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INTERVAL EXCHANGE TRANSFORMATION EXTENSION OF A SUBSTITUTION DYNAMICAL SYSTEM

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Let α be a free group automorphism on F_N with maximal index and Σ_α the associated attracting subshift. We prove that there exists an Interval Exchange Transformation on the circle whose coding factorizes onto Σ_α. In the case when there is an explicit construction of the $\mathbb{R}$-tree associated to α, we construct algorithmically such IET. We conclude by giving examples.

Keywords: Symbolic dynamics; substitutive dynamical systems; free group automorphisms; free group actions; real trees.

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1. Introduction

In the '80s, Gérard Rauzy [33] studied the Tribonacci substitution and constructed a self-similar exchange of pieces on the so-called Rauzy fractal (see Fig. 1) as a geometric representation of the symbolic substitutive system; see [31] for more details.

At the same time, Arnoux and Yoccoz introduced and studied in [5] the so-called Arnoux–Yoccoz interval exchange transformation (IET), a self-similar IET with a cubic dilatation factor, namely the same cubic number as the main eigenvalue of the Tribonacci substitution. As observed by Arnoux [4] those dynamical systems are intimately related. Mainly: the natural coding of the Arnoux–Yoccoz IET factorizes onto the symbolic subsitutive Tribonacci dynamical system. This relationship is somehow summarized in the striking picture of a Peano curve drawn in the Rauzy fractal (see http://iml.univ-mrs.fr/galerie/dac/pix/Arnoux_pix1/index.htm) which represents the semi-conjugacy. This relation seems very specific to this particular example; a similar construction is possible for a specific class of substitutions with the same structure (Arnoux–Rauzy) but no other example were known.
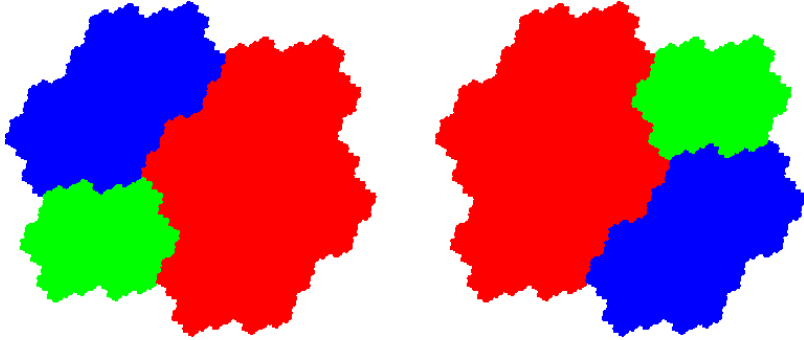
Fig. 1.   Rauzy fractal and exchange of pieces associated to the Tribonacci substitution.

We tackle here the question of whether such a construction is possible in a much more general setting and to which extent.

1.1. *A special case*

Our starting point is the construction of a new example of a comparable picture in the case of a rather different substitution:

$$\alpha : a \rightarrow ab$$
$$b \rightarrow c$$
$$c \rightarrow a.$$

On the one hand, denote Σ_α the language of the shift orbit of the unique fixed point of α. On the other hand, let ℓ be the real root of the polynomial $P(X) = X^3 - X^2 - 1$; consider the IET transformation defined by the length vector

$$\lambda = (\, \ell^3 \quad \ell \quad \ell^3 \quad \ell \quad \ell^2 \quad 1 \quad \ell^2 \quad 1 \quad \ell^2 \quad \ell \,),$$

the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 9 & 3 & 4 & 8 & 6 & 7 & 2 & 0 & 1 & 5 \end{pmatrix},$$

and denote $\Sigma_{(\lambda,\pi)}$ the language of the interval exchange transformation $F_{(\lambda,\pi)}$ on the interval $I = [0, \|\lambda\|_1]$. Finally, let $\Pi : \{0, 1, \ldots, 9\}^* \rightarrow \{a, b, c\}^*$ be the projection $\{3, 4, 5, 9\} \mapsto a$, $\{1, 2, 0\} \mapsto b$ and $\{6, 7, 8\} \mapsto c$. Our starting point is the following result:

Theorem 1.1.

$$\Pi(\Sigma_{(\lambda,\pi)}) = \Sigma_\alpha.$$

This result says that the natural symbolic dynamics of the IET factorizes on the symbolic dynamical system associated with the given substitution. Once λ and π are given, the proof of this result is rather straightforward. Indeed, the IET $F_{(\lambda,\pi)}$

is self-induced. One may induce on a well-chosen subset of I, namely $[\ell + \ell^3, 5\ell^2 + 2\ell + 4]$, the complement of $I_0 \cup I_1 \cup I_2$; then, check that the induction is described by a substitution which itself factorizes on α through Π.

But the difficulty is in finding such pair (λ, π) given a substitution. We are going to propose an algorithm to construct such IET for a wide class of substitutions. But before explaining this in more details, let us say a bit more about this specific example.

From the general theory of Rauzy fractals, since α is a Pisot substitution, one can build a geometric representation of the underlying symbolic system (Σ_α, S). There is a partition of a compact domain R_α of the plane (called Rauzy fractal, see Fig. 2) into three pieces and a piecewise translation of the pieces which is conjugate to (Σ_α, S) (this is a conjugacy in measure; topologically it is a symbolic extension of the exchange of pieces). Moreover, it is possible to define a map $\Phi : I \to R_\alpha$ mapping the interval onto the Rauzy fractal (i.e. a space-filling curve). Observe that by construction, $\Phi(I_i) \subset R_{\Pi(i)}$ and $R_a = \cup_{i \in \Pi^{-1}(a)} \Phi(I_i)$. We will not study the geometric properties of this map but the experimental picture is shown on Fig. 3 and may help to visualize the semi-conjugacy.
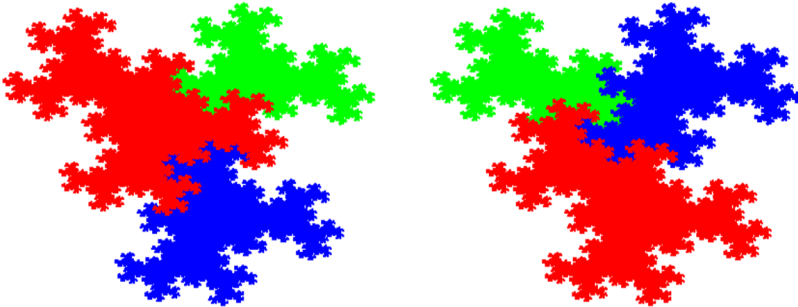

Fig. 2.   Rauzy fractal and exchange of pieces associated to α.
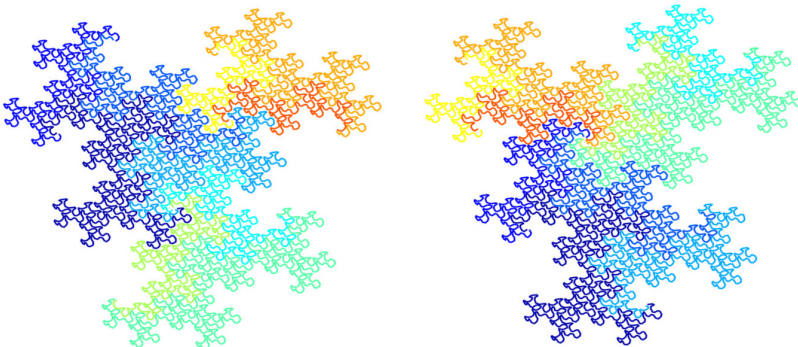

Fig. 3.   (Color online) Rough approximation of the Peano curve $\Phi(I)$. Colors are put according to the partitions of I before and after applying the IET.

1.2. *Symbolic dynamics and geometric representations*

In order to present our more general construction, we recall basic notions of symbolic dynamics.

1.2.1. *Geometric representations*

For a dynamical system (X, T) and a finite partition of X, $\mathcal{I} = \{I_i | i \in \mathcal{A}\}$, we define the *coding function* $\nu_{\mathcal{I}} : X \to \mathcal{A}$ by $\nu_{\mathcal{I}}(x) = i$ if $x \in I_i$, and we define the *itinerary* of a point $x \in X$ under T with respect to the partition $\mathcal{I}$ as the sequence $\varphi_{\mathcal{I}}(x) = (\nu_{\mathcal{I}}(T^n x))_{n \in \mathbb{N}}$. If the system is recurrent, the closure Σ of the set of itineraries of points in X is characterized by the set of its finite words, the *language*; it is a subshift. The usual shift map S acts on Σ by $S((\omega_n)_{n \in \mathbb{N}}) = (\omega_{n+1})_{n \in \mathbb{N}}$, defining the *symbolic dynamics* of X with respect to $\mathcal{I}$. Conversely, we will say that (X, T) is a *geometric factor* of the symbolic dynamical system (Σ, S) through partition $\mathcal{I}$. When there is a T-invariant measure on X, we will say this factor is a *geometric representation* if the itinerary map is almost surely one-to-one. Trivially, the definition of a symbolic system yields a geometric representation on a Cantor set $\Sigma \subset \mathcal{A}^{\mathbb{N}}$. For some classes of symbolic dynamical systems, it has been fruitful to look for geometric factors and representations as translations on Abelian groups (torus, solenoids, . . .). More generally, we shall look for geometric representations among piecewise isometries.

1.2.2. *Interval exchange transformations*

Let X be a metric space and $\mathcal{I}$ a partition of X with d elements. We call *piecewise isometry* a map T on X which is bijective on X and such that the restriction $T_{|I}$ of T to any $I \in \mathcal{I}$ is an isometry. A class of example is that of *Interval Exchange Transformations*: piecewise (positive) isometries of the interval. Those can be characterized by the list of the length of the intervals of the partition (a vector $\lambda \in \mathbb{R}_+^d$) together with a permutation π of $\{1, \ldots, d\}$. The IET $F_{\lambda, \pi}$ maps I onto itself, permuting the d intervals $\{I_1, \ldots, I_d\}$ in $\mathcal{I}$ of lengths given by λ according to π. We denote by $\Sigma_{(\lambda, \pi)}$ both the language and the subshift associated with the symbolic dynamics of the dynamical system $(X, F_{\lambda, \pi})$ with respect to the partition $\mathcal{I}$. Alternatively the IET is characterized by the list of the intervals and a translation vector attached to each element of the partition; observe that the bijectivity condition then yields a relation between the length of the intervals and the translation vectors. Also recall that they can be seen as first-return maps of geodesic flows on flat surfaces. Families of examples (self induced ones) arise from sections of stable/unstable foliations of pseudo-Anosov maps of surfaces (see [19] for the general theory of pseudo-Anosov maps and [36] for their relations with interval exchange transformations).

1.2.3. *Substitutive dynamical systems*

If Y is a subset of X, and $x \in Y$, we define the first-return time of x as $n_x = \inf\{n \in \mathbb{N}^* | T^n(x) \in Y\}$ and the induced map T_Y of T on Y (or first-return map) as the map $x \mapsto T^{n_x}(x)$. The dynamical system (X, T) is *self-induced* if there is a subset $Y \subset X$ and a bijective map $h : X \to Y$ called *renormalization map* such that h conjugates the induced map T_Y on Y with T itself: $T_Y \circ h = h \circ T$.

Given a finite set $\mathcal{A}$ (the *alphabet*), a substitution σ on the alphabet $\mathcal{A}$ is a non-erasing morphism on the free monoid $\mathcal{A}^*$; it is characterized by the images of letters and it naturally extends to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$. It is *primitive* if the image of any letter under some iterate of σ contains every letters. Primitive substitutions have a finite number of periodic points in $\mathcal{A}^{\mathbb{N}}$; the closure of the orbit under the usual shift map of any of these periodic points defines the symbolic dynamical system (Σ_σ, S) associated to the substitution (independently on the chosen periodic point). By construction, (Σ_σ, S) is self-induced. This class of systems are the prototypical self-induced systems. For a more detailed description, see [31].

The *matrix of a substitution* σ is the $|\mathcal{A}| \times |\mathcal{A}|$ matrix whose entries indexed by $\mathcal{A}$ are the number of each letter in the image of each letter. Such positive matrix has a Perron–Frobenius eigenvalue. The substitution is *Pisot unimodular* if the other eigenvalues are of modulus strictly smaller than one and the discriminant is equal to one. The construction of the Rauzy fractal associated to a primitive Pisot unimodular substitution (see [17]) is a standard way to obtain a geometric representation of the symbolic substitutive dynamical system as an exchange of pieces on a compact subset $\mathcal{R}$ of $\mathbb{R}^{|\mathcal{A}|-1}$. It is self-induced and the renormalization map is a similarity. Self-induced IET yield substitutions (renormalization maps are homothety or piecewise contractions). When $|\mathcal{A}| = 2$, Rauzy fractals yield IET on two intervals: essentially rotations on the circle. When $|\mathcal{A}| = 3$, the exchange of pieces on the Rauzy fractal is a two-dimensional object.

However, a natural question is whether a given primitive substitution codes an IET. That is to say: does it have a geometric representation as an IET. The article [27] gives an algorithm to check if it is the case. Anyhow, there are obvious arguments showing it is not always the case. For instance if the characteristic polynomial of the matrix is not reciprocal (see, for example, [34]), the substitution cannot code an IET; indeed, substitutions associated with self-induced IET are related with pseudo-Anosov maps of surfaces and inherit geometric constraints; they must preserve a symplectic form. For the Tribonacci substitution, the characteristic polynomial is obviously not reciprocal. However, although there is no IET geometric representation for it, Arnoux's work ([5, 4]) shows that an IET may be a geometric factor of the substitutive system generated by the Tribonacci substitution in the sense that its symbolic dynamics "projects" onto the dynamics of the Tribonacci substitution.

1.2.4. σ-structure

Substitutions can be used to recover the symbolic dynamics of self-induced dynamical systems. Specifically:

Definition 1.2. Let (X, T) be a dynamical system, let σ be a substitution on the alphabet $\mathcal{A}$, and let $\mathcal{I} = \{I_i | i \in \mathcal{A}\}$ be a partition of X indexed by $\mathcal{A}$. We say that the dynamical system has σ-structure with respect to the partition $\mathcal{I}$ if there exist a subset $Y \subset X$, and a bijection $h : X \to Y$, such that:

(1) The map h conjugate T to T_Y: $T = h^{-1} \circ T_Y \circ h$.
(2) The first-return time is constant, denoted by n_i, on each set $J_i = h(I_i)$.
(3) The sets $T^k(J_i)$, for $i \in \mathcal{A}$ and $0 \leq k < n_i$, form a partition which refines $\mathcal{I}$.
(4) For $x \in J_i$, the finite coding $(\nu_{\mathcal{I}}(T^k x))_{0 \leq k < n_i}$ with respect to the partition $\mathcal{I}$ is given by the word $\sigma(i)$.

The original definition of σ-structure can be found in [3]. If a system has σ-structure with respect to a partition, it is semi-conjugate (by the itinerary map $\varphi_{\mathcal{I}}$) to the symbolic dynamical system associated to σ; if the sequence of partitions obtained by iterating the map h is generating (for example if the sequences of diameters of the elements of the partitions tend to 0), this semi-conjugacy $\varphi_{\mathcal{I}}$ is in fact one-to-one.

1.3. A more general construction

Our strategy to obtain a more general construction uses yet another type of geometric representations issued from a more group theoretic point of view.

1.3.1. Strategy

Indeed, substitutions as free monoid morphisms are also (positive) free group morphisms and hence an invertible substitution is the same as a positive free group automorphism. When such automorphism is furthermore irreducible (with irreducible powers, *iwip*, see precise definition in Sec. 2.1), a construction due to Gaboriau, Jaeger, Levitt and Lustig ([20]) yields an $\mathbb{R}$-tree (the *repelling tree*) together with an action of the free group by isometries and an action of the automorphism by homothety. Some more work ([12]) shows that this object may be (in different ways depending on the basis) reduced to a compact $\mathbb{R}$-tree (called the *heart*) together with a partial system of isometries. There is a partition of the heart such that the coding of the bi-infinite orbits of the system of partial isometries essentially coincides with the language of the fixed point of the substitution. In general, the subset of the heart corresponding to these bi-infinite sequences (called the limit set) is strictly smaller than the heart itself (it might be a Cantor set). Under an additional assumption (maximal index, see definition in Sec. 2.3), the heart and

the limit set coincide. In this case, the system of isometries is (essentially, i.e. up to boundary points) an exchange of pieces; we work in this setting. If the limit set is a simplicial tree (distance between branching points bounded below), then this picture yields a geometric representation which is essentially an IET on an interval made of the union of the branches and we immediately obtain a geometric representation as an IET. This case is known as the *geometric* case. Hence we focus on the *parageometric* case where there are countably many branching points (we give a complete definition at the end of Sec. 2.3). In this case, there should be *no* geometric representation of the symbolic system as an IET.

1.3.2. *Main result*

The central result presented here is that under assumptions (invertibility, irreducibility, parageometric) on the substitution that can be checked algorithmically, there exists an extension of the symbolic system that is an IET. The following formulation is in terms of free group automorphism:

Theorem 1.3. *Let α be a positive iwip and parageometric free group automorphism on $F_\mathcal{A}$ (the free group with basis $\mathcal{A}$). There is an alphabet $\tilde{\mathcal{A}}$, a projection $\Pi : \tilde{\mathcal{A}} \to \mathcal{A}$, a length vector $\lambda = \{\lambda_a; a \in \tilde{\mathcal{A}}\}$ and a permutation π of $\tilde{\mathcal{A}}$ (and hence an IET) such that*

$$\Pi(\Sigma_{(\lambda,\pi)}) = \Sigma_\alpha.$$

We recall that a positive automorphism is an invertible substitution; Σ_α must be understood in this sense. We stress that the assumptions on the automorphism are theoretically possible to check following an algorithm, even if, in practice it is not so easy. However, as we shall see, we are not able to construct explicitly the interval exchange transformation algorithmically from the automorphism; indeed, we need an explicit construction of the partial system of isometries acting on the heart of the $\mathbb{R}$-tree associated with the automorphism which is not known in general but only for some classes of examples.

1.3.3. *Idea of the proof*

Our idea is to build an extension of the exchange of pieces on the limit set. We consider the $\mathbb{R}$-tree. We put a cyclic order at each branching point in such a way that this order is in some sense compatible with the dynamics. Roughly, it can be seen as "drawing" the tree in the plane. Given an order it makes sense to construct a *contour* of the real tree. That is a circle "around" the tree. One can think that we follow branches of the tree being on one "side" of it; when we reach a branching point we proceed, following another branch (which one is determined by the cyclic order). This is essentially the same as a height function for a rooted tree. Each branch is seen twice; each branching point is seen a number of times equal to its

degree; only leaves are seen once. To formalize this we use the fibered space made of the tree together with, at each point, the set of connected components of the tree minus the point. We check in Sec. 3 that under a few technical conditions the construction we propose yields a circle. In Sec. 4, we analyze the properties of the (full, non-compact) $\mathbb{R}$-tree $\mathcal{T}$ associated with our automorphism; we show that we can choose a cyclic order which is "coherent" with the actions of both translations (action of the free group) and homothety (action of the automorphism). In Sec. 5, we restrict the actions to the so-called heart T of the tree $\mathcal{T}$. Instead of an F_N-action, we recover a $\mathbb{Z}$-action on a compact space, namely an exchange of pieces, f. The counterpart is that we introduce singularities and that the action on connected components becomes trickier. We also restrict the order to this subtree. In Sec. 6, we prove that the technical conditions required in Sec. 3 are fulfilled by the $\mathbb{R}$-tree T. Hence we can construct the contour of this compact tree with respect to this order and we can naturally extend the action of the exchange of pieces to a piecewise map $\tilde{f}$ on the circle (seen as a union of intervals). We check that this map is indeed an IET. At this stage, we are already in a position to prove Theorem 1.3.

1.3.4. *Extension of the substitution*

It appears that the IET $\tilde{f}$ is self-induced; the object of the next three sections is to understand completely its self-similar structure. Hence, we have to define an extension of the automorphism on $\mathcal{A}$, understand its effect and its relation with the induction. Indeed, there is a slight difficulty to ensure we obtain a proper coding: the IET $\tilde{f}$ may be defined on a partition coarser than the one needed to obtain the symbolic dynamics in terms of the substitution. Let us mention that this difficulty also appeared in Arnoux's work and was only solved in this specific case in [1]. Our most complete result states:

Theorem 1.4. *Let α be a positive iwip parageometric free group automorphism on $F_{\mathcal{A}}$. There is an alphabet $\tilde{\mathcal{A}}$, a projection $\Pi : \tilde{\mathcal{A}} \to \mathcal{A}$, a substitution $\tilde{\alpha}$ on $\tilde{\mathcal{A}}$, a partition $\mathcal{I} = \{I_a; a \in \tilde{\mathcal{A}}\}$ of $I = [0,1]$ and an IET $\tilde{f}$ of $(I, \mathcal{I})$ such that $\alpha \circ \Pi = \Pi \circ \tilde{\alpha}$, $\tilde{f}$ has $\tilde{\alpha}$-structure for $\mathcal{I}$ and $\Pi(\Sigma_{\tilde{\alpha}}) = \Sigma_{\alpha}$.*

In Sec. 7, we observe that the extension of the automorphism acts as a piecewise homothety: it yields "new" singularities. In Sec. 8, we show that the induction of the IET on the extension of the natural inducing zone (in the tree) is conjugated with the IET itself (by the piecewise contraction). Finally, in Sec. 9, we show that this guarantees the $\tilde{\alpha}$-structure announced.

1.4. *Explicit constructions*

Our construction is explicit, provided we have a good knowledge of the combinatorial structure of the repelling tree of the automorphism α. To illustrate this

fact we explain in Sec. 10 how to construct the IET given a tree substitution construction of this real tree. Determination of the IET is straightforward only with a good knowledge of the tree itself: in general we do not have an algorithm to determine completely the tree from the automorphism but special classes are extensively described in [24] and [25].

We take this advantage to treat specific examples. Firstly, in Sec. 11, we show the results arising from the so-called Tribonacci substitution. For this substitution, there are two choices for cyclic the order; the first one yields the Arnoux–Yoccoz interval exchange transformation while the second one yields a rather different interval exchange transformation; we believe this new example is interesting in itself. Then, in Sec. 12, we treat in details the case presented in the introduction; this does illustrate the construction and gives a proof (not the most direct) of Theorem 1.1.

1.5. *Perspectives*

This work was motivated by an example. We obtain a rather general result that yields plenty of examples. It is not clear how important it is to develop a general theory. However, it would certainly be interesting to explore the situation when the automorphism is *not* of maximal index. In this situation, it still makes sense to consider the contour of the heart and to extend the system of isometries of the $\mathbb{R}$-tree to a system of isometries of the interval. We imagine that this point of view should yield an Interval Translation (partial system of isometries) whose symbolic dynamics is related to the substitution dynamical system. But, it is not at all clear that our strategy to build the contour would work; for instance, it seems that *a priori*, the Hausdorff dimension may vary locally on the heart; furthermore the argument giving a finite partition of the tree fails.

As for the other assumptions, things should be easier but maybe not so interesting: irreducibility is a natural assumption (otherwise we would have to decompose). On the one hand, positivity does not seem to be crucial but description of the induction scheme is not completely clear without this assumption; in group theoretic terms, one has to find a traintrack decomposition and the graph may not be the standard one. On the other hand, a substitution that would not be invertible should not yield an interval exchange transformation so easily — but we have no idea what it should yield.

Another question is to analyze the extension $\tilde{\alpha}$ of the automorphism. It should be geometric (by construction). How is its inverse related with the inverse of the initial automorphism α?

Finally, observe that the maximal index assumption makes sense in the Culler–Vogtmann space. Instead of an automorphism we would start with an $\mathbb{R}$-tree together with a partial action of a free group; our construction could make sense and yield an IET.

2. Automorphisms of the Free Group

2.1. *Notations*

Let F_N be the free group on N (≥ 2) generators and let ∂F_N be its Gromov boundary. The *double boundary* $\partial^2 F_N$ is defined by

$$\partial^2 F_N = (\partial F_N \times \partial F_N)\backslash\Delta,$$

where Δ is the diagonal.

We will always assume the free group to be endowed with a basis. The free group F_N endowed with a basis $\mathcal{A} = \{a_0, \ldots, a_{N-1}\}$ will simply be referred to as $F_{\mathcal{A}}$. The set of inverse letters is denoted by $\mathcal{A}^{-1} = \{a_0^{-1}, \ldots, a_{N-1}^{-1}\}$. We consider $F_{\mathcal{A}}$ to be the set of finite words $v = v_0 v_1 \cdots v_p$ with letters in $(\mathcal{A} \cup \mathcal{A}^{-1})$ satisfying, for all i, $0 \leq i < p$, $v_i \neq v_{i+1}^{-1}$: the word v is *reduced*. The length of the word v is $|v| = p + 1$. The identity element of $F_{\mathcal{A}}$ is identified with the empty word ϵ. Similarly, $\partial F_{\mathcal{A}}$ will be the set of reduced words $V = (V_i)_{i \in \mathbb{N}}$ with letters in $(\mathcal{A} \cup \mathcal{A}^{-1})$. A word $u \in F_{\mathcal{A}}$ is a *prefix* of a word $V \in F_{\mathcal{A}} \cup \partial F_{\mathcal{A}}$ if $u_i = V_i$ for $0 \leq i \leq |u| - 1$. The word u is a *suffix* of $v \in F_{\mathcal{A}}$ if $u_i = v_i$ for $|v| - |u| \leq i \leq |v| - 1$.

The free group $F_{\mathcal{A}}$ acts continuously on $\partial F_{\mathcal{A}}$ by left translation: if $u = u_0 \cdots u_p \in F_{\mathcal{A}}$ and $V = V_0 V_1 \cdots \in \partial F_{\mathcal{A}}$, then $uV = u_0 \cdots u_{p-i-1} V_{i+1} \cdots V_p \cdots$, where $V_0 \cdots V_i = u_k^{-1} \cdots u_{k-i}^{-1}$ is the longest common prefix of u^{-1} and V, is in $\partial F_{\mathcal{A}}$.

An automorphism α of $F_{\mathcal{A}}$ is *positive* if for all $a \in \mathcal{A}$, all the letters of $\alpha(a)$ are in $\mathcal{A}$. The automorphism α induces a homeomorphism $\partial\alpha$ on $\partial F_{\mathcal{A}}$, and a homeomorphism $\partial^2\alpha$ on $\partial^2 F_{\mathcal{A}}$.

We say an automorphism α of $F_{\mathcal{A}}$ is *iwip* (that is, irreducible with irreducible power) if no proper free factor of $F_{\mathcal{A}}$ is fixed by α. It should be noted that a positive iwip automorphism of $F_{\mathcal{A}}$ is *primitive* in the following sense: for any $a \in \mathcal{A}$, there exists an integer k such that $\alpha^k(a)$ contains every letters of $\mathcal{A}$. In this case, the restriction of α to positive words is obviously a primitive substitution. The reader is referred to [2] for a helpful discussion on the iwip and primitivity properties.

2.2. *The attracting subshift*

We define the shift map S on $\partial^2 F_{\mathcal{A}}$:

$$S : \partial^2 F_{\mathcal{A}} \to \partial^2 F_{\mathcal{A}}$$

$$(X, Y) \mapsto (Y_0^{-1} X, Y_0^{-1} Y),$$

where Y_0 is the first letter of Y.

Let α be a positive automorphism of $F_{\mathcal{A}}$. Let a be a letter of $\mathcal{A}$. The primitivity condition implies that we can find an integer k such that $\alpha^k(a) = pas$ where p and s are non-empty words of $F_{\mathcal{A}}$ with letters in $\mathcal{A}$. Now define

$$X = \lim_{n \to +\infty} p^{-1} \alpha^k(p^{-1}) \alpha^{2k}(p^{-1}) \cdots \alpha^{nk}(p^{-1}),$$

$$Y = \lim_{n \to +\infty} as\alpha^k(s)\alpha^{2k}(s) \cdots \alpha^{nk}(s).$$

The *attracting subshift* of α is defined by

$$\Sigma_\alpha = \overline{\{S^n(X,Y); n \in \mathbb{Z}\}}.$$

The attracting subshift only depends on α and not on the choice of the letter a nor on the integer k. The map S is a homeomorphism on Σ_α. The projection of Σ_α on the first (respectively, second) coordinate will be denoted Σ_α^- (respectively, Σ_α^+). We observe that seeing α as a substitution, the restriction of Σ_α^+ to positive letters is indeed exactly the subshift Σ_α associated to the substitution α as defined in Sec. 1.2. The reader is referred to [32] for an analysis of substitutive dynamical systems.

2.3. *The index of an automorphism*

The index of an automorphism α of the free group F_N is defined in [20] by:

$$ind_{inn}(\alpha) = rk(Fix(\alpha)) + \frac{1}{2}a(\alpha) - 1,$$

where $Fix(\alpha) = \{u \in F_N; \alpha(u) = u\}$ and $a(\alpha)$ is the set of equivalence classes of attracting fixed points of ∂F_N: $a(\alpha) = (Att(\partial\alpha))/Fix(\alpha)$.

The set of conjugacy $i_w : u \mapsto w^{-1}uw$ of F_N is denoted $Inn(F_N)$. Recall that the outer class Φ of α is the set of automorphisms $\{i_w \circ \alpha; i_w \in Inn(F_N)\}$. Two automorphisms φ and ψ of Φ are *isogredient* if there is $w \in F_N$ such that $\varphi = i_w \circ \psi \circ i_{w^{-1}}$. The index of the outer class Φ of α is defined by:

$$ind_{\text{out}}(\alpha) = \sum_{[\varphi]} \max(0, ind_{inn}(\varphi)),$$

where the sum is taken over all isogredience classes of Φ.

Finally, we define

$$ind_{\max}(\alpha) = \max_k (ind_{\text{out}}(\alpha^k)).$$

This last index is the only one of interest in this paper. From now on, when referring to the index of an automorphism α, we will actually refer to $ind_{\max}(\alpha)$ and we will only write $ind(\alpha)$. The following result was obtained in [20].

Theorem 2.1. ([20]) *Let α be an automorphism of F_N. Then we have:*

$$ind(\alpha) \leq N - 1.$$

Definition 2.2. An automorphism α of the free group $F_{\mathcal{A}}$ is parageometric if it has maximal index (i.e. $ind(\alpha) = N - 1$) while its inverse does not have maximal index.

We stress that this property can be checked by following an algorithm (see [26]).

3. Contour of an $\mathbb{R}$-Tree

An $\mathbb{R}$-*tree* T is a geodesic and 0-hyperbolic metric space: for all $x, y \in T$, there exists a unique arc $[x, y]$ joining x and y. For all $x \in T$, we denote $\mathcal{C}_x$ the list of connected components of $T \backslash \{x\}$. We call *degree* of x the cardinality $|\mathcal{C}_x|$ of $\mathcal{C}_x$ and say that x is a *branching point* if $|\mathcal{C}_x| > 2$. We denote $]]x, y[[$ the connected component of $T \backslash \{x, y\}$ containing $]x, y[$ and $[[x, y]] =]]x, y[[\cup \{x, y\}$. For all $x, y \in T$, $x \neq y$, we denote $C_x(y)$ the element of $\mathcal{C}_x$ containing y. Observe that if $C \in \mathcal{C}_x$, then the topological boundary of C in T is reduced to $\{x\}$.

Throughout all of Secs. 3.1 and 3.2, we consider T to be an $\mathbb{R}$-tree satisfying the following properties:

(H_0) T is bounded and complete.

(H_1) The set $\mathcal{B}_T$ of branching points (degree greater than 2) is countable and dense. The set of degrees of branching points is bounded (in particular, points have finite degree).

(H_2) T is endowed with a measure μ on the Borel σ-field which is such that for any $x \in T$ and any sequence $x_n \to x$, $\mu(]]x, x_n[[) \to 0$; in particular, it is non-atomic. We also assume that the measure does not charge arcs: for all $x, y \in T$, $\mu([x, y]) = 0$, but charges all significant subtree: for all $x \in T$ and all $C \in \mathcal{C}_x$, $\mu(C) > 0$.

There will be another assumption (H_3): density of branching points "on both sides"; we need more notations to state it, see further.

3.1. *Cyclic order*

Let $x \in T$. Observe that the set $\mathcal{C}_x$ is finite (property (H_1)). We provide each point x with a cyclic order by defining a cyclic permutation σ_x on the set $\mathcal{C}_x$. Fixing an element C of $\mathcal{C}_x$, the map σ_x provides an ordered sequence $C, \sigma_x(C), \sigma_x^2(C), \ldots, \sigma_x^{|\mathcal{C}_x|-1}(C)$ and we have $\sigma_x^{|\mathcal{C}_x|}(C) = C$.

This makes sense essentially for branching points (where $|\mathcal{C}_x| > 2$) since a cyclic order on one or two elements is nothing. Given three elements A, B, C of $\mathcal{C}_x$ with B distinct from A and C, we say that B is between A and C and write $A \prec B \prec C$ if $B = \sigma^i(A)$ and $C = \sigma^j(A)$ for some $1 < i < j \leq |\mathcal{C}_x|$. Of course, either $A \prec B \prec C$ or $C \prec B \prec A$. Also observe that $A \prec B \prec A$.

Remark 3.1. Observe that fixing an order at each branching point is the same as drawing the tree on a surface.

We assume furthermore:

(H_3) For all x and y in T with $x \neq y$, there is $z \in]x, y[$ and $C \in \mathcal{C}_z$ (recall we have $\mu(C) > 0$ from (H_2)) with $C_z(x) \prec C \prec C_z(y)$.

3.2. *Contour of a tree*

We consider the fibered set

$$\tilde{T} = \{(x, C); x \in T, C \in \mathcal{C}_x\}.$$

Given $\tilde{x} \in \tilde{T}$, we denote $\pi(\tilde{x})$ its first coordinate (in T) and $C(\tilde{x})$ (in $\mathcal{C}_x$) its second one. We assume a cyclic order σ_x is defined for any point x of the tree T. The aim of this section is to construct a topology on $\tilde{T}$ using these orders. For any point $\tilde{x}$ of $\tilde{T}$, we set $C_i(\tilde{x}) = \sigma_x^i(C(\tilde{x}))$.

Let $\tilde{x} = (x, C)$ and $\tilde{y} = (y, C')$ be two points of $\tilde{T}$. Firstly we assume that with $x \neq y$. For any point z of $]x, y[$ (recall this is the open geodesic arc joining x and y in T), we define

$$\mathcal{C}_z^+(\tilde{x}, \tilde{y}) = \{C \in \mathcal{C}_z; C_z(x) \prec C \prec C_z(y)\}.$$

Observe that this definition does not depend on the choice of the lifts of x and y in $\tilde{T}$; however, this notation allows us to extend the definition to the boundary points x and y. Namely, we define $\mathcal{C}_x^+(\tilde{x}, \tilde{y}) = \{C \in \mathcal{C}_x; C(\tilde{x}) \prec C \prec C_x(y)\}$ and $\mathcal{C}_y^+(\tilde{x}, \tilde{y}) = \{C \in \mathcal{C}_y; C_y(x) \prec C \prec \sigma_y(C(\tilde{y}))\}$ (notice that $C(\tilde{y})$ is included). To settle the case when $x = y$, we put $\mathcal{C}_x^+(\tilde{x}, \tilde{y}) = \mathcal{C}_y^+(\tilde{x}, \tilde{y}) = \{C \in \mathcal{C}_x; C(\tilde{x}) \prec C \prec \sigma_y(C(\tilde{y}))\}$. One can check that with this definition, $\mathcal{C}_x^+(\tilde{x}, \tilde{x}) = \emptyset$.

For any point $z \in [x, y]$, we set $\mathcal{C}_z^-(\tilde{x}, \tilde{y}) = \mathcal{C}_z^+(\tilde{y}, \tilde{x})$.

One should think of these definitions as a mean to define the sides of a segment: if we were to walk from x to y along the geodesic arc $[x, y] \subset T$, crossing a branching point z would have us leave trees to the "left" (the trees of $\mathcal{C}_z^-(\tilde{x}, \tilde{y})$) and trees to the "right" (the trees of $\mathcal{C}_z^+(\tilde{x}, \tilde{y})$).

We recall that the tree $]]x, y[[= C_x(y) \cap C_y(x)$ contains $]x, y[$. We split this tree into a "left" ($\epsilon = -1$) and a "right" ($\epsilon = +1$) part. We set

$$\mathcal{C}^\epsilon(x, y) = \bigcup_{z \in]x, y[} \mathcal{C}_z^\epsilon(\tilde{x}, \tilde{y}),$$

$$\mathcal{C}^\epsilon(\tilde{x}, \tilde{y}) = \bigcup_{z \in [x, y]} \mathcal{C}_z^\epsilon(\tilde{x}, \tilde{y}),$$

$$]]x, y[[^\epsilon = \bigcup_{C \in \mathcal{C}^\epsilon(x, y)} C$$

and

$$[[\tilde{x}, \tilde{y}]]^\epsilon = \bigcup_{C \in \mathcal{C}^\epsilon(\tilde{x}, \tilde{y})} C.$$

Observe that these unions are in fact denumerable since $\mathcal{C}_z^\epsilon(\tilde{x}, \tilde{y})$ is empty except maybe if z is a branching point or equal to x or y. Note that $]]x, y[[^+ =]]y, x[[^-$ and

that $]]x, y[[=]x, y[\cup]]x, y[[^+ \cup]]x, y[[^-$. Let us stress that these sets are subsets of T even if they depend on points of $\tilde{T}$.

We are going to endow $\tilde{T}$ with a metric. For this purpose, we define $d_+(\tilde{x}, \tilde{y}) = \mu([[\tilde{x}, \tilde{y}]]^+)$, which can be decomposed as follows:

$$d_+(\tilde{x}, \tilde{y}) = \sum_{C \in \mathcal{C}^+(\tilde{x}, \tilde{y})} \mu(C)$$

$$= \sum_{C \in \mathcal{C}_x^+(\tilde{x}, \tilde{y})} \mu(C) + \sum_{z \in]x, y[\cap \mathcal{B}_T} \sum_{C \in \mathcal{C}_z^+(\tilde{x}, \tilde{y})} \mu(C) + \sum_{C \in \mathcal{C}_y^+(\tilde{x}, \tilde{y})} \mu(C).$$

Observe that since, for all $\tilde{x}$ and $\tilde{y}$ distinct elements of $\tilde{T}$, $T = [x, y] \cup [[\tilde{x}, \tilde{y}]]^- \cup [[\tilde{x}, \tilde{y}]]^+$, and since, in view of Assumption $(H2)$, $\mu([x, y]) = 0$, the following identity holds:

$$d_+(\tilde{x}, \tilde{y}) + d_+(\tilde{y}, \tilde{x}) = \mu(T).$$

Lemma 3.2. *Let $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ be three distinct points of $\tilde{T}$. One of the following identities holds*:

(i) $d_+(\tilde{x}, \tilde{y}) = d_+(\tilde{x}, \tilde{z}) + d_+(\tilde{z}, \tilde{y})$
(ii) $d_+(\tilde{y}, \tilde{x}) = d_+(\tilde{y}, \tilde{z}) + d_+(\tilde{z}, \tilde{x}).$

Proof. Let $\tilde{x} = (x, C(\tilde{x}))$, $\tilde{y} = (y, C(\tilde{y}))$ and $\tilde{z} = (z, C(\tilde{z}))$ be three distinct points of $\tilde{T}$.

Case 1. Firstly we consider the special case when $z \in [x, y]$. We assume that $C(\tilde{z}) \in \mathcal{C}_z^+(\tilde{x}, \tilde{y})$ or $x \in C(\tilde{z})$; otherwise $C(\tilde{z}) \in \mathcal{C}_z^-(\tilde{x}, \tilde{y})$ or $y \in C(\tilde{z})$ so we exchange the roles of $\tilde{x}$ and $\tilde{y}$ in the remainder of the proof and prove (ii) instead of (i). In this case, $\mathcal{C}_z^+(\tilde{x}, \tilde{y})$ is the disjoint union of $\mathcal{C}_z^+(\tilde{x}, \tilde{z})$ and $\mathcal{C}_z^+(\tilde{z}, \tilde{y})$ (one of them may be empty) and the disjoint decomposition,

$$\mathcal{C}^+(\tilde{x}, \tilde{y}) = \mathcal{C}^+(\tilde{x}, \tilde{z}) \cup \mathcal{C}^+(\tilde{z}, \tilde{y})$$

immediately yields (i).

Case 2. Now we consider the case $z \notin [x, y]$. Let $\underline{z} = [x, y] \cap [y, z] \cap [z, x]$ and assume that $C_{\underline{z}}(z) \in \mathcal{C}_{\underline{z}}^+(\tilde{x}, \tilde{y})$; otherwise we exchange the roles of $\tilde{x}$ and $\tilde{y}$ in the remainder of the proof and prove (ii) instead of (i). We set $\tilde{\underline{z}}_- = (\underline{z}, \sigma_{\underline{z}}^{-1}(C_{\underline{z}}(z)))$ and $\tilde{\underline{z}}_+ = (\underline{z}, C_{\underline{z}}(z))$ so that,

$$\mathcal{C}^+(\tilde{x}, \tilde{y}) = \mathcal{C}^+(\tilde{x}, \tilde{\underline{z}}_-) \cup \{C_{\underline{z}}(z)\} \cup \mathcal{C}^+(\tilde{\underline{z}}_+, \tilde{y}).$$

Observe that $\underline{z} \in [x, z]$, $C(\tilde{\underline{z}}_-) \in \mathcal{C}_{\underline{z}}^+(\tilde{x}, \tilde{z})$ or $x \in C(\tilde{\underline{z}}_-)$ so that, using Case 1 gives $\mathcal{C}^+(\tilde{x}, \tilde{z}) = \mathcal{C}^+(\tilde{x}, \tilde{\underline{z}}_-) \cup \mathcal{C}^+(\tilde{\underline{z}}_-, \tilde{z})$; and also that $\underline{z} \in [z, y]$ and $z \in C(\tilde{\underline{z}}_+)$, so that, $\mathcal{C}^+(\tilde{z}, \tilde{y}) = \mathcal{C}^+(\tilde{z}, \tilde{\underline{z}}_+) \cup \mathcal{C}^+(\tilde{\underline{z}}_+, \tilde{y}).$

To conclude, we decompose $C_{\underline{z}}(z)$ into the branch $]\underline{z}, z]$ and all the components attached to it:

$$C_{\underline{z}}(z) =]\underline{z}, z] \cup \left(\bigcup_{C \in \mathcal{C}^+(\underline{\tilde{z}}_-, \tilde{z})} C \right) \cup \left(\bigcup_{C \in \mathcal{C}^+(\tilde{z}, \underline{\tilde{z}}_+)} C \right).$$

Since, by Assumption $(H2)$, the measure μ does not charge intervals, (i) follows.

Observe that in the cases $x \in [z, y]$ and $y \in [x, z]$, we have $\underline{z} = x$ and $\underline{z} = y$, respectively. Also observe that the situation $x = y = z$ is covered by Case 1. Finally, let us stress the fact that the same kind of decomposition also yields $d_+(\tilde{z}, \tilde{x}) = d_+(\tilde{z}, \tilde{y}) + d_+(\tilde{y}, \tilde{x})$ and $d_+(\tilde{y}, \tilde{z}) = d_+(\tilde{y}, \tilde{x}) + d_+(\tilde{x}, \tilde{z})$ in situation (i) and the corresponding identities in situation (ii), i.e. $d_+(\tilde{x}, \tilde{z}) = d_+(\tilde{x}, \tilde{y}) + d_+(\tilde{y}, \tilde{z})$ and $d_+(\tilde{z}, \tilde{y}) = d_+(\tilde{z}, \tilde{x}) + d_+(\tilde{x}, \tilde{y})$. $\qquad\square$

We observe that d_+ is not symmetric and bounded by $\mu(T)$, we set $d(\tilde{x}, \tilde{y}) = \min(d_+(\tilde{x}, \tilde{y}), d_+(\tilde{y}, \tilde{x}))$ and show that this defines a metric on $\tilde{T}$.

Lemma 3.3. *The function d is a distance.*

Proof. We check the characteristic properties of a distance:

(Symmetry) For all $\tilde{x}$ and $\tilde{y}$ in $\tilde{T}$, $d(\tilde{x}, \tilde{y}) = \min(d_+(\tilde{x}, \tilde{y}), d_+(\tilde{y}, \tilde{x})) = d(\tilde{y}, \tilde{x})$.

(Separation) Let $\tilde{x} = (x, C)$ and $\tilde{y} = (y, C')$ be in $\tilde{T}$ with $\tilde{x} \neq \tilde{y}$. If $x = y$, then $C \neq C'$ so $\mathcal{C}^+(\tilde{x}, \tilde{y})$ contains at least C' (and $\mathcal{C}^+(\tilde{y}, \tilde{x})$ contains at least C) which are of positive measure in view of Assumption $(H2)$. If $x \neq y$, then Assumption (H_3) says that $\mathcal{C}^+(x, y)$ and $\mathcal{C}^+(y, x)$ are nonempty; it is of positive measure in view of Assumption $(H2)$. In both cases, the sum defining $d_+(\tilde{x}, \tilde{y})$ has at least one positive term and hence is nonzero. It follows that $d_+(\tilde{x}, \tilde{y}) = 0$ implies $\tilde{x} = \tilde{y}$. Conversely, for all $\tilde{x} \in \tilde{T}$, $\mathcal{C}^+(\tilde{x}, \tilde{x})$ is empty so $d_+(\tilde{x}, \tilde{x})$ is a sum on an empty set. It follows that $d(\tilde{x}, \tilde{x}) = d_+(\tilde{x}, \tilde{x}) = 0$.

(Triangular inequality) We make use of Lemma 3.2. Assume for instance that we are in situation (i); otherwise we exchange the roles of $\tilde{x}$ and $\tilde{y}$. So that $d_+(\tilde{x}, \tilde{y}) = d_+(\tilde{x}, \tilde{z}) + d_+(\tilde{z}, \tilde{y})$, and, at the same time, $d_+(\tilde{z}, \tilde{x}) = d_+(\tilde{z}, \tilde{y}) + d_+(\tilde{y}, \tilde{x})$ and $d_+(\tilde{y}, \tilde{z}) = d_+(\tilde{y}, \tilde{x}) + d_+(\tilde{x}, \tilde{z})$.

If $d_+(\tilde{x}, \tilde{z}) \leq d_+(\tilde{z}, \tilde{x})$ and $d_+(\tilde{y}, \tilde{z}) \leq d_+(\tilde{z}, \tilde{y})$, then $d_+(\tilde{x}, \tilde{y}) = d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y})$ and hence, taking the minimum, we obtain the desired inequality: $d(\tilde{x}, \tilde{y}) \leq d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y})$.

Otherwise, if, for instance, $d_+(\tilde{x}, \tilde{z}) > d_+(\tilde{z}, \tilde{x})$, we write that $d_+(\tilde{z}, \tilde{x}) = d_+(\tilde{z}, \tilde{y}) + d_+(\tilde{y}, \tilde{x})$ to claim that $d(\tilde{y}, \tilde{x}) \leq d_+(\tilde{y}, \tilde{x}) \leq d_+(\tilde{z}, \tilde{x}) = \min(d_+(\tilde{x}, \tilde{z}), d_+(\tilde{z}, \tilde{x})) = d(\tilde{x}, \tilde{z})$, and, we conclude. We proceed symmetrically when $d_+(\tilde{y}, \tilde{z}) > d_+(\tilde{z}, \tilde{y})$. $\qquad\square$

We endow $\tilde{T}$ with the metric d.

Lemma 3.4. *The metric space $(\tilde{T}, d)$ is a circle.*

Proof. Let $\tilde{o} = (\pi(\tilde{o}), C(\tilde{o})) \in \tilde{T}$ and consider the map $\psi : \tilde{T} \to \{z \in \mathbb{C}; |z| = 1\}$, $\tilde{x} \mapsto \exp(2i\pi d_+(\tilde{o}, \tilde{x})/\mu(T))$. We claim that it is a homeomorphism.

(i) Injectivity. Let $\tilde{x}$ and $\tilde{y}$ in $\tilde{T}$ be such that $\psi(\tilde{x}) = \psi(\tilde{y})$. It must be that $d_+(\tilde{o}, \tilde{x}) \equiv d_+(\tilde{o}, \tilde{y})$ modulo $\mu(T)$. Since $0 \leq d_+(\tilde{o}, \tilde{x}) < \mu(T)$, we must have $d_+(\tilde{o}, \tilde{x}) = d_+(\tilde{o}, \tilde{y})$. In view of Lemma 3.2, it implies that either $d_+(\tilde{x}, \tilde{y}) = 0$ or $d_+(\tilde{y}, \tilde{x}) = 0$ and hence that $d(\tilde{x}, \tilde{y}) = 0$. We conclude that $\tilde{x} = \tilde{y}$.

(ii) Surjectivity. Let s be in $(0, 1)$; we simply prove there exists a point $\tilde{z}$ of $\tilde{T}$ such that $d_+(\tilde{o}, \tilde{z}) = s$. This is achieved by constructing a sequence $(\tilde{z}_n)_n$ of points of $\tilde{T}$ converging to $\tilde{z}$ and such that $(d_+(\tilde{o}, \tilde{z}_n))_n$ goes to s.

Choose a point x in $C(\tilde{o})$. Recall (property (H_2)) that if $(x_n)_n$ is a sequence of T with limit $\pi(\tilde{o})$, then $\mu(]]\pi(\tilde{o}), x_n[[) \to 0$. Choosing the sequence $(x_n)_n$ so that $x_n \in [\pi(\tilde{o}), x]$ for any $n \in \mathbb{N}$ allows us to pick a point $z_0 \neq \pi(\tilde{o})$ of T such that $\tilde{z}_0 = (z_0, C_{z_0}(\pi(\tilde{o})))$ satisfies $d_+(\tilde{o}, \tilde{z}_0) = t < s$. Define k as the unique integer such that

$$\sum_{1 \leq i < k} \mu(C_i(z_0)) \leq s - t \leq \sum_{1 \leq i \leq k} \mu(C_i(z_0)),$$

where $C_i(z_0) = \sigma_{z_0}^i(C_{z_0}(\pi(\tilde{o})))$. Choose any point y in $C_k(z_0)$. If $C_k(z_0) \neq C_{z_0}(\pi(\tilde{o}))$, pick any point $\tilde{y}$ of $\pi^{-1}(\{y\})$. If $C_k(z_0) = C_{z_0}(\pi(\tilde{o}))$, choose a point $\tilde{y} = (y, C(\tilde{y}))$ in $]]\tilde{z}_0, \tilde{o}[[^+$. In both cases, we have $d_+(\tilde{o}, \tilde{y}) > d_+(\tilde{o}, \tilde{z}_0)$. Again, thanks to property (H_2), we can choose a point $\tilde{z}_1 \in \tilde{T}$ such that $\pi(\tilde{z}_1) \in [z_0, y]$ and $t < d_+(\tilde{o}, \tilde{z}_1) < s$. Iterating the process would yield a sequence $(\tilde{z}_n)_n$ of points of $\tilde{T}$ such that $(d_+(\tilde{o}, \tilde{z}_n))_n$ is increasing and bounded by s; but this sequence may not directly go to s. However, the argument shows that the supremum $\sup\{t \leq s : \exists \tilde{z} \in \tilde{T}, d_+(\tilde{o}, \tilde{z}) = t\}$ is equal to s.

(iii) Continuity. Simply observe that for any two points $\tilde{x}, \tilde{y}$ of $\tilde{T}$, the length in $\{z \in \mathbb{C}; |z| = 1\}$ of the geodesic arc joining $\psi(\tilde{x})$ and $\psi(\tilde{y})$ is $2\pi d(\tilde{x}, \tilde{y})/\mu(T)$. This shows continuity of both the map and the inverse map. $\square$

3.3. *Contour of a finite forest*

We will require to work in the more general setting of finite forests (see Theorem 5.8 and the ensuing discussion); a *finite forest* is a finite union of $\mathbb{R}$-tree. In this section, we slightly adapt the definitions given above in order to include the case of a finite forests.

Let T be a finite forest and let $\mathfrak{T}$ be its convex hull. If x and y are two points of T, the arc $[x, y]$ may not be included in T, but it is included in $\mathfrak{T}$. For any point x, define

$$\mathcal{C}_x = \{C \cap T; C \text{ is a connected component of } \mathfrak{T}\backslash\{x\} \text{ such that } C \cap T \neq \emptyset\}.$$

We consider the set of discontinuities of T:

$$\mathcal{S}_\mathcal{D} = \{x \in T; \exists\, y \in T;]x, y[\cap T = \emptyset\}.$$

The degree of a point x is the number of elements C of $\mathcal{C}_x$ such that $x \in \overline{C}$. Note that it is exactly $|\mathcal{C}_x|$ for the points of $T \backslash \mathcal{S}_\mathcal{D}$ but not for the points of $\mathcal{S}_\mathcal{D}$.

In order for the definitions and notations of the previous section to make sense in this case, we need to consider the branching points of $\mathfrak{T}$ that do not belong to T, as these appear naturally when working with an arc $[x, y]$ if x and y do not belong to the same connected component of T. We denote $\mathcal{B}_\mathfrak{T}$ the set of branching points of $\mathfrak{T}$.

Assume now that T satisfies (H_0) and (H_1): we stress that the set of branching points is dense in T and not $\mathfrak{T}$. Also assume that there is a measure μ satisfying (H_2) on $\mathfrak{T}$ (and not only T). Define a cyclic order σ_z at each branching point z of $\mathfrak{T}$ (again, not only T) such that (H_3) is satisfied for any x, y belonging to a common connected component of T. Also define the fibered set

$$\tilde{T}_0 = \{(x, C); x \in T, C \in \mathcal{C}_x\}.$$

For any two points $\tilde{x} = (x, C)$ and $\tilde{y} = (y, C')$ of $\tilde{T}_0$ and any point z of $]x, y[$ we define

$$\mathcal{C}_z^+(\tilde{x}, \tilde{y}) = \{C \in \mathcal{C}_z; C_z(x) \prec C \prec C_z(y)\}$$

exactly as above. Note, however, that this time, the point z may not be a point of T. The sets $\mathcal{C}_x^+(\tilde{x}, \tilde{y})$ and $\mathcal{C}_y^+(\tilde{x}, \tilde{y})$ are the same as before and we set

$$\mathcal{C}^\epsilon(x, y) = \bigcup_{z \in \mathcal{B}_\mathfrak{T} \cap]x, y[} \mathcal{C}_z^\epsilon(\tilde{x}, \tilde{y}) \quad \text{and} \quad \mathcal{C}^\epsilon(\tilde{x}, \tilde{y}) = \bigcup_{z \in \mathcal{B}_\mathfrak{T} \cap [x, y]} \mathcal{C}_z^\epsilon(\tilde{x}, \tilde{y})$$

(and this time we consider the branching points of $\mathfrak{T}$). The sets $]]x, y[[^\epsilon$ and $[[\tilde{x}, \tilde{y}]]^\epsilon$ are also as above.

Finally, we define $d_+(\tilde{x}, \tilde{y}) = \mu([[\tilde{x}, \tilde{y}]]^+)$ for any points $\tilde{x}, \tilde{y}$ of $\tilde{T}_0$, and we set $d(\tilde{x}, \tilde{y}) = \min(d_+(\tilde{x}, \tilde{y}), d_+(\tilde{y}, \tilde{x}))$. Obviously, d is not a distance since there exist (as long as T has more than one connected component) points $\tilde{x}$ and $\tilde{y}$ of $\tilde{T}_0$ such that $[[\tilde{x}, \tilde{y}]]^+$ is empty. The contour of the finite forest T is then defined as the metric space $(\tilde{T}, d)$ where $\tilde{T} = \tilde{T}_0/\sim$ and we have $\tilde{x} \sim \tilde{y}$ when $d(\tilde{x}, \tilde{y}) = 0$. It is again a circle.

4. The Repelling Tree of a Free Group Automorphism

An important property of the trees we are going to consider is that they are endowed with an action of the free group F_N by isometries. Such an action is *minimal* if there is no proper F_N-invariant subtree, *small* (see [14]) if no edge stabilizer contains a free non-Abelian subgroup, and *very small* (see [13]) if it is small and if for any nontrivial $g \in F_N$, the fixed subtree $Fix(g)$ is equal to $Fix(g^p)$ for $p \geq 2$ and $Fix(g)$ is isometric to a subset of $\mathbb{R}$.

Recall that the Outer space CV_N can be seen as the space of isometric action of F_N on simplicial (the distance between two branching points is bounded below) $\mathbb{R}$-trees. It was shown in [6] that the boundary ∂CV_N consists exactly of very small actions of F_N on (not necessarily simplicial) $\mathbb{R}$-trees. The reader is referred to [35] for a nice survey on the Outer space.

4.1. *The repelling tree*

It is shown in [29] that any iwip automorphism α of the free group F_N has north–south dynamic on compactified Outer space $CV_N \cup \partial CV_N$: there exist an attracting tree $T_\alpha^+ \in \partial CV_N$ and a repelling tree $T_\alpha^- \in \partial CV_N$ which are both invariant under the action of α. A construction of such trees is given in [20] in a more general case. In this paper, we will only consider the repelling tree of an iwip automorphism; the basic properties of this tree are compiled in the following theorem.

Theorem 4.1. ([20, 29]) *Let α be an iwip automorphism of the free group F_N. The repelling tree T_α^- is such that*:

- *the action of F_N by isometries on T_α^- is minimal, very small, and has dense orbits,*
- *the action of α on T_α^- is seen as a contracting homothety $\boldsymbol{\alpha}$ with factor $\lambda_{\boldsymbol{\alpha}} < 1$; the map $\boldsymbol{\alpha}$ satisfies, for all $x \in T_\alpha^-$ and all $w \in F_N$,*

$$\alpha(w) \cdot \boldsymbol{\alpha}(x) = \boldsymbol{\alpha}(w.x).$$

In addition to this theorem, we state the following result for future use.

Theorem 4.2. ([21]) *Let T be an $\mathbb{R}$-tree with a minimal, very small action of F_N by isometries. The number of orbits of branching points is bounded above by $2N - 2$.*

From now on, declaring an iwip automorphism α will implicitly declare its repelling tree T_α^-, and its contracting homothety $\boldsymbol{\alpha}$ with factor $\lambda_{\boldsymbol{\alpha}} < 1$.

Let α be an iwip automorphism of F_N. We consider the metric completion $\overline{T_\alpha^-}$ of T_α^-. The free group F_N also acts on $\overline{T_\alpha^-}$ by isometries (although some of the properties of the action on T_α^- are not carried onto $\overline{T_\alpha^-}$), and the homothety $\boldsymbol{\alpha}$ on T_α^- naturally extends to a homothety on $\overline{T_\alpha^-}$; the latter will also be denoted by $\boldsymbol{\alpha}$. It should be noted that the number of F_N-orbits of branching points is the same in T_α^- and $\overline{T_\alpha^-}$. Indeed, the set $\overline{T_\alpha^-} \backslash T_\alpha^-$ only contains points of degree 1. The rest of the section is dedicated to describing a cyclic order on $\overline{T_\alpha^-}$ compatible with the actions of both F_N and $\boldsymbol{\alpha}$.

4.2. *A cyclic order for the repelling tree*

We still consider α to be an iwip automorphism; the metric completion of its repelling tree is simply denoted $\mathcal{T}$. Choose a point $x \in \mathcal{T}$. Note that for any point

$y = g \cdot x \in \mathcal{T}$, $g \in F_N$, and for any connected component $C \in \mathcal{C}_x$, the set $g \cdot C$ belongs to $\mathcal{C}_y$ and we have

$$\boldsymbol{\alpha}(g \cdot C) = \alpha(g) \cdot \boldsymbol{\alpha}(C).$$

Our point here is that the action of $\boldsymbol{\alpha}$ on connected components depends on the F_N-orbits of points, rather than the points themselves. Since there is only a finite number of orbits of branching points, we may choose a power d_0 of $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}^{d_0}$ (globally) maps any F_N-orbit of branching points to itself, and a multiple d of d_0 such that for any branching point $x \in \mathcal{T}$ and any $C \in \mathcal{C}_x$, we have $\boldsymbol{\alpha}^d(C) = \gamma \cdot C$ for some $\gamma \in F_N$.

We assume $d = 1$ from now on. Cyclic orders on $\mathcal{T}$ we will consider are built as follows: for any F_N-orbit of branching points, we choose a representative point x. We assign x a cyclic order σ_x and define, for any $g \in F_N$, a cyclic order at the point $g \cdot x$ by setting $\sigma_{g \cdot x}(g \cdot C) = g \cdot \sigma_x(C)$ for any $C \in \mathcal{C}_x$.

5. The Limit Set of a Positive iwip Automorphism

In [12], the authors define, for a given basis $\mathcal{A}$ (with $|\mathcal{A}| = N$) of the free group, the limit set $\Omega_\mathcal{A}$ and the heart $K_\mathcal{A}$ of an $\mathbb{R}$-tree T with an isometric, minimal, very small F_N-action that has dense orbits. The idea behind these definitions is to obtain a complete picture of the F_N-action on T by looking at the induced, partial action of F_N on a fundamental, compact domain of its metric completion $\overline{T}$. The heart $K_\mathcal{A}$ of T is simply the convex hull of the limit set $\Omega_\mathcal{A}$; it is a compact $\mathbb{R}$-tree. The limit set may or may not be equal to the heart; in any case, the limit set $\Omega_\mathcal{A}$ can be seen as the attractor of a system of partial isometries on $K_\mathcal{A}$ (see [8] for a very helpful example and [15] for a more general theory).

5.1. *Limit set*

Let α be a positive iwip automorphism of $F_\mathcal{A}$.

In [29], the authors define a map $Q : \partial F_\mathcal{A} \to \overline{T_\alpha^-} \cup \partial T_\alpha^-$ (where ∂T_α^- is the Gromov boundary of T_α^-) which gives a geometric interpretation of the action of $F_\mathcal{A}$ on its boundary $\partial F_\mathcal{A}$. The map Q is $F_\mathcal{A}$-equivariant (for all $w \in F_\mathcal{A}$, and all $Y \in \partial F_\mathcal{A}$, we have $Q(wY) = wQ(Y)$), onto, and satisfies the following property.

Proposition 5.1. ([29]) *Let Y be a point in $\partial F_\mathcal{A}$; for any point $x \in \overline{T_\alpha^-}$, if $(w_n)_n$ is a sequence of elements of $F_\mathcal{A}$ converging to Y, and if the sequence $(w_n \cdot x)_n$ converges to a point $y \in \overline{T_\alpha^-}$, then $y = Q(Y)$.*

The map Q is never one-to-one (see Proposition 5.5). However, a strong property was proven in [9].

Theorem 5.2. ([9]) *There are finitely many orbits of points of $\overline{T_\alpha^-} \cup \partial T_\alpha^-$ with more than two Q-preimages.*

Recall that α generates an attracting subshift $\Sigma_\alpha \subset \partial^2 F_\mathcal{A}$ whose projection on the second coordinate is $\Sigma_\alpha^+ \subset \partial F_\mathcal{A}$.

Definition 5.3. ([12]) The *limit set* $\Omega_\mathcal{A}^-$ of the repelling tree T_α^- is the subset of $\overline{T_\alpha^-}$ defined by $\Omega_\mathcal{A}^- = Q(\Sigma_\alpha^+)$.

Remark 5.4. This definition is somewhat nonstandard. Limit sets were first defined in [12] using the dual lamination of an $\mathbb{R}$-tree (see [11]). It is however known from [28] that the above definition is equivalent.

It will become necessary for us to work with bi-infinite trajectories. The transition is made by the following proposition, which also gives insight on Theorem 5.2.

Proposition 5.5. *For any point (X, Y) of Σ_α, we have $Q(X) = Q(Y)$.*

Remark 5.6. Note that the result is actually known: proofs can be found in [22] and [15]. Here, we give a proof using a combinatorial approach.

Proof. Let a be an element of $\mathcal{A}$. Since α is iwip, there exists, for any $n \in \mathbb{N}$, an integer k_n such that the word $\alpha^{k_n}(a)$ contains $(X_0 X_1 \cdots X_n)^{-1}(Y_0 Y_1 \cdots Y_n)$ as a subword. Assume that for any $n \in \mathbb{N}$, we have: $\alpha^{k_n}(a) = (X_0 X_1 \cdots X_n u_{(n)})^{-1}$ $(Y_0 Y_1 \cdots Y_n v_{(n)})$ where $u_{(n)}$ and $v_{(n)}$ are possibly empty words of $F_\mathcal{A}$. Define the sequences $x_{(n)} = (X_0 X_1 \cdots X_n u_{(n)})_n$ and $y_{(n)} = (Y_0 Y_1 \cdots Y_n v_{(n)})_n$ of F_N; they converge to X and Y respectively.

Recall that $\boldsymbol{\alpha}$ is a homothety with factor $\lambda_{\boldsymbol{\alpha}} < 1$; suppose that the point $P \in \overline{T_\alpha^-}$ is fixed by $\boldsymbol{\alpha}$. Both $(x_n \cdot P)_n$ and $(y_n \cdot P)_n$ are sequences of the compact $\overline{T_\alpha^-} \cup \partial T_\alpha^-$. There exists an increasing map γ such that the sequences $(x_n \cdot P)_{\gamma(n)}$ and $(y_n \cdot P)_{\gamma(n)}$ converge, and Proposition 5.1 tells us that they converge to $Q(X)$ and $Q(Y)$ respectively. Finally, we have

$$|x_{\gamma(n)} \cdot P, y_{\gamma(n)} \cdot P| = |P, \alpha^{k_{\gamma(n)}}(a) \cdot P| = \lambda_{\boldsymbol{\alpha}}^{k_{\gamma(n)}} |P, aP|$$

(where $|\cdot, \cdot|$ is the distance on T) which converges to 0 as n goes to $+\infty$. □

We now consider the map Q^2, equivariant for the partial action of $F_\mathcal{A}$ on Σ_α and onto, defined by:

$$Q^2 : \Sigma_\alpha \to \Omega_\mathcal{A}^-$$
$$(X, Y) \mapsto Q(Y).$$

Following from Theorem 5.2, there are only finitely many orbits of points of $\Omega_\mathcal{A}^-$ with more than one Q^2-preimage. Moreover, we prove the following lemma for future use.

Lemma 5.7. *Any point of T has a finite number of Q^2-preimages.*

Proof. Let y be a point in T and $Stab(y)$ the set of words $w \in F_\mathcal{A}$ such that $w \cdot y = y$. We denote by $\partial Stab(y)$ the Gromov boundary of $Stab(y)$ and

$Q_r^{-1}(\{y\}) = Q^{-1}(\{y\})\backslash\partial Stab(y)$. Following [9, Sec. 5.2], we define $ind_Q(y) = \#(Q_r^{-1}(\{y\})/Stab(y)) + 2rank(Stab(y)) - 2$. It follows from [9, Theorem 5.3] that $ind_Q(y) \leq 2|\mathcal{A}| - 2$, which implies that $\#(Q_r^{-1}(\{y\})/Stab(y))$ is finite. Now suppose there is a point $Y \in \Sigma_\alpha^+ \cap Q_r^{-1}(\{y\})$ such that the set $O_y(Y) = \{wY; w \in Stab(y)\} \cap \Sigma_\alpha^+$ contains an infinite number of points. Then we can find a sequence (Y_n) of points of $O_y(Y)$ converging in $\Sigma_\alpha^+ \cap \partial Stab(y)$. This set is, however, empty as α is iwip; an easy way to support this is to check that for some $a \in \mathcal{A}$ and $k \in \mathbb{N}$, no point of $\partial Stab(y)$ can contain the word $\alpha^k(a)$. $\square$

The system of partial isometries on $\Omega_{\mathcal{A}}^-$ induced by the $F_{\mathcal{A}}$-action on $\overline{T_\alpha^-}$ can be described in a somewhat combinatorial fashion. For any element a of $\mathcal{A}$, we consider the isometry induced by the element a^{-1}:

$$a^{-1} : \{Q^2(X,Y); (X,Y) \in \Sigma_\alpha, Y_0 = a\} \rightarrow \{Q^2(a^{-1}X, a^{-1}Y); (X,Y) \in \Sigma_\alpha, Y_0 = a\}$$

$$x \mapsto a^{-1} \cdot x.$$

Our construction requires the limit set $\Omega_{\mathcal{A}}^-$ to be a finite forest (in the sense of Sec. 3.3). Such a property is equivalent to having an automorphism with maximal index: this is algorithmically checkable for positive automorphisms (see [26]).

Theorem 5.8. ([10]) *Let α be an iwip automorphism of F_N and let $\mathcal{A}$ be a basis of F_N. The limit set $\Omega_{\mathcal{A}}^-$ of the repelling tree T_α^- is a finite forest if and only if $ind(\alpha) = N - 1$.*

Note that conditions which ensure connectedness of the limit set are unknown at this point. We conjecture that positivity of the automorphism is a sufficient condition. Since we have no proof, we treat the case of finite forests.

5.2. *Exchange of pieces*

We now assume that α is a positive iwip automorphism of $F_{\mathcal{A}}$ with maximal index; the limit set of its repelling tree associated to the basis $\mathcal{A}$ is simply denoted T. The attracting subshift (see Sec. 2.2) of α is denoted Σ_α. Recall that the map $Q^2 : \Sigma_\alpha \rightarrow T$ is onto. For any $a \in \mathcal{A}$, we define

- $\Sigma_\alpha(a) = \{(X,Y) \in \Sigma_\alpha, Y_0 = a\}$,
- $\Sigma_\alpha(a^{-1}) = \{(X,Y) \in \Sigma_\alpha, X_0 = a^{-1}\}$,
- $T_a = Q^2(\Sigma_\alpha(a))$,
- $T_{a^{-1}} = Q^2(\Sigma_\alpha(a^{-1})) = a^{-1} \cdot T_a$,

where a^{-1} is the isometry induced by the $F_{\mathcal{A}}$-action on $\overline{T_\alpha^-}$. It follows from the compactness of the sets $\Sigma_\alpha(a)$ (respectively, $\Sigma_\alpha(a^{-1})$) that T_a (respectively, $T_{a^{-1}}$) are compact subsets of T for any $a \in \mathcal{A}$. We now give an alternative useful definition of the sets T_a and deduce that they are finite forests.

Proposition 5.9. *For any $a \in \mathcal{A}$, we have $T_a = \{x \in T; a^{-1} \cdot x \in T\}$. Moreover, T_a is a finite forest.*

Proof. The inclusion $T_a \subset \{x \in T; a^{-1} \cdot x \in T\}$ is a direct consequence of the equivariance of Q^2. Let z be a point of T such that $a^{-1} \cdot z$ is also in T. We can choose a small enough compact $\mathbb{R}$-tree t_z containing z and such that $a^{-1} \cdot t_z \subset T$. Any point x of $a^{-1} \cdot t_z$ has at least two Q-preimages (Proposition 5.5); moreover, if $a \cdot x$ is not a point of T_a, then there is $b \in \mathcal{A}$, $b \neq a$ and $Y \in \Sigma_\alpha^+$ with $Y_0 = b$ such that $Q(Y) = a \cdot x$ and $Q(a^{-1}Y) = x$, giving x a third Q-preimage. It follows from Theorem 5.2 that there must be a subset of T_a that is dense in t_z. Compactness of T_a implies that every point of t_z, including z, is in T_a.

To prove that T_a is a finite forest, we simply deduce from the first part of the proof that if two points x and y of T_a are such that we have both $[x, y] \subset T$ and $[a^{-1} \cdot x, a^{-1} \cdot y] \subset T$, then $[x, y] \subset T_a$. Since T is a finite forest, this happens for all but a finite number of pairs. □

It immediately follows that $T_{a^{-1}} = \{x \in T; a \cdot x \in T\}$ and that $T_{a^{-1}}$ is a finite forest for any $a \in \mathcal{A}$. It will be essential for our construction that the sets $\{T_a, a \in \mathcal{A}\}$ and $\{T_{a^{-1}}, a \in \mathcal{A}\}$ be partitions of T up to a finite number of points.

Proposition 5.10. *For all $a, b \in \mathcal{A}$, $a \neq b$, the set $T_a \cap T_b$ (respectively, $T_{a^{-1}} \cap T_{b^{-1}}$) contains at most a finite number of points.*

Proof. Since T_a and T_b are finite forests, the set $T_a \cap T_b$ can only contain an infinite number of points if at least one of its connected components is a nontrivial compact $\mathbb{R}$-tree. This contradicts Theorem 5.2. □

Observe that if T only has one connected component (it is an $\mathbb{R}$-tree), then each T_a is also connected and the sets $T_a \cap T_b$ contain at most one point.

Let $\mathcal{S}$ and $\mathcal{S}^-$ be the sets of forward and backward singularities defined by

$$\mathcal{S} = \bigcup_{\substack{a,b \in \mathcal{A} \\ a \neq b}} (T_a \cap T_b) \quad \text{and} \quad \mathcal{S}^- = \bigcup_{\substack{a,b \in \mathcal{A} \\ a \neq b}} (T_{a^{-1}} \cap T_{b^{-1}}).$$

We need to take care of the fact that T may have more than one connected component. We consider the set of discontinuities of T to be another set of singularities:

$$\mathcal{S}_\mathcal{D} = \{x \in T; \exists \, y \in T;]x, y[\cap T = \emptyset\}.$$

The reader is referred to Remark 7.4 for some insight on those additional singularities.

If x is a point of $T \backslash \mathcal{S}$, then there is a unique $a \in \mathcal{A}$ such that $x \in T_a$. The system of partial isometries gives rise to a map $f : T \backslash \mathcal{S} \to T$ defined, for every x by $f(x) = a^{-1} \cdot x$ if $x \in T_a$. We now define the set

$$\mathcal{S}^* = \mathcal{S} \cup f^{-1}(\mathcal{S}^-) \cup \mathcal{S}_\mathcal{D}.$$

Notice that the sets $\mathcal{S}$, $\mathcal{S}^-$ and $\mathcal{S}_\mathcal{D}$ are finite. Moreover, it is easy to see that for any point of $f(T \backslash \mathcal{S})$, the set of f-preimages cannot contain more than $|\mathcal{A}|$ elements. We deduce that $\mathcal{S}^*$ is also a finite set.

We state now straightforward result for future use:

Lemma 5.11. *For all tree $V \subset T$ with $V \cap (\mathcal{S}^*) = \emptyset$, the map $f_{|V}$ is a homeomorphism that is also an isometry.*

5.3. Induced actions of $F_{\mathcal{A}}$ and α on the connected components of T

Recall that the limit set T is a subset of the metric completion $\mathcal{T}$ of the repelling tree of α. For any point x of $\mathcal{T}$, the set $\mathcal{C}_x^{\mathcal{T}}$ contains the connected components of $\mathcal{T} \backslash \{x\}$ and we assume a cyclic order $\sigma_x^{\mathcal{T}}$ has been defined as in Sec. 4.2. For any $x \in T$, we define $\mathcal{C}_x = \{C \cap T; C \in \mathcal{C}_x^{\mathcal{T}}$ and $C \cap T \neq \emptyset\}$ and we denote σ_x the cyclic order on $\mathcal{C}_x$ induced by $\sigma_x^{\mathcal{T}}$. Observe that there are points $x \in T$ such that $|\mathcal{C}_x^{\mathcal{T}}| > |\mathcal{C}_x|$: indeed when we cut $\mathcal{T}$, cutting points lose degree and hence have less connected components around.

Observe that there exist points $x \in T_a$, $a \in \mathcal{A}$, such that $|\mathcal{C}_{a^{-1}.x}| \neq |\mathcal{C}_x|$. Specifically, for any $a \in \mathcal{A}$ and any $x \in T_a$, we have $|\mathcal{C}_{a^{-1}.x}| \leq |\mathcal{C}_x|$ (respectively, $|\mathcal{C}_{a^{-1}.x}| \geq |\mathcal{C}_x|$) if $x \in \mathcal{S}$ (respectively, $x \in f^{-1}(\mathcal{S}^-)$) and $|\mathcal{C}_{a^{-1}.x}| = |\mathcal{C}_x|$ otherwise.

Let x be a point of T and let g be a word of $F_{\mathcal{A}}$ such that $g \cdot x$ is still in T. For any $C \in \mathcal{C}_x$, there exist $V \in \mathcal{C}_x^{\mathcal{T}}$ such that $C = V \cap T$; we define $g \star C = g \cdot V \cap T$ and $\alpha_\star(C) = \alpha(V) \cap \alpha(T)$. Note that $g \star C$ (respectively, $\alpha_\star(C)$) may very well be empty (since we may have $|\mathcal{C}_{g \cdot x}| < |\mathcal{C}_x|$), and as such, not belong to $\mathcal{C}_{g \cdot x}$.

6. IET on the Contour

We still consider α to be a positive iwip automorphism of $F_{\mathcal{A}}$ with maximal index, T is the limit set (associated to the basis $\mathcal{A}$) of its repelling tree, $f : T \backslash \mathcal{S}^* \to T$ (where $\mathcal{S}^*$ are singularities) is the map induced by the system of partial isometries on T, and a cyclic order σ_x is defined on each point x of T as in Secs. 4.2 and 5.3. We are going to extend f to an IET on the contour $\tilde{T}$ of T.

6.1. Contour of the limit set

First of all, we have to check that the assumptions under which we know how to construct the contour (see Sec. 3) are fulfilled.

6.1.1. Assumption (H_0)

Follows from compactness of Σ_α^+ and Proposition 5.1.

6.1.2. Assumption (H_1)

The automorphism requires additional properties in order for (H_1) to be satisfied. In fact, considering α is a homothety, it is easy to see that the limit set T can have 0, 1, or infinitely many branching points. Our construction could work in any

of these settings (with a few minor tweaks), but we obviously would not benefit from it in the first two cases. It is proven in [10] that (when the index of α is maximal) there are infinitely many branching points in T if and only if the index of the inverse automorphism α^{-1} is not maximal. We assume we are in this case from now on; the automorphism α is then called *parageometric* (compare [20, 23, 10]). Density of branching points in T is a consequence of the density of F_N-orbits in $\mathcal{T}$ (Theorem 4.1). Countability follows immediately from Theorem 4.2. Finally, we deduce that the set of degrees of points of T is bounded from the construction of $\mathcal{T}$ given in [20].

6.1.3. *Assumption* (H_2)

Let $\mathcal{H}^\delta$ be the Hausdorff measure associated to the Hausdorff dimension δ of T. We refer to [18] for an introduction to these concepts. The Hausdorff measure is obviously invariant by the system of isometries. The understanding of this measure is greatly simplified by the theorem of [15] stated below. Recall that (Σ_α, S) and (Σ_α^+, S) (where Σ_α is the attracting subshift, Σ_α^+ is the projection on its second coordinate, and S is the shift map (Sec. 2.2)) are minimal and uniquely ergodic dynamical systems ([32]).

Remark 6.1. It follows that the positive orbit(s) of any point of T under the action of the system of partial isometries is dense in T.

Define μ_α as the unique probability measure invariant by S.

Theorem 6.2. ([15]) *Let T_0 be any subset of T and let Σ_0 be its Q^2-preimage. Then $\mathcal{H}^\delta(T_0)$ is equal (up to a multiplicative constant) to $\mu_\alpha(\Sigma_0)$.*

We assume that α is parageometric. We get from [23] that the expanding factor λ_α of (the train-track map ([7]) representing the outer class of) α is different from the expanding factor $\lambda_{\alpha^{-1}}$ of (the train-track map representing the outer class of) α^{-1}. It is proven in [15] that the Hausdorff dimension δ of T is given by

$$\delta = \frac{\ln(\lambda_\alpha)}{\ln(\lambda_{\alpha^{-1}})}.$$

As T is a finite forest (with an infinite number of points), it contains an interval, and its Hausdorff dimension must be ≥ 1. As α is parageometric, we have $\delta > 1$. We deduce that $\mathcal{H}^\delta$ does not charge any point or arc.

Suppose that there is a point $x \in T$ and a sequence $(x_n)_n$ of T converging to x such that $\lim_{n \to +\infty} \mathcal{H}^\delta(]]x, x_n[[) > 0$. We will have similar results for every point of (one of) the positive orbit of x (we can simply avoid the singularity problem by taking a subsequence). As $\mathcal{H}^\delta(T)$ is finite, this can only happen if the positive orbit of x is periodic, which contradicts its density in T.

Finally, it is easy to see that any (non-empty, nontrivial) ball in T is charged by $\mathcal{H}^\delta$. Choose an integer k so that $\boldsymbol{\alpha}^k(T)$ is small enough to fit in the ball, then use

the density in T of the positive orbit of the fixed point of $\boldsymbol{\alpha}$. If $f^n(\boldsymbol{\alpha}^k(T))$ contains a singularity x for some $n \in \mathbb{N}$, we can simply take the intersection of $f^n(\boldsymbol{\alpha}^k(T))$ with one of the connected component of $T \backslash \{x\}$ and carry on. We conclude that any $C \in \mathcal{C}_x$ for any $x \in T$ is charged.

6.1.4. *Assumption* (H_3)

We prove here that for any positive iwip parageometric automorphism of the free group, (H_3) is satisfied regardless of the cyclic order chosen (as long as it is chosen according to Secs. 4.2 and 5.3).

Observe that, regardless of the number of connected components of T, if (H_3) is not satisfied, there exist two points $x \neq y$ belonging to the same connected component of T ($[x, y] \subset T$) such that for all branching point z in $[x, y]$, $\sigma_z(C_z(x)) = C_z(y)$.

Lemma 6.3. *Let $x \neq y$ belong to T such that $[x, y] \subset T$ and for all branching point z in $[x, y]$, $\sigma_z(C_z(x)) = C_z(y)$. Then, the return map of f to $[x, y]$ is an IET.*

From this lemma we deduce that if (H_3) is not satisfied and hence such points x and y exist, then f itself is an IET (on a finite union of intervals) and the number of branching is indeed finite (minimality). More precisely: the orbit of the interval $[x, y]$ is a finite union of intervals and its closure is the whole limit set. So we are back to a situation already treated.

Proof. We consider the interval $I = [x, y]$ in the tree T; we want to define the return map of f to I.

Step 1. We claim that the images of I can split only a finite number of times before coming back to I because, in order to split, it must meet a singularity and there are finitely many of them. To say it formally, we claim that there is a finite partition $\{I_1, \ldots, I_d\}$ of I such that for all $i \in \{1, \ldots, d\}$ and all $n \in \mathbb{N}$ such that for all $j < n$, $f^j(I_i) \cap I$ does not contain an interval, there is $g = g(i, n) \in F_N$ with, for all $u \in I_i$, $f^n(u) = gu$.

Let us prove the claim. We say that a point $z \in I$ is regular if there is $\epsilon > 0$ such that if $I_z^\epsilon = I \cap B(z, \epsilon)$ and n is small enough so that for all $1 \leq j < n$, $f^j(I_z^\epsilon) \cap I$ does not contain an interval; we then have, for all $u \in I_z^\epsilon$, $f^n(u) = gu$ for some $g \in F_N$. Observe that if $z \in I$ is not regular then, there must exist j (take the smallest) such that $f^j(z) = s$ is a singularity; for all $\epsilon > 0$ (small enough), $f^j(I_z^\epsilon)$ is an interval around s (and for $1 \leq j' \leq j$, $f^{j'}(I_z^\epsilon)$ does not overlap I). Assume s is of degree d; there are at most $\lfloor d/2 \rfloor$ non-overlapping intervals containing s; hence there are at most $\lfloor d/2 \rfloor$ such points z. This remark immediately implies that there are only finitely many non-regular points. Connected components of regular points yield the desired partition.

Step 2. Consider one of these pieces; i.e. let $i \in \{1, \ldots, d\}$. We claim that there is $n \in \mathbb{N}$ such that $f^n(I_i) \cap I$ contains an interval. Otherwise the orbit of I_i would

never get cut; hence we would have an interval for which the coding is constant. This is impossible since the partition is generating and the self-similarity shows that cylinders get smaller and smaller (with respect to the distance, not only the measure).

Step 3. Let $i \in \{1, \ldots, d\}$ and n be the smallest integer such that $f^n(I_i) \cap I$ contains an interval. We claim that $f^n(I_i) \cap [[x, y]] = f^n(I_i) \cap [x, y]$; or, in other words that, either $f^n(I_i) \subset I$, or, $f^n(I_i)$ contains one end of I and I contains one end of $f^n(I_i)$. Still in other words, noting $f^n(I_i) = [x_i, y_i]$, either $f^n(I_i) \cap I = [x_i, y_i]$, or $f^n(I_i) \cap I = [x, y_i]$, or $f^n(I_i) \cap I = [x_i, y]$. Observe that this last alternative can occur only for two indices, since I has only two end points. That is where we use the fact that (H_3) is not satisfied: consider the boundary points of $f^n(I_i) \cap I$ (one on the side of x, the other on the side of y). We claim that these boundary points are either the boundary points of $f^n(I_i)$ or x or y. Otherwise there would be a branching point z of T, $z \in I_i$ such that one of $f^n([x_i, z])$ and $f^n([z, y_i])$ is in I and the other is not (while still being in $[[x, y]]$). But this cannot happen since,

(0) recall that there is some $g \in F_N$ such that on I_i, $f^n(z) = g \cdot z$;
(i) by definition of the cyclic order, for all $C \in \mathcal{C}_z$, $\sigma_{g \cdot z}(g \star C) = g \star \sigma_z(C)$;
(ii) since z and $g \cdot z$ belong to I, $\sigma_z(C_z(x)) = C_z(y)$ and $\sigma_{g \cdot z}(C_{g \cdot z}(x)) = C_{g \cdot z}(y)$;
(iii) hence applying (i), $\sigma_{g \cdot z}(g \star C_z(x)) = g \star \sigma_z(C_z(x))$;
(iv) we deduce using (ii) that $g \star C_z(x) = C_{g \cdot z}(x)$, if and only if $g \star C_z(y) = C_{g \cdot z}(y)$.

It follows that the only possibilities are those proposed.

Step 4. Except for maybe two intervals that shall get cut at the boundaries of I, the other ones come back completely in I after n_i steps. For the parts that get cut and finish outside of I, we apply again the same arguments: they will eventually intersect I and "fill" the remaining space (but if the boundaries are already occupied, they will entirely come back inside I). Finally, we have proved that the return map to I is a piecewise isometry of an interval; it is bijective; hence it is an interval exchange transformation. □

6.2. *The IET*

We define $\tilde{\mathcal{S}}^* = \pi^{-1}(\mathcal{S}^*)$ (see Sec. 5.2). For all $\tilde{x} = (x, C) \in \tilde{T} \backslash \tilde{\mathcal{S}}^*$ and a such that $x \in T_a$, we set

$$\tilde{f}(\tilde{x}) = (a^{-1} \cdot x, a^{-1} \star C).$$

We extend $\tilde{f}$ at all singularities so that it is d_+-continuous in the following sense: For any $\tilde{x} \in \tilde{\mathcal{S}}^*$ and any sequence $(\tilde{x}_n)_n$ such that $\lim_{n \to +\infty} d_+(\tilde{x}_n, \tilde{x}) = 0$, we set $\tilde{f}(\tilde{x}) = \lim_{n \to +\infty} \tilde{f}(\tilde{x}_n)$.

Proposition 6.4. *The map $\tilde{f}$ is an interval exchange transformation on the circle.*

Proof. Observe any connected component of $T\backslash\mathcal{S}^*$ is included in T_a for some a, and the connected components of $\tilde{T}\backslash\tilde{\mathcal{S}}^*$ are (open) intervals. Denote $I_1, \ldots, I_d$ these intervals. We simply prove that for all $1 \leq k \leq d$, the restriction of $\tilde{f}$ to I_k is a translation.

Choose a connected component I_k of $\tilde{T}\backslash\tilde{\mathcal{S}}^*$ and let $a \in \mathcal{A}$ be such that the projection of I_k is in T_a. Consider two points $\tilde{x}$ and $\tilde{y}$ of I_k; up to switching $\tilde{x}$ and $\tilde{y}$, we can assume $d_+(\tilde{x}, \tilde{y}) < |I_k|$. Let C be an element of $\mathcal{C}^+(\tilde{x}, \tilde{y})$ (see Sec. 3.2). Observe that C does not contain any point of $\mathcal{S}^*$ (otherwise I_k would contain a point of $\tilde{\mathcal{S}}^*$) and deduce that we have $a^{-1} \star C = a^{-1} \cdot C$.

Since the degree of any point $z \in [\pi(\tilde{x}), \pi(\tilde{y})]$ is equal to the degree of $a^{-1} \cdot z$ (see discussion in Sec. 5.3) and since the cyclic order was chosen according to Sec. 4.2, we obtain

$$d_+(\tilde{x}, \tilde{y}) = \sum_{C \in \mathcal{C}^+(\tilde{x}, \tilde{y})} \mathcal{H}^\delta(C) = \sum_{C \in \mathcal{C}^+(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y}))} \mathcal{H}^\delta(C) = d_+(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})). \qquad \square$$

Note that the translation is given by $d_+(\tilde{x}, \tilde{f}(\tilde{x}))$ for any $\tilde{x}$ of I_k.

At this stage, we have constructed an IET suitable for the proof of Theorem 1.3. Let $\tilde{\mathcal{A}} = \{1, \ldots, d\}$ and Π map $k \in \tilde{\mathcal{A}}$ to the letter $a \in \mathcal{A}$ such that the projection of I_k is in T_a. Let λ and π be the length vector and the permutation associated with $\tilde{f}$. The theorem follows from the fact that, by construction, the dynamics of $\tilde{f}$ on $\tilde{T}$ factors on that of the exchange of pieces on T.

7. Homothety

We keep the notations and assumptions of the previous section. Recall the automorphism α acts on the limit set T as a contracting homothety $\boldsymbol{\alpha}$ with factor $\lambda_{\boldsymbol{\alpha}}$. In this section, we study the homothety $\boldsymbol{\alpha}$ and define a lift $\tilde{\boldsymbol{\alpha}}$ on $\tilde{T}$.

We denote R the boundary of $\boldsymbol{\alpha}(T)$ in T:

$$R = \boldsymbol{\alpha}(T) \cap \overline{T\backslash\boldsymbol{\alpha}(T)}.$$

Proposition 7.1. *The boundary R of $\boldsymbol{\alpha}(T)$ in T is a finite set.*

Proof. Let z be a point of R. Let $(z_n)_n$ and $(z_n')_n$ be two sequences of T converging to z and such that for all $n \in \mathbb{N}$, $z_n \in \boldsymbol{\alpha}(T)$ and $z_n' \in T\backslash\boldsymbol{\alpha}(T)$. The first step is to use Theorem 5.2 to prove that there is only a finite number of orbits that can contain z. The transition to the attracting subshift is made with the following lemma.

Lemma 7.2. ([15]) *For any point $Z \in \Sigma_\alpha$, we have $Q^2(\partial^2\alpha(Z)) = \boldsymbol{\alpha}(Q^2(Z))$.*

We deduce we can choose a sequence $(Z_n)_n$ of $\partial^2\alpha(\Sigma_\alpha)$ and a sequence $(Z_n')_n$ of $\Sigma_\alpha\backslash\partial^2\alpha(\Sigma_\alpha)$ such that, for every $n \in \mathbb{N}$, $Q^2(Z_n) = z_n$ and $Q^2(Z_n') = z_n'$. Both sets $\partial^2\alpha(\Sigma_\alpha)$ and $\Sigma_\alpha\backslash\partial^2\alpha(\Sigma_\alpha)$ are easily seen as closed (and therefore compact), as deciding if a point (X, Y) is in one or the other comes down to studying the first

few letters of X and Y. We can then assume that the sequences $(Z_n)_n$ and $(Z'_n)_n$ converge, and Proposition 5.1 gives us two distinct Q^2-preimages for z.

Theorem 5.2 then tells us that there is only a finite number of orbits that can contain z. To conclude the proof, we need to check that only a finite number of points of the same orbit can be in $Q^2(\partial^2\alpha(\Sigma_\alpha)) \cap Q^2(\Sigma_\alpha\backslash\partial^2\alpha(\Sigma_\alpha))$. Let z be a point in R. Suppose we have a sequence $(z_n)_n$ of pairwise distinct points that are both on the orbit of z and in R. From Lemma 5.7, we can assume that there are two sequences $(Z_n)_n$ and $(Z'_n)_n$ of points of Σ_α such that all Z_n (respectively, Z'_n) are on the same S-orbit (recall that S is the shift map), and for all $n \in \mathbb{N}$, we have $Z_n \in \partial^2\alpha(\Sigma_\alpha)$, $Z'_n \in \Sigma_\alpha\backslash\partial^2\alpha(\Sigma_\alpha)$ and $Q^2(Z_n) = Q^2(Z'_n) = z_n$. This can only happen if for any $n \in \mathbb{N}$, the points Z_n and Z'_n agree on one of their coordinates. It follows that for any integer p, we will find an integer m such that for $Z_m = (X_{(m)}, Y_{(m)})$ and $Z'_m = (X'_{(m)}, Y'_{(m)})$, the first p letters of $X_{(m)}$ (respectively, $Y_{(m)}$) are equal to the first p letters of $X'_{(m)}$ (respectively, $Y'_{(m)}$). This contradicts the fact that for all $n \in \mathbb{N}$, we have $Z_n \in \partial^2\alpha(\Sigma_\alpha)$ and $Z'_n \in \Sigma_\alpha\backslash\partial^2\alpha(\Sigma_\alpha)$. □

For any positive integer n, we denote

$$R_n = \boldsymbol{\alpha}^n(T) \cap \overline{T\backslash\boldsymbol{\alpha}^n(T)}.$$

We will require (Sec. 9) $|R_n|$ to be uniformly bounded. Define m as the smallest integer such that $R_m \cap R_{2m} = \emptyset$ (such an integer exists because $\boldsymbol{\alpha}$ is a contracting homothety and the fixed point of $\boldsymbol{\alpha}$ cannot be in any R_n).

Proposition 7.3. *For any $n \geq m$, we have $|R_n| = |R_m|$.*

Proof. First, observe that for any positive integer n, we have $|R_{n+1}| \geq |R_n|$ since R_{n+1} contains at least the images by $\boldsymbol{\alpha}$ of the elements of R_n. Hence, we only need to prove that for any integer $k > 0$, we have $|R_{(k+1)m}| \leq |R_{km}|$. Let x be a point of $R_{(k+1)m}$: there exist $C \in \mathcal{C}_x$ and a neighborhood V of x such that $V \cap C$ is a non-empty subset of $T\backslash\boldsymbol{\alpha}^{(k+1)m}(T)$. Observe that x is a point of $\boldsymbol{\alpha}^{km}(T)$ but not of R_n by our hypothesis, and deduce $V \cap C \subset \boldsymbol{\alpha}^{km}(T)$ and $\boldsymbol{\alpha}^{-m}(V \cap C) \subset T\backslash\boldsymbol{\alpha}^{km}(T)$ if V is small enough. The closure of $\boldsymbol{\alpha}^{-m}(V \cap C)$ contains $\boldsymbol{\alpha}^{-m}(x)$ and we deduce $\boldsymbol{\alpha}^{-m}(x) \in R_{km}$. □

From now on we will simply assume that $R_1 \cap R_2 = \emptyset$, which implies $|R_n| = |R_1|$ for any positive integer n. Observe that it also implies $R_{n+1} = \boldsymbol{\alpha}(R_n)$ for any positive integer n. We set $R = R_1$.

Remark 7.4. The set $\mathcal{S}_{\mathcal{D}}$ of discontinuities of T (see Sec. 3.3) is a subset of $\boldsymbol{\alpha}^{-1}(R)$. This is easily seen since for any two points x and y of T such that $]x, y[\cap T$ is empty, we have $]\boldsymbol{\alpha}(x), \boldsymbol{\alpha}(y)[\subset T$ (making $\boldsymbol{\alpha}(x)$ and $\boldsymbol{\alpha}(y)$ points of R).

Observe that $|\mathcal{C}_x| \leq |\mathcal{C}_{\boldsymbol{\alpha}(x)}|$ if $x \in \boldsymbol{\alpha}^{-1}(R)$ and $|\mathcal{C}_x| = |\mathcal{C}_{\boldsymbol{\alpha}(x)}|$ otherwise. The homothety $\boldsymbol{\alpha}$ lifts to the homothety $\tilde{\alpha}$ defined for any point $(x, C) \in \tilde{T}$

by $\tilde{\boldsymbol{\alpha}}(x, C) = (\boldsymbol{\alpha}(x), \boldsymbol{\alpha}_\star(C))$ (see Sec. 5.3). We observe that our choice of cyclic order (Secs. 4.2 and 5.3) implies that, for all $x \in T \backslash \boldsymbol{\alpha}^{-1}(R)$ and all $V \in \mathcal{C}_x$, $\sigma_{\boldsymbol{\alpha}(x)}(\boldsymbol{\alpha}_\star(V)) = \boldsymbol{\alpha}_\star(\sigma_x(V))$.

Proposition 7.5. *The set $\tilde{\boldsymbol{\alpha}}(\tilde{T})$ is a union of intervals; the map $\tilde{\boldsymbol{\alpha}}$ is a piecewise uniform contraction with factor $\lambda_{\boldsymbol{\alpha}}$.*

Proof. Observe that the set $\tilde{T} \backslash \pi^{-1}(\boldsymbol{\alpha}^{-1}(R))$ is a union of intervals, say $K_1, \ldots, K_D$. We simply prove the restriction of $\tilde{\boldsymbol{\alpha}}$ to (the interior of) these subintervals is a uniform contraction. The proof is very analogous to that of Proposition 6.4.

Choose a connected component K_k of $\tilde{T} \backslash \pi^{-1}(\boldsymbol{\alpha}^{-1}(R))$. Consider two points $\tilde{x}$ and $\tilde{y}$ of K_k; up to switching $\tilde{x}$ and $\tilde{y}$, we can assume $d_+(\tilde{x}, \tilde{y}) < |K_k|$.

Since the degree of any point $z \in [\pi(\tilde{x}), \pi(\tilde{y})]$ is equal to the degree of $\boldsymbol{\alpha}(z)$ and since the cyclic order was chosen according to Secs. 4.2 and 5.3, we obtain

$$d_+(\tilde{\boldsymbol{\alpha}}(\tilde{x}), \tilde{\boldsymbol{\alpha}}(\tilde{y})) = \sum_{C \in \mathcal{C}^+(\tilde{\boldsymbol{\alpha}}(\tilde{x}), \tilde{\boldsymbol{\alpha}}(\tilde{y}))} \mathcal{H}^\delta(C) = \sum_{C \in \mathcal{C}^+(\tilde{x}, \tilde{y})} \mathcal{H}^\delta(\boldsymbol{\alpha}_\star(C))$$

(see Secs. 3.2 and 5.3 for the definitions of $\mathcal{C}^+(\tilde{x}, \tilde{y})$ and $\boldsymbol{\alpha}_\star$). For any $C \in \mathcal{C}^+(\tilde{x}, \tilde{y})$, the set $\boldsymbol{\alpha}_\star(C)$ cannot contain any point of R (otherwise K_k would contain a point of $\pi^{-1}(\boldsymbol{\alpha}^{-1}(R))$) which implies $\boldsymbol{\alpha}_\star(C) = \boldsymbol{\alpha}(C)$. Also, the equality $\mathcal{H}^\delta(\boldsymbol{\alpha}(V)) = \lambda_{\boldsymbol{\alpha}} \mathcal{H}^\delta(V)$ for any measurable set $V \subset T$ is a direct consequence of the definition of the Hausdorff measure, and we deduce $d_+(\tilde{\boldsymbol{\alpha}}(\tilde{x}), \tilde{\boldsymbol{\alpha}}(\tilde{y})) = \lambda_{\boldsymbol{\alpha}} d_+(\tilde{x}, \tilde{y})$. $\square$

8. Induction

Let T_1 be the set of points z of $\boldsymbol{\alpha}(T)$ such that there is an integer $k(z)$ satisfying:

- for all $0 \leq k < k(z)$, $f^k(z) \in T \backslash \mathcal{S}$,
- for all $0 \leq k \leq k(z)$, $f^k(z) \in T \backslash R$,
- $k(z)$ is the smallest positive integer such that $f^{k(z)}(z) \in \boldsymbol{\alpha}(T)$.

Let $f_1 : T_1 \to \boldsymbol{\alpha}(T)$ be the first-return map on $\boldsymbol{\alpha}(T)$ induced by f.

Proposition 8.1. *For all $z \in T_a$ such that $\boldsymbol{\alpha}(z) \in T_1$, we have: $f_1(\boldsymbol{\alpha}(z)) = \alpha(a^{-1})\boldsymbol{\alpha}(z)$.*

Proof. Let z be a point in T_a, and $Z \in \Sigma_\alpha(a)$ a Q^2-preimage of z. The point $Q^2(\partial^2 \alpha(Z))$ is in $\boldsymbol{\alpha}(T)$ and so is the point $Q^2(\partial^2 \alpha(a^{-1}Z)) = (\alpha(a))^{-1}Q^2(\partial^2 \alpha(Z))$. We assume $\boldsymbol{\alpha}(z) = Q^2(\partial^2 \alpha(Z))$ is a point of T_1, and we prove $k(z) = |\alpha(a)|$.

Suppose $\alpha(a) = vw$, and v is a proper prefix of $\alpha(a)$ such that $v^{-1}Q^2(\partial^2 \alpha(Z)) \in \boldsymbol{\alpha}(T)$. The point $v^{-1}\partial^2 \alpha(Z)$ is obviously in Σ_α, as $\partial^2 \alpha(Z)$ is in $\Sigma_\alpha(\alpha(a))$. If $v^{-1}\partial^2 \alpha(Z)$ is also a point of $\partial^2 \alpha(\Sigma_\alpha)$, then both $\alpha^{-1}(v)$ and $\alpha^{-1}(w)$ are non-empty words of $F_\mathcal{A}$ with letters in $\mathcal{A}$; this contradicts $\alpha(a) = vw$. If the point $v^{-1}\partial^2 \alpha(Z)$

is not in $\partial^2\alpha(\Sigma_\alpha)$, then $Q^2(v^{-1}\partial^2\alpha(Z))$ has another Q^2-preimage $Z' \in \partial^2\alpha(\Sigma_\alpha)$. In that case, the point $Q^2(v^{-1}\partial^2\alpha(Z))$ is in R, which contradicts $\boldsymbol{\alpha}(z) \in T_1$. $\quad\square$

Corollary 8.2. *For all $z \in T\backslash\mathcal{S}$ such that $\boldsymbol{\alpha}(z) \in T_1$, we have $f_1(\boldsymbol{\alpha}(z)) = \boldsymbol{\alpha}(f(z))$.*

We prove this conjugacy lifts to a conjugacy on $\tilde{T}$. Recall that the map $\tilde{f}$ is defined for any point (x, C) of $\tilde{T}\backslash\tilde{\mathcal{S}}^*$ by $\tilde{f}(x, C) = (a^{-1} \cdot x, a^{-1} \star C)$ if $x \in T_a$, and was extended to all singularities so that it is d_+-continuous (see Sec. 6). Also recall that the map $\tilde{\boldsymbol{\alpha}}$ is defined for all $(x, C) \in \tilde{T}$ by $\tilde{\boldsymbol{\alpha}}(x, C) = (\boldsymbol{\alpha}(x), \boldsymbol{\alpha}_\star(C))$ (see Sec. 7).

We define the set $\tilde{T}_1$ as the set of points (x, C) of $\tilde{T}$ such that $x \notin \mathcal{S}$ and $\boldsymbol{\alpha}(x) \in T_1$. We note that the set $\tilde{T}\backslash\tilde{T}_1$ is finite.

Proposition 8.3. *The first-return map $\tilde{f}_1$ induced by $\tilde{f}$ on $\tilde{\boldsymbol{\alpha}}(\tilde{T})$ satisfies, for any point $\tilde{x} \in \tilde{T}_1$,*

$$\tilde{f}_1(\tilde{\boldsymbol{\alpha}}(\tilde{x})) = \tilde{\boldsymbol{\alpha}}(\tilde{f}(\tilde{x})).$$

Proof. We observe that if $\tilde{x} \in \tilde{T}_1$, then the maps $f_1 \circ \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \circ f$ are bijections on sufficiently small neighborhoods of $\pi(\tilde{x})$.

Let $\tilde{x} = (x, C)$ be a point of $\tilde{T}_1$ and $\tilde{y} = (y, D) = \tilde{\boldsymbol{\alpha}}(\tilde{x})$. We check that the return time $\tilde{k}(\tilde{y}) = \min\{i > 0; \tilde{f}^i(\tilde{y}) \in \tilde{\boldsymbol{\alpha}}(\tilde{T})\}$ is equal to $k(y)$. For any $0 \leq i \leq k(y)$, we have $\tilde{f}^i(\tilde{y}) = (f^i(y), C_i)$ for some $C_i \in \mathcal{C}_{f^i(y)}$. Hence, $\tilde{f}^i(\tilde{y}) \in \tilde{\boldsymbol{\alpha}}(\tilde{T})$ implies $f^i(y) \in \boldsymbol{\alpha}(T)$ and we deduce $k(y) \leq \tilde{k}(\tilde{y})$. Since $\tilde{x} \in \tilde{T}_1$, the point $f^{k(y)}(y)$ belongs to $\boldsymbol{\alpha}(T)\backslash R$, which implies that $(f^{k(y)}(y), C) \in \tilde{\boldsymbol{\alpha}}(\tilde{T})$ for all C in $\mathcal{C}_{f^{k(y)}(y)}$. We deduce $\tilde{f}^{k(y)}(\tilde{y}) \in \tilde{\boldsymbol{\alpha}}(\tilde{T})$ and $k(y) = \tilde{k}(\tilde{y}) = k$.

Recall that since $x \notin \mathcal{S}$, there exists a unique element a of $\mathcal{A}$ such that $x \in T_a$. We obtain

$$\tilde{\boldsymbol{\alpha}} \circ \tilde{f}(\tilde{x}) = \tilde{\boldsymbol{\alpha}}(a^{-1} \cdot x, a^{-1} \star C) = (\boldsymbol{\alpha}(a^{-1} \cdot x), \boldsymbol{\alpha}_\star(a^{-1} \star C))$$

and since $\boldsymbol{\alpha}(x) \in T_1$, we deduce from Proposition 8.1 that we also have

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(\tilde{x}) = \tilde{f}_1(\boldsymbol{\alpha}(x), \boldsymbol{\alpha}_\star(C)) = (\alpha(a^{-1}) \cdot \boldsymbol{\alpha}(x), \alpha(a^{-1}) \star \boldsymbol{\alpha}_\star(C)).$$

Observe that $a^{-1} \star C$ is non-empty since $x \notin \mathcal{S}$, the sets $\boldsymbol{\alpha}_\star(a^{-1} \star C)$ and $\boldsymbol{\alpha}_\star(C)$ are both non-empty (the image by $\boldsymbol{\alpha}_\star$ of any C of any $\mathcal{C}_z$ ($z \in T$) is in fact non-empty) and $\alpha(a^{-1}) \star \boldsymbol{\alpha}_\star(C)$ is non-empty since $f^i(\boldsymbol{\alpha}(x)) \notin \mathcal{S}$ for any $0 \leq i < k$ ($\boldsymbol{\alpha}(x) \in T_1$). Along with Proposition 8.1, this is enough to conclude $\tilde{f}_1(\tilde{\boldsymbol{\alpha}}(\tilde{x})) = \tilde{\boldsymbol{\alpha}}(\tilde{f}(\tilde{x}))$. $\quad\square$

9. Coding the IET

In this section we prove Theorem 1.4. We consider the partition defined by the connected components of $T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$. We use it to construct an IET and we show that its coding can be obtained with the attracting subshift of an automorphism.

The reader is referred to Secs. 5.2 and 7 for the definitions of singularities.

Lemma 9.1. *Let x be a point of $T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$. The unique element a of $\mathcal{A}$ such that $x \in T_a$ is such that the point $\boldsymbol{\alpha}(x)$ satisfies*

- *for any $0 \le k < |\alpha(a)|$, $f^k(\boldsymbol{\alpha}(x)) \notin \mathcal{S}^*$,*
- *for any $0 \le k \le |\alpha(a)|$, $f^k(\boldsymbol{\alpha}(x)) \notin R$.*

Proof. Let X be the set of points y of $\boldsymbol{\alpha}(T)$ such that there is a point $x \in T_a\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$ such that $y = \boldsymbol{\alpha}(x)$ and y is not on the orbit of a point with more than one Q^2-preimage. Recall (from Theorem 5.2) that there is only a finite number of orbits of points with more than one Q^2-preimage. The set X is dense in $\boldsymbol{\alpha}(T_a)$ and any point y of X satisfies

- for any $0 \le k < |\alpha(a)|$, $f^k(y) \notin \mathcal{S}^*$,
- for any $0 \le k \le |\alpha(a)|$, $f^k(y) \notin R$,
- for any $0 < k < |\alpha(a)|$, $f^k(y) \notin \boldsymbol{\alpha}(T)$

(as any point of $\mathcal{S}, \mathcal{S}^-$ (recall $\mathcal{S}^* = \mathcal{S} \cup f^{-1}(\mathcal{S}^-)$) or R has more than one Q^2-preimage).

Choose a point $x \in T_a\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$ and suppose x has degree d. For each C_i ($1 \le i \le d$) of $\mathcal{C}_x$, choose a point $x_i \in C_i \cap \boldsymbol{\alpha}^{-1}(X)$. Since $x \notin \boldsymbol{\alpha}^{-1}(R)$, the point $\boldsymbol{\alpha}(x)$ also has degree d. Assuming $\alpha(a) = v_0 v_1 \cdots v_p$, finding an integer $0 \le h < |\alpha(a)|$ such that $f^h(\boldsymbol{\alpha}(x)) \in \mathcal{S}$ means for one of the x_i, the point $f^h(\boldsymbol{\alpha}(x_i))$ is not in T_{v_h} which is impossible. Finding an integer $0 \le h < |\alpha(a)|$ such that $f^h(\boldsymbol{\alpha}(x)) \in R$ means the points $\boldsymbol{\alpha}(x_i)$ do not have the same return time (to $\boldsymbol{\alpha}(T)$), which is also impossible (Proposition 8.1). Moreover, if $f^{|\alpha(a)|}(\boldsymbol{\alpha}(x)) \in R$ then one of the x_is is not in $\boldsymbol{\alpha}(T)$; again impossible.

We have just proven that $x \in T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$ implies $f(x) \notin \boldsymbol{\alpha}^{-1}(R)$. Working with $f(x)$ and applying a similar reasoning, we obtain that for any $0 \le k < |\alpha(a)|$, we have $f^{-k}(\boldsymbol{\alpha}(x)) \notin \mathcal{S}^-$, and we conclude. $\qquad\square$

The map $f : T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R)) \to f(T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R)))$ is obviously a bijection. An important consequence of the above lemma is that the set $\boldsymbol{\alpha}^{-1}(R)$ is a subset of both $(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$ and $T\backslash f(T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R)))$. The property is carried onto the circle, and it is essential to the construction of a proper IET; in particular, the transition from an IET on the circle to a regular IET cannot be done without such points as they allow us to choose an origin (the point 0).

Finally, the following corollary will allow for a substitutive definition of the IET's coding.

Corollary 9.2. *For any point $x \in T_a\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$ and for any $0 \le k \le |\alpha(a)|$, we have $f^k(\boldsymbol{\alpha}(x)) \notin \boldsymbol{\alpha}^{-1}(R)$.*

Proof. Lemma 9.1 tells us that for any $x \in T\backslash(\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R))$, the point $f(x)$ is not in $\boldsymbol{\alpha}^{-1}(R)$. Hence, all that is left to prove is $\boldsymbol{\alpha}(x) \notin \boldsymbol{\alpha}^{-1}(R)$.

Define R_2 as the boundary of $\boldsymbol{\alpha}^2(T)$ in T:

$$R_2 = \boldsymbol{\alpha}^2(T) \cap \overline{T \backslash \boldsymbol{\alpha}^2(T)}.$$

If $\boldsymbol{\alpha}^2(x)$ is a point of R then there exist $C \in \mathcal{C}_{\boldsymbol{\alpha}^2(x)}$ and a neighborhood V of $\boldsymbol{\alpha}^2(x)$ such that $V \cap C$ is a non-empty subset of $T \backslash \boldsymbol{\alpha}(T)$ and consequently of $T \backslash \boldsymbol{\alpha}^2(T)$. The point $\boldsymbol{\alpha}^2(x)$ belongs to $\boldsymbol{\alpha}^2(T)$ which makes it a point of R_2. This contradicts the hypothesis $R \cap R_2 = \emptyset$ made previously (see Proposition 7.3 and the ensuing discussion). $\qquad\square$

Proof of Theorem 1.4. We are now in a position to prove the main result of this paper. Pick any lift $\tilde{o} \in \tilde{T}$ of any point of $\boldsymbol{\alpha}^{-1}(R)$ and consider the set $\tilde{T}$ to be the interval $I = [0; \mathcal{H}^\delta(T)[$ (recall that δ is the Hausdorff dimension of T and $\mathcal{H}^\delta$ is its associated Hausdorff measure), so that each point of $\tilde{T}$ is the point $d_+(\tilde{o}, \tilde{x})$ of I. Name $\tilde{\mathcal{L}}$ the lift of $\mathcal{S}^* \cup \boldsymbol{\alpha}^{-1}(R)$ and consider the partition $\tilde{\mathcal{P}}$ of I defined by the connected components of $\tilde{T} \backslash \tilde{\mathcal{L}}$. For any $a \in \mathcal{A}$, we denote p_a the number of intervals of $\tilde{\mathcal{P}}$ which are lifts of subsets of T_a. Define the alphabet $\tilde{\mathcal{A}} = \{\tilde{a}_i; a \in \mathcal{A} \text{ and } 1 \leq i \leq p_a\}$ and assume $\tilde{\mathcal{P}} = \{\tilde{I}_{\tilde{a}_i}; a \in \mathcal{A} \text{ and } 1 \leq i \leq p_a\}$. Define, for $\tilde{a}_i \in \mathcal{A}$, $\Pi(\tilde{a}_i) = a$.

Thanks to Lemma 9.1 and Corollary 9.2 we can define the first-return map $\tilde{f}_1$ associated to $\tilde{f}$ on $\tilde{\boldsymbol{\alpha}}(\tilde{T} \backslash \tilde{\mathcal{L}})$ (see Sec. 8 and Proposition 8.3); in particular, it is such that for any interval $\tilde{I}_{\tilde{a}_i}$, there exists a positive integer $k_{\tilde{a}_i}$ such that

- for any $0 \leq h < k_{\tilde{a}_i}$, there exists $u_h \in \tilde{\mathcal{A}}$ such that $\tilde{f}^h(\tilde{\boldsymbol{\alpha}}(\tilde{I}_{\tilde{a}_i})) \subset \tilde{I}_{u_h}$,
- $\tilde{f}^{k_{\tilde{a}_i}}(\tilde{\boldsymbol{\alpha}}(\tilde{I}_{\tilde{a}_i})) = \tilde{f}_1(\tilde{\boldsymbol{\alpha}}(\tilde{I}_{\tilde{a}_i}))$.

For any $\tilde{a}_i$, define $\tilde{\alpha}(\tilde{a}_i) = u_0 \cdots u_{k_{\tilde{a}_i}-1}$ where u_h is the letter of $\tilde{\mathcal{A}}$ such that $\tilde{f}^h(\tilde{\boldsymbol{\alpha}}(\tilde{I}_{\tilde{a}_i})) \subset \tilde{I}_{u_h}$ for any $0 \leq h < k_{\tilde{a}_i}$. Hence $\tilde{f}$ has $\tilde{\alpha}$-structure for $\tilde{\mathcal{P}}$.

10. Using a Tree Substitution

Tree substitutions are an efficient tool to construct self-similar (in the sense of [30]) $\mathbb{R}$-trees. The reader is referred to [24] for a combinatorial and metric analysis of tree substitutions. Here, we use them to construct limit sets of certain automorphisms (see [24] and [25] for detailed examples). While not providing great details about these transformations, we wish to explain how constructing a limit set using a tree substitution gives an easy way to determine the interval exchange transformations defined previously.

10.1. *Tree substitutions and limit sets*

An oriented graph $G = (\mathcal{V}, \mathcal{E})$ is defined by a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. A path in G is a list $(v_0, \ldots, v_p)$ of vertices such that for any $0 \leq i < p$, we have either $(v_i, v_{i+1}) \in \mathcal{E}$ or $(v_{i+1}, v_i) \in \mathcal{E}$. A cycle is a path $(v_0, \ldots, v_p)$ with $v_0 = v_p$.

Given an alphabet $\mathcal{A}_\tau$, an $\mathcal{A}_\tau$-labeled simplicial tree $T^s = (\mathcal{V}, \mathcal{E})$ (s stands for simplicial) is a connected (there is a path between any two vertices) graph with no cycles and where $\mathcal{E}$ is a subset of $\mathcal{V} \times \mathcal{V} \times \mathcal{A}_\tau$. Let Υ be the set of $\mathcal{A}_\tau$-labeled trees (all vertices are assumed to be taken in a common arbitrary uncountable set). A tree substitution is a map $\tau : \Upsilon \to \Upsilon$; it is defined on trees $X_a = (\{x, y\}, \{(x, y, a)\})$ containing only one edge (with the condition that x and y are also vertices of $\tau(X_a)$) and extended to all trees of Υ by taking the union of the images of the edges. An example is given Fig. 4.

Let T be a connected limit set of the repelling tree of an iwip positive automorphism. Given a simplicial tree $T_0^s = (\mathcal{V}, \mathcal{E})$, we say that $\nu_0 : \mathcal{V} \to T$ is an embedding map if for any $v \in \mathcal{V}$ the degree of v in T_0^s (the number of adjacent edges) and the degree of $\nu_0(v)$ in T (the number of connected components of $T \backslash \{\nu_0(v)\}$) are equal. The convex hull in T of the set $\nu_0(\mathcal{V})$ is the embedding of T_0^s associated to ν_0. We say that T can be described by (τ, T_0^s) where τ is a tree substitution and T_0^s is a simplicial tree if there exists a sequence $(\nu_n)_n$ of embedding maps $\nu_n : \tau^n(T_0) \to T$ such that

- for any $n \in \mathbb{N}$ and any vertex v of $\tau^n(T_0^s)$, we have $\nu_{n+1}(v) = \nu_n(v)$ (recall that the vertices of $\tau^n(T_0^s)$ are also vertices of $\tau^{n+1}(T_0^s)$ by definition of tree substitutions),
- for any $n \in \mathbb{N}$ and any $a \in \mathcal{A}_\tau$, if (x_1, x_2, a) and (y_1, y_2, a) are edges of $\tau^n(T_0^s)$, then $|\nu_n(x_1), \nu_n(x_2)| = |\nu_n(y_1), \nu_n(y_2)|$,
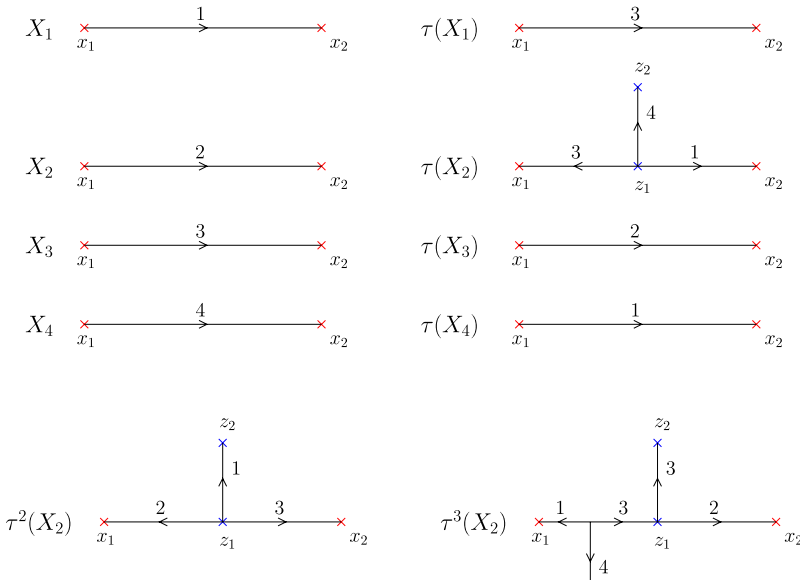
Fig. 4. A tree substitution over the alphabet $\{1, 2, 3, 4\}$.

- the sequence $(T_n)_n$ (where T_n is the embedding of $\tau^n(T_0^s)$ associated to ν_n) converges (with respect to the Hausdorff distance) to T.

10.2. *Cyclic order and interval lengths*

We assume the limit set T is described by (τ, T_0^s). A cyclic order on the branching points of T can then simply be defined by setting a cyclic order on the branching points of the simplicial trees $\tau(X_a)$ (with $X_a = (\{x,y\}, \{(x,y,a)\})$) for any $a \in \mathcal{A}_\tau$. This can be achieved in a combinatorial way by defining a positive and a negative side to the tree substitution. We define the oriented substitution $\tilde{\tau}$ associated to τ. For each letter $a \in \mathcal{A}_\tau$, the image $\tilde{\tau}(a^+)$ (respectively, $\tilde{\tau}(a^-)$) of a^+ is defined by crossing the tree $\tau(X_a)$ from x to y positive (respectively, negative) side (arbitrarily chosen) first; following an edge (x_1, x_2, b) from x_1 to x_2 (respectively, from x_2 to x_1) yields the letter b^+ (respectively, b^-).

For example, the following oriented substitution may be associated to the tree substitution of Fig. 4 with the cyclic order induced by the planar representation:

$$\tilde{\tau} : 1^+ \to 3^+$$
$$1^- \to 3^-$$
$$2^+ \to 3^- 4^+ 4^- 1^+$$
$$2^- \to 3^+ 1^-$$
$$3^+ \to 2^+$$
$$3^- \to 2^-$$
$$4^+ \to 1^+$$
$$4^- \to 1^-$$

In order for the cyclic order to be carried onto T properly, we assume that it is preserved by the embedding maps at each branching points.

If T contains p distinct orbits of branching points with degrees $d_1, d_2, \ldots, d_p$, then $(\prod_{i=1}^{p}(d_i - 1)!)$ distinct cyclic orders may be constructed as in Secs. 4.2 and 5.3. Observe however that all may not be reached when using a tree substitution, since there may be a branching point v of $\tau(X_a)$ for some $a \in \mathcal{A}_\tau$ that will correspond to more than one orbit of branching points. It is also possible that two distinct branching points v_1 and v_2 of $\tau(X_a)$ and $\tau(X_b)$ (possibly with $a = b$) would correspond to a common orbit; in that case, the cyclic orders at these points must be compatible.

We still consider that T is described by (τ, T_0^s), and we assume that $(\nu_n)_n$ is the associated sequence of embedding maps. We define the incidence matrix $M_{\tilde{\tau}}$ of an oriented substitution $\tilde{\tau}$ associated to τ. For any $a, b \in \mathcal{A}_\tau$ and any $\epsilon_1, \epsilon_2 \in \{+, -\}$, the entry $M_{\tilde{\tau}}(a^{\epsilon_1}, b^{\epsilon_2})$ is defined by the number of occurrences of a^{ϵ_1} in $\tilde{\tau}(b^{\epsilon_2})$. Observe that Perron–Frobenius theorem guarantees the existence of a real eigenvalue λ of maximal modulus. Let us further assume that the associated eigenvector has positive entries. Positivity guarantees that this eigenvalue is the same as the

Perron–Frobenius eigenvalue of M_τ. Define $\mathbf{V}$ as the positive left eigenvector of $M_{\tilde{\tau}}$ associated to λ. Observe that $(\mathbf{V}(a^+) + \mathbf{V}(a^-))_{a \in \mathcal{A}_\tau}$ is the eigenvector of M_τ associated with λ. It follows that, for any $a \in \mathcal{A}_\tau$, if n_a is the number of edges of T_0^s labeled a,

$$\sum_{a \in \mathcal{A}_\tau} n_a(\mathbf{V}(a^+) + \mathbf{V}(a^-)) = \mathcal{H}^\delta(T),$$

where δ is the Hausdorff dimension of T and $\mathcal{H}^\delta$ its associated Hausdorff measure.

Proposition 10.1. *For any $n \in \mathbb{N}$, any edge (x_1, x_2, a) of $\tau^n(T_0^s)$ and any $\epsilon \in \{+, -\}$, we have*

$$\mathcal{H}^\delta(]\nu_n(x_1), \nu_n(x_2)[[^\epsilon) = \lambda^{-n}\mathbf{V}(a^\epsilon).$$

Proof. It is straightforward that $\lambda^{-n}(\mathbf{V}(a^+) + \mathbf{V}(a^-))$ gives the Hausdorff measure of the tree between two points for which the edge between these points is labeled a at stage n. When we apply τ to an edge, we can distinguish edges on the left, on the right and along the initial edge. Now observe that when we apply the oriented substitution (to an oriented edge) we recover edges on one side with both orientations (those corresponding to branches started on a on this side) and edges along the initial edge with only one of the two possible orientations. Denote B_{a^ϵ} and C_{a^ϵ} the respective sets of edges and observe that, by property (H_3), the subset B_{a^ϵ} should never be empty (otherwise work with an iterate of α). We can decompose

$$\mathbf{V}(a^\epsilon) = \lambda^{-1} \sum_{b \in B_{a^\epsilon}} (\mathbf{V}(b^+) + \mathbf{V}(b^-)) + \lambda^{-1} \sum_{c^* \in C_{a^\epsilon}} \mathbf{V}(c^*)$$

$$= \lambda^{-1} \sum_{b \in B_{a^\epsilon}} \mathcal{H}^\delta(T_b) + \lambda^{-1} \sum_{c^* \in C_{a^\epsilon}} \mathbf{V}(c^*).$$

We can iterate this decomposition (on the oriented edges in C_{a^ϵ}). Since the set of branching edges is never empty, the remaining term will decrease to 0. Hence, iterating this decomposition, we make that appearance of the Hausdorff measure of all trees on the side ϵ of edge a; in the limit, we recover the sum of the Hausdorff measures of all those trees which are the connected components defining our distance. $\square$

11. Example 1

Recall that the Tribonacci substitution is given by:

$$\alpha : a \mapsto ab$$
$$b \mapsto ac$$
$$c \mapsto a.$$

We mentioned in the Introduction the so-called Arnoux–Yoccoz interval exchange transformation. Following [1], we can write it in a partition suitable for induction: This partition seemed to be *ad hoc*; it is not very surprising to observe that our more general viewpoint yields the same partition. We present shortly the way it appears following our scheme. But, along the procedure we have to choose an order and it appears that for this substitution, we have an alternative possibility. Following the procedure with this alternative order, we will obtain another IET associated with the same substitution.

11.1. *Planar order and the Arnoux–Yoccoz IET*

To illustrate our construction, we first show how the Arnoux–Yoccoz IET arises from the construction, of the tree substitution. In [24] a tree substitution which yields the Tribonacci Rauzy fractal is proposed. This tree substitution τ is shown in Fig. 5. The initial tree is $\tau^2(X_A)$.

Distinguishing positive and negative sides we get the substitution:

$$\tilde{\tau} : A^+ \to D^+ C^+$$
$$A^- \to D^- B^- B^+ C^-$$
$$B^+ \to D^- E^+ E^- A^+$$
$$B^- \to D^+ A^-$$
$$C^+ \to B^+$$
$$C^- \to B^-$$
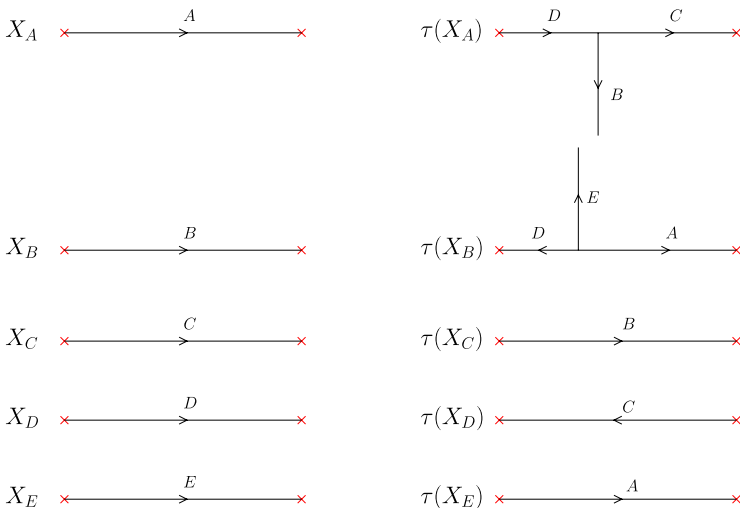
Fig. 5. Tree substitution and cyclic ordering corresponding to the Arnoux–Yoccoz IET.
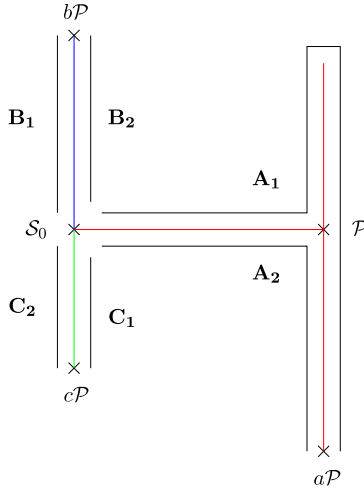
Fig. 6.   (Color online) Partition of the contour.

$$D^+ \to C^-$$

$$D^- \to C^+$$

$$E^+ \to A^+$$

$$E^- \to A^-.$$

The set $\mathcal{S}$ of forward singularities (see Sec. 5.2) only contains the point $\mathcal{S}_0$ and the set $\mathcal{S}^-$ of backward singularities (see again Sec. 5.2) contains only $\mathcal{P}$ (Fig. 6). The point $\mathcal{S}_0$ has three lifts on the contour and the set $f^{-1}(\mathcal{S}^-)$ (where f is the map induced by the system of isometries on $T\backslash\mathcal{S}$) contains three points, each of which has one lift. We obtain a partition in six intervals A_1, A_2, B_1, B_2, C_1 and C_2.

This partition is shown in Fig. 6. The drawing is substantially simplified in order for the circle and its partition to appear clearly. The sets T_a, T_b and T_c (recall T_i (see Sec. 5.2) is the set of points x of T such that $i^{-1} \cdot x$ is also in T for any $i \in \mathcal{A}$) are reduced to simple trees and are colored red, blue and green respectively.

Recall (Secs. 4.2 and 7) that we must choose a power k of α so that the induced homothety $\boldsymbol{\alpha}$ on T satisfies:

- any orbit of branching point is globally mapped to itself by α^k,
- the cyclic order is preserved by α^k,
- the boundaries of $\boldsymbol{\alpha}^k(T)$ and $\boldsymbol{\alpha}^{2k}(T)$ in T are disjoint.

The first two conditions are satisfied for $k = 1$ and the third for $k = 3$. Define R as the boundary of $\boldsymbol{\alpha}^3(T)$ in T. Each point of $\boldsymbol{\alpha}^{-3}(R)$ has a unique lift on the contour, and these lifts divide A_1, A_2 and B_1 into $(A_1^1, A_1^2), (A_2^1, A_2^2)$ and (B_1^1, B_1^2) respectively.
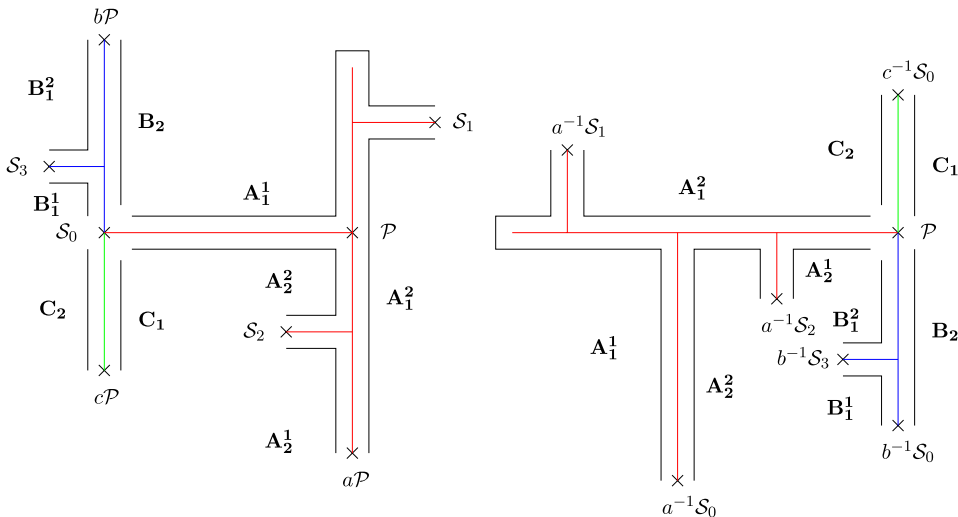
Fig. 7. (Color online) Interval exchange transformation induced by the exchange of pieces.

In Fig. 7, we give the simplest possible representation of the tree, the exchange of pieces, and the induced interval exchange transformation; it is simply the convex hull in T of all the necessary singularities. The exchange of pieces is color coded: the isometries map the sets T_a (red), T_b (blue) and T_c (green) of the tree on the left to their counterparts on the tree on the right. The differences between the two representations of the tree is due to the crudeness of the drawing; getting a proper picture would require to draw the limit tree. Also note that we have: $\mathcal{S}_3 = a^{-1}\mathcal{S}_0, \mathcal{S}_1 = c^{-1}\mathcal{S}_0$ and $\mathcal{S}_2 = b^{-1}\mathcal{S}_0$.

This yields exactly the Arnoux–Yoccoz IET. The reader is referred to [1] for comparison, but should however note the following:

- due to different conventions, the IET of [1] should be read from right to left in order to match ours,
- the factor α used in [1] is exactly ℓ^{-1}, where ℓ is the real number solution of $X^3 - X^2 - X - 1 = 0$.
- the IET is defined up to a multiplicative constant.

$$\begin{pmatrix} B_1^2 & B_2 & A_1^1 & A_1^2 & A_2^1 & A_2^2 & C_1 & C_2 & B_1^1 \\ \ell^3 & \ell^4 & \ell^3 + \ell^2 & \ell^4 & \ell^3 & \ell^4 + \ell^2 & \ell^3 & \ell^3 & \ell^2 + \ell \\ A_1^1 & A_1^2 & C_2 & C_1 & B_2 & B_1^1 & B_1^2 & A_2^1 & A_2^2 \\ \ell^3 + \ell^2 & \ell^4 & \ell^3 & \ell^3 & \ell^4 & \ell^2 + \ell & \ell^3 & \ell^3 & \ell^4 + \ell^2 \end{pmatrix}$$

Remark 11.1. We use the tree substitution to determine the lengths of the interval. We do not give an explicit computation for this example. However, the methods are detailed in great lengths for the example of Sec. 12.
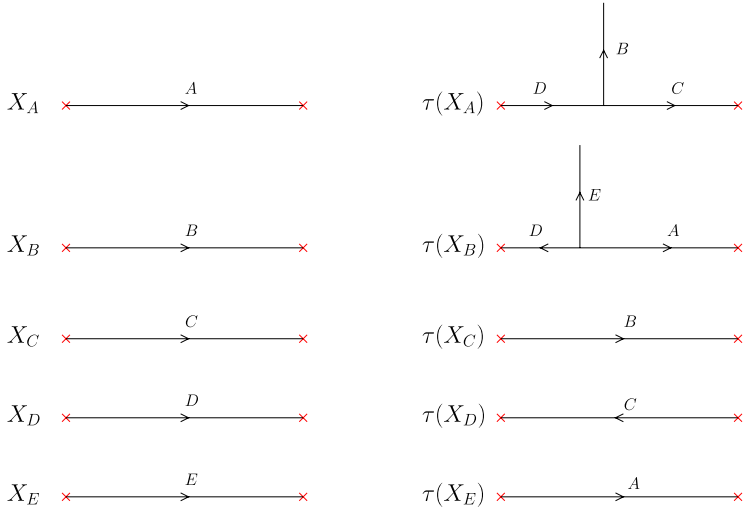
Fig. 8.   Another cyclic order on the tree.

11.2. *Another IET with the same factor*

Let us now do the same construction but changing the order at the branching point in the image of X_A. The tree substitution using the new order is shown in Fig. 8.

This changes the substitution only for the image of A^+ and A^-:

$$\tilde{\tau} : A^+ \to D^+ B^+ B^- C^+$$
$$A^- \to D^- C^-$$
$$B^+ \to D^- E^+ E^- A^+$$
$$B^- \to D^+ A^-$$
$$C^+ \to B^+$$
$$C^- \to B^-$$
$$D^+ \to C^-$$
$$D^- \to C^+$$
$$E^+ \to A^+$$
$$E^- \to A^-$$

The sets $\mathcal{S}$ and $\mathcal{S}^-$ are the same as before and we obtain the partition of Fig. 9 with the new cyclic order.

This time, the lifts of the three points of $\boldsymbol{\alpha}^{-3}(R)$ divide A_1 into (A_1^1, A_1^2, A_1^3) and B_1 into (B_1^1, B_1^2), and we obtain the interval exchange transformation of Fig. 10 with $\mathcal{S}_3 = a^{-1}\mathcal{S}_0, \mathcal{S}_1 = c^{-1}\mathcal{S}_0$ and $\mathcal{S}_2 = b^{-1}\mathcal{S}_0$.

If ℓ is the real number solution of $X^3 - X^2 - X - 1 = 0$, the IET is defined by its length vector (note that the length of the interval below is exactly twice the
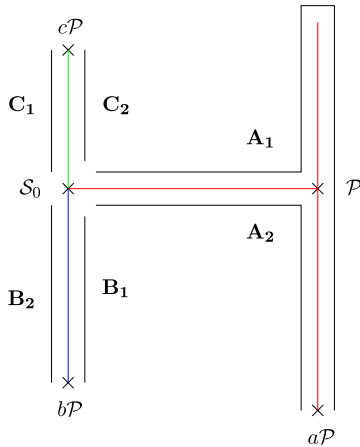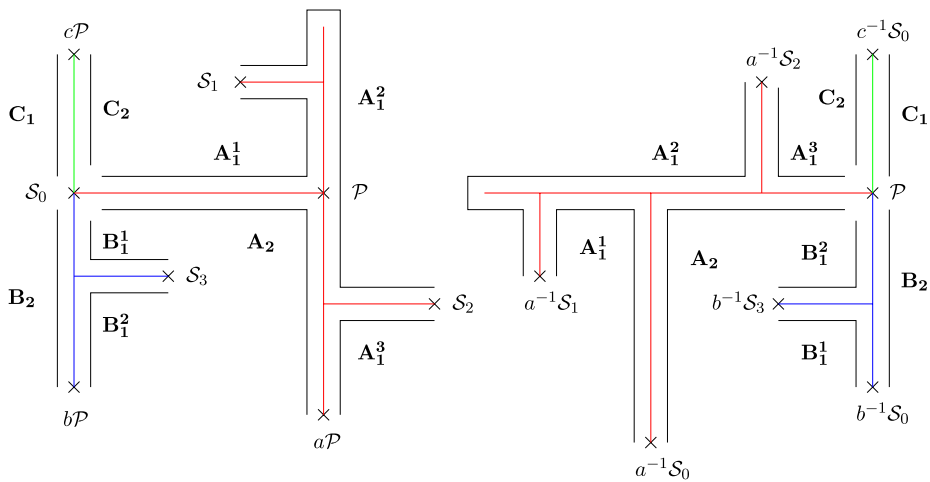
Fig. 9. Partition of the contour.

Fig. 10. Interval exchange transformation induced by the exchange of pieces.

length of the interval used in the previous section)

$$\begin{pmatrix} B_1^2 & B_2 & C_1 & C_2 & A_1^1 & A_1^2 & A_1^3 & A_2 & B_1^1 \\ \ell^5 + \ell^4 & \ell^2 + \ell & 3\ell^3 + \ell^2 & \ell + 1 & \ell^4 + \ell^2 & \ell^5 + \ell^3 & \ell^5 + \ell^4 & \ell^3 + \ell^2 & \ell^3 + \ell \end{pmatrix},$$

and its permutation

$$\begin{pmatrix} B_1^2 & B_2 & C_1 & C_2 & A_1^1 & A_1^2 & A_1^3 & A_2 & B_1^1 \\ A_1^1 & A_1^2 & A_1^3 & C_2 & C_1 & B_2 & B_1^1 & B_1^2 & A_2 \end{pmatrix}.$$

As in the standard case, the points of $\boldsymbol{\alpha}^{-3}(R)$ are singularities of both the original partition and its image; hence the exchange may be defined on a union of

three intervals K_1, K_2, K_3 as follows:

$$
\begin{pmatrix}
K_1 & . & K_2 & . & K_3 \\
B_1^2 B_2 C_1 C_2 A_1^1 & . & A_1^2 & . & A_1^3 A_2 B_1^1 \\
A_1^1 A_1^2 A_1^3 C_2 & . & C_1 B_2 & . & B_1^1 B_1^2 A_2
\end{pmatrix}.
$$

To draw a parallel between the two transformations, let us observe that they can both be defined as exchanges of six intervals on a circle; in both cases the circle is cut into three intervals A, B and C with respective proportions: ℓ^3, ℓ^2 and ℓ. Then, each interval is cut into two pieces. The dynamics can be seen as a (simultaneous) transposition of those pieces composed with a permutation (and rotation) of the three intervals. For the standard Arnoux–Yoccoz, the length vector and the circular permutation are:

$$
\lambda = (\,\ell^5 \quad \ell^5 \quad \ell^4 \quad \ell^4 \quad \ell^3 \quad \ell^3\,) \quad \text{and}
$$

$$
\tilde{\pi} =
\begin{pmatrix}
A_1 & A_2 & C_1 & C_2 & B_1 & B_2 \\
A_2 & A_1 & C_2 & C_1 & B_2 & B_1
\end{pmatrix},
$$

while, for the other one, they are (again the circle below is twice as long as the circle above):

$$
\lambda = (\,\ell^2(3\ell^3 + \ell^2) \quad \ell^2(\ell + 1) \quad \ell(3\ell^3 + \ell^2) \quad \ell(\ell + 1) \quad 3\ell^3 + \ell^2 \quad \ell + 1\,) \quad \text{and}
$$

$$
\tilde{\pi} =
\begin{pmatrix}
A_1 & A_2 & B_1 & B_2 & C_1 & C_2 \\
A_2 & A_1 & C_2 & C_1 & B_2 & B_1
\end{pmatrix}.
$$

Remark 11.2. Let us stress the fact that here we give circular permutations: they do not define proper interval exchange transformations because the information about the rotation is missing; for instance, we could specify the positions of the left end points.

12. Example 2

We consider the alphabet $\mathcal{A} = \{a, b, c\}$ and the substitution

$$
\alpha : a \to ab
$$
$$
b \to c
$$
$$
c \to a.
$$

We are going to show how the strategy described in this paper yields an interval exchange transformation that factorizes onto the substitution dynamical system associated with this substitution and hence prove Theorem 1.1. The following Sec. 12.1 gives the steps of the proof while the next two sections provide technical details on how to compute the interval exchange characteristics (combinatorics and lengths) from the substitution itself.
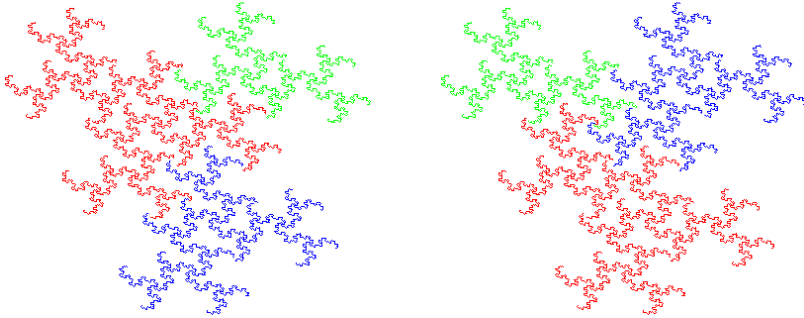
Fig. 11.   (Color online) The exchange of pieces on (an approximation of) T.

12.1. *Proof of Theorem 1.1*

The limit set of the repelling tree associated to the basis $\mathcal{A}$ is defined as in Sec. 5; we denote it by T. Since there is only one orbit of branching points in T (see [25]), the choice of cyclic order at this point (of degree 3) determines the whole picture; it is shown in Fig. 11 in correspondence with the Rauzy fractal. Observe that this choice only determines the global orientation of the picture and does not affect the combinatorics (if the order was reversed at each branching point, the tree would just be described in the opposite direction). The picture shows T_a, T_b and T_c as red, blue and green respectively.

The dynamics of the piecewise isometry is well known and one can essentially guess the combinatorics of the interval exchange transformation on Fig. 11. The set $\mathcal{S}$ of forward singularities (see Sec. 5.2) contains only the two points $\mathcal{S}_0$ and $\mathcal{S}_1$ (Fig. 12), each of which has two lifts on the contour, the set $\mathcal{S}^-$ (see again Sec. 5.2) of backward singularities contains only the point $\mathcal{P}$, which has three lifts. The partial action of the free group on T induces a map f on $T\backslash\mathcal{S}$. Observe that the set $f^{-1}(\mathcal{S}^-)$ contains three points (with only one lift). These singularities yield a partition of the contour of T in seven intervals, $A_1, A_2, C_1, C_2, A_3, B_1$ and B_2.

This partition is shown in Fig. 12. The drawing is substantially simplified in order for the circle and its partition to appear clearly. The sets T_a, T_b and T_c are reduced to simple trees but their respective colors are the ones from Fig. 11. The singularities $\mathcal{S}_0$ and $\mathcal{S}_1$ both belong to $\mathcal{S}$, and the point $\mathcal{P}$ is only point of $\mathcal{S}^-$. This decomposition gives a first idea of the interval exchange transformation.

Since 3 is the smallest integer k for which $\boldsymbol{\alpha}^k(T)$ and $\boldsymbol{\alpha}^{2k}(T)$ are disjoint (see Sec. 7), we define R as the boundary of $\boldsymbol{\alpha}^3(T)$ in T. Each point of $\boldsymbol{\alpha}^{-3}(R)$ has a unique lift on the contour, and these lifts divide A_1, B_1 and C_1 into $(A_1^1, A_1^2), (B_1^1, B_1^2)$ and (C_1^1, C_1^2) respectively. Note that aside from the definition of the set R, the entire process (the interval exchange, the induction and the coding) does not require us to work with $\boldsymbol{\alpha}^3$; only orbit- and order-preserving properties are required, and we can simply work with $\boldsymbol{\alpha}$.
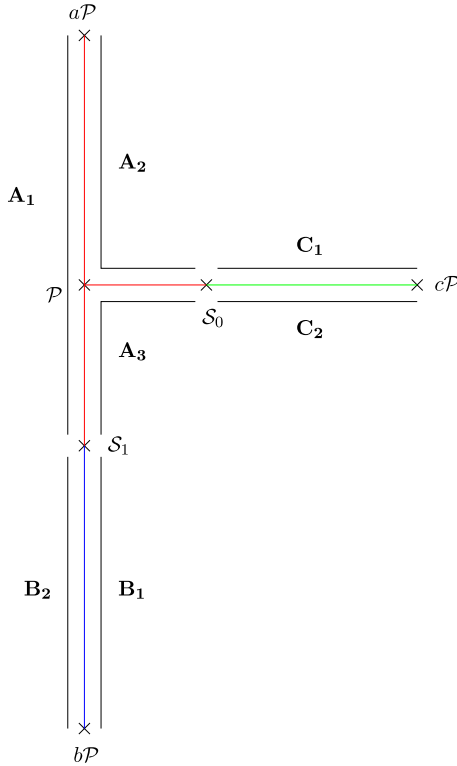
Fig. 12.   Partition of the contour.

It remains to determine the lengths of the intervals and the positions of the singularities; this work is detailed in Secs. 12.2 and 12.3.

In Fig. 13, we give the simplest possible representation of the tree, the exchange of pieces, and the induced interval exchange transformation; it is simply the convex hull in T of all the necessary singularities. The exchange of pieces is color coded: the isometries map the sets T_a (red), T_b (blue) and T_c (green) of the tree on the left to their counterparts on the tree on the right. The differences between the two trees is due to the crudeness of the drawing; getting a proper picture would require to draw the limit tree.

Remark 12.1.

- The position of the singularities cannot be immediately deduced from these combinatorial considerations. As it requires additional definitions and results, we postpone their construction to Sec. 12.3.
- We will use the following facts which follow from the analysis in Sec. 12.3:
  - $\mathcal{S}_{i+1} = \boldsymbol{\alpha}^{-1}(\mathcal{S}_i)$ for $0 \leq i \leq 3$,
  - $b^{-1}\mathcal{S}_1 = \mathcal{S}_3$, $a^{-1}\mathcal{S}_0 = \mathcal{S}_4$ and $c^{-1}\mathcal{S}_0 = \mathcal{S}_2$.
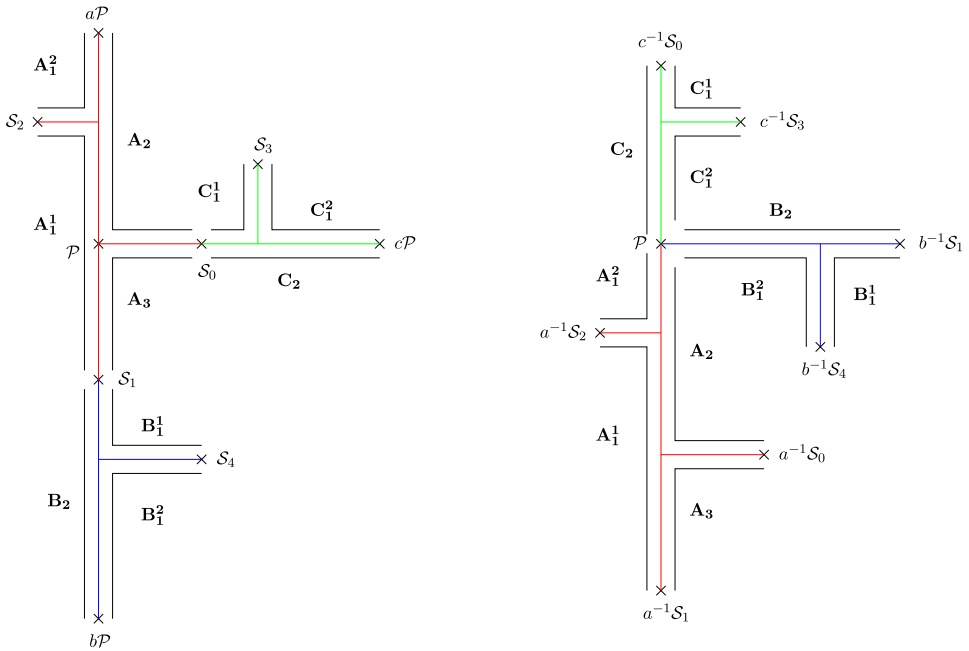
Fig. 13.   (Color online) Interval exchange transformation induced by the exchange of pieces.

We obtain the interval exchange transformation determined by the following permutation and length vectors (the computation of the lengths uses the tree substitution and is explained in the next section).

$$
\begin{pmatrix}
B_1^2 & B_2 & A_1^1 & A_1^2 & A_2 & C_1^1 & C_1^2 & C_2 & A_3 & B_1^1 \\
\ell^3 & \ell & \ell^3 & \ell & \ell^2 & 1 & \ell^2 & 1 & \ell^2 & \ell \\
A_3 & A_1^1 & A_1^2 & C_2 & C_1^1 & C_1^2 & B_2 & B_1^1 & B_1^2 & A_2 \\
\ell^2 & \ell^3 & \ell & 1 & 1 & \ell^2 & \ell & \ell & \ell^3 & \ell^2
\end{pmatrix}
$$

where ℓ is the real number solution of $X^3 - X^2 - 1 = 0$. The length vector is obviously defined up to a (positive) multiplicative constant. Observe that the total length of the intervals $|A| = \ell^3 + 2\ell^2 + \ell = \ell^4 + 2\ell^2$, $|B| = \ell^3 + 2\ell$ and $|C| = \ell^2 + 2$ are in ratio ℓ^2, ℓ and 1 as one could expect. With this normalization, the total length of the interval is $\ell^4 + \ell^3 + 3\ell^2 + 2\ell + 2 = 5\ell^2 + 3\ell + 4$.

In the spirit of the end of Sec. 11, we could present this IET on the circle using a partition into seven intervals. It would be defined by a length interval and a circular permutation (again a global rotation would be needed for a proper definition):

$$
\lambda = (\ell + \ell^3 \quad \ell^2 \quad 1 + \ell^2 \quad 1 \quad \ell^2 \quad \ell + \ell^3 \quad \ell) \quad \text{and}
$$

$$
\tilde{\pi} = \begin{pmatrix}
A_1 & A_2 & C_1 & C_2 & A_3 & B_1 & B_2 \\
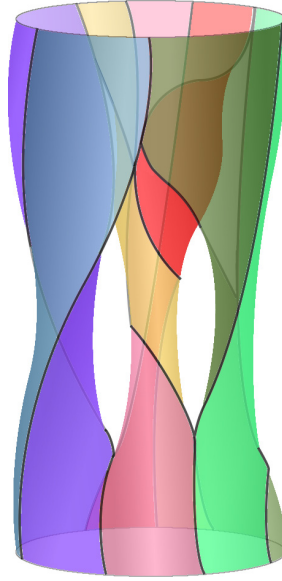A_2 & A_3 & A_1 & C_2 & C_1 & B_2 & B_1
\end{pmatrix}.
$$

Fig. 14. (Color online) This picture shows how the combinatorics of our IET could be realized as a section of a flow on a surface of genus 3. On the bottom circle, $A_3, B_1, B_2, A_1, A_2, C_1, C_2$ are respectively in pink, green, dark green, yellow, red, blue and purple.

We present the picture of Fig. 14 (with no justifications) which shows how the interval exchange transformation on the circle may be realized as a section of a flow on a surface.

As explained in Sec. 9, the points of $\boldsymbol{\alpha}^{-3}(R)$ are singularities of both the original partition and its image. The exchange can hence be defined on a union of three intervals K_1, K_2, K_3 as follows:

$$
\begin{pmatrix}
K_1 & . & K_2 & . & K_3 \\
B_1^2 B_2 A_1^1 & . & A_1^2 A_2 C_1^1 & . & C_1^2 C_2 A_3 B_1^1 \\
A_3 A_1^1 A_1^2 C_2 & . & C_1^1 C_1^2 B_2 & . & B_1^1 B_1^2 A_2
\end{pmatrix}.
$$

The interval exchange transformation is defined everywhere except on the singularities of the interval, and is denoted $\tilde{f}$.

The induction works as follows. Let $\alpha_1 \in K_1, \alpha_2 \in K_2, \alpha_3 \in K_3$ be the lifts on the contour of T of the fixed point $\mathcal{P}$ of $\boldsymbol{\alpha}$. Define $h_1 : K_1 \to K_1, h_2 : K_2 \to K_2, h_3 : K_3 \to K_3$ as the homotheties with common factor ℓ^{-1} and centers $\alpha_1, \alpha_2, \alpha_3$ respectively. The homothety $\boldsymbol{\alpha}$ of T lifts to the map $\tilde{\boldsymbol{\alpha}}$ of the interval defined as:

$$
\tilde{\boldsymbol{\alpha}} : x \mapsto h_1(x) - \alpha_1 + \alpha_3 = \ell^{-1}(x - \alpha_1) + \alpha_3 \quad \text{if } x \in K_1
$$
$$
x \mapsto h_2(x) - \alpha_2 + \alpha_1 = \ell^{-1}(x - \alpha_2) + \alpha_1 \quad \text{if } x \in K_2
$$
$$
x \mapsto h_3(x) - \alpha_3 + \alpha_2 = \ell^{-1}(x - \alpha_3) + \alpha_2 \quad \text{if } x \in K_3.
$$

Observe the points $\alpha_1, \alpha_2, \alpha_3$ are fixed by $\boldsymbol{\alpha}^3$. We denote by $\tilde{f}_1$ the first return map induced by $\tilde{f}$ on $\tilde{\boldsymbol{\alpha}}(I)$. We now place the points $\alpha_1, \alpha_2, \alpha_3$ in the interval in order for $\tilde{\boldsymbol{\alpha}}$ and $\tilde{f}_1$ to be perfectly explicit. Assuming I has length L and left end point 0, we have $\alpha_1 = \ell^5, \alpha_2 = |K_1| + \ell^3$ and $\alpha_3 = |K_1| + |K_2| + \ell^4$ (see the next section for the computation). Indeed, the point α_1 divides the interval A_1^1 into two parts of lengths $(\ell^2, 1)$, the point α_2 divides A_2 into two parts of lengths $(1 + \ell^{-1}, \ell^{-2})$ and the point α_3 divides A_3 into two parts of lengths (ℓ, ℓ^{-1}). We deduce the set $\tilde{\boldsymbol{\alpha}}(I)$ is (up to singularities) the interval $I \backslash (B_1^1 \cup B_1^2 \cup B_2)$.

We now ensure the self-similar structure by checking that the equality $\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{\alpha}} \circ \tilde{f}$ holds for each of the ten subintervals. We make profuse use of the equality $\ell^3 = \ell^2 + 1$, which also gives $\alpha_1 = 2\ell^2 + \ell + 