quadrangles forms an* $S(3, 4, 8)$. *An orbit under* $\mathrm{PSL}_2(\mathbb{F}_7)$ *of triangles forms an* $S(2, 3, 7)$ *(here, the action is defined in lemma 6).*

To be more explicit, recall that a $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of triangles is but a line in corollary 5. Starting from such an orbit, one rebuilds quadrangles by adding ∞ to every triangle and by adding to the collection the complements of these quadrangles. For instance, starting from the orbit containing 013, one builds the following $S(3, 4, 8)$:

$$\begin{array}{ccccccc} \infty013 & \infty124 & \infty235 & \infty346 & \infty045 & \infty156 & \infty026 \\ 2456 & 0356 & 0146 & 0125 & 1236 & 0234 & 1345. \end{array}$$

Proof. The underlying set of the $S(3, 4, 8)$ is $\mathbb{P}^1(\mathbb{F}_7)$; blocks are quadrangles in a given $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit. To fix notations, let us consider that of $\{\infty, 0, 1, 3\}$. Let $\{a, b, c\}$ be a 3-set. By lemma 1, there exists $h \in \mathrm{PSL}_2(\mathbb{F}_7)$ that maps $\{a, b, c\}$ to $\{\infty, 0, 1\}$. Then $\{a, b, c, h^{-1}(3)\}$ is an equianharmonic quadrangle in the same $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit as $\{\infty, 0, 1, 3\}$. If $\{a, b, c\}$ is included in some block $\{a, b, c, d\}$, then $h(d) = [\infty, 0, 1, h(d)] = [a, b, c, d] \in \{3, 5\}$. Since $\{\infty, 0, 1, 5\}$ is not a block (they are not in the same line in corollary 5, see lemma 6), one has $h(d) = 3$, which proves the unicity of a block containing $\{a, b, c\}$.

Now, the underlying set of the $S(2, 3, 7)$ is $\mathbb{F}_7$; blocks are triangles in a fixed $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit, say, that of 013. A block is in particular a triple $\{a, b, c\}$ such that $\{\infty, a, b, c\}$ is equianharmonic. Hence, given a pair $\{a, b\}$ in $\mathbb{F}_7$, the unique block containing $\{a, b\}$ is obtained by erasing ∞ from the unique equianharmonic triangle in the $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of $\{\infty, 0, 1, 3\}$ that contains $\{\infty, a, b\}$. ■

Remark. The passage from quadrangles to triangles follows the classical construction of an $S(t - 1, k - 1, n - 1)$ out of an $S(t, k, n)$ by selecting blocks containing a fixed point (here, ∞), then erasing it.

5. ALTERNATIVE PRESENTATIONS. We give two other ways to recover the Fano plane from $\mathrm{PGL}_2(\mathbb{F}_7)$. Proofs of these elementary results appear in [**3**].

"Nilpotent conic". The group $\mathrm{GL}_2(\mathbb{F}_7)$ acts by conjugation on the space $\mathfrak{gl}_2(\mathbb{F}_7)$ of 2×2 matrices (its Lie algebra). The action factors through $\mathrm{PGL}_2(\mathbb{F}_7)$. Since trace is preserved, the subspace $\mathfrak{sl}_2(\mathbb{F}_7) = \ker(\mathrm{tr})$ is stable. The determinant is a quadratic form on $\mathfrak{sl}_2(\mathbb{F}_7)$. Let $Q = -\det$. By the Cayley-Hamilton theorem, the isotropic cone of Q is the set $\mathcal{N}$ of nilpotent matrices. Let $\mathcal{C}$ be the corresponding conic in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{F}_7))$ defined by the equation $Q = 0$.

Say a line L in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{F}_7))$ is *tangent* (resp. *secant*, resp. an *exterior*) to $\mathcal{C}$ if $L \cap \mathcal{C}$ is a unique point (resp. a pair of distinct points, resp. empty). A point p outside $\mathcal{C}$ is *exterior* (resp. *interior*) if p belongs to two (resp. no) tangents. Recall that *polarity* (with respect to $\mathcal{C}$) between a point and a line is the projective counterpart of orthogonality (with respect to Q) between a line and a plane. It turns out that the polar of an exterior (resp. interior) point is a secant (resp. an exterior line). A triangle is *self-polar* if the polar of every vertex is the opposite side. Two triangles are *adjacent* if they share a common vertex: this makes the set of self-polar triangles into a graph. The action of $\mathrm{PGL}_2(\mathbb{F}_7)$ induces an action on all these geometric objects and on the graph.

Proposition 9. *There are* 14 *self-polar triangles whose vertices are interior points. The corresponding graph is the Heawood graph.*

This result slightly improves on [**5**, § 7], in which another quadratic form is used, since here, the Heawood graph comes with a natural action of $\mathrm{PGL}_2(\mathbb{F}_7)$.

Elementary 2**-subgroups of** $\mathrm{PSL}_2(\mathbb{F}_7)$**.** In fact, the projective plane $\mathbb{P}(\mathfrak{sl}_2(\mathbb{F}_7))$ is the set of involutions in $\mathrm{PGL}_2(\mathbb{F}_7)$. For $A \in \mathrm{GL}_2(\mathbb{F}_7)$, let $[A] \in \mathrm{PGL}_2(\mathbb{F}_7)$ be the homography defined by A. Then $[A]$ is an involution in $\mathrm{PGL}_2(\mathbb{F}_7)$ if and only if A^2 is a scalar matrix. By the Cayley-Hamilton theorem, this means that $\mathrm{tr}(A) = 0$, hence the claim. Moreover, one can identify involutions in $\mathrm{PSL}_2(\mathbb{F}_7)$ and interior points of $\mathcal{C}$.

Since $\det(A) = (\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))/2$ for every 2×2 matrix A, the polar form of Q on $\mathfrak{sl}_2(\mathbb{F}_7)$ is the bilinear form $(A, B) \mapsto \mathrm{tr}(AB)$, two elements $A, B \in \mathrm{GL}_2(\mathbb{F}_7) \cap \mathfrak{sl}_2(\mathbb{F}_7)$ are orthogonal with respect to Q if and only if $[AB]$ is an involution. This means that two involutions are orthogonal as points in $\mathbb{P}\,\mathfrak{sl}_2(\mathbb{F}_7)$ if and only if they commute. In other terms, the vertices of an self-polar triangles form, together with the neutral elements, a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.
E.g.: The group $\mathfrak{K}$ given by the classes of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & -3 \\ 1 & -3 \end{pmatrix}$.

From subgroups to equilateral triangles. A subgroup of $\mathrm{PSL}_2(\mathbb{F}_7)$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ decomposes $\mathbb{P}^1(\mathbb{F}_7)$ into two orbits of cardinal 4 that turn out to be equianharmonic. Since all of them are conjugated under $\mathrm{PGL}_2(\mathbb{F}_7)$, it is enough (and easy) to check it in a single example. This procedure induces a one-to-one correspondence between self-polar triangles in $\mathbb{P}(\mathfrak{sl}_2(\mathbb{F}_7))$ and equilateral triangles in $\mathbb{F}_7$.
E.g.: The orbits of the subgroup $\mathfrak{K}$ defined above are $\{\infty, 0, 1, 3\}$ and $\{2, 4, 5, 6\}$. To check this, compute the action of all elements of $\mathfrak{K}$ on ∞ and take the complement.

Conclusion. In this article, we havec seen several geometric realizations of combinatorial structures related to the Fano plane–the Heawood graph, a tiling of a torus by seven hexagons, the Steiner systems $S(2, 3, 7)$ and $S(3, 4, 8)$. It is amusing that some kind of special triangles is involved in each construction.

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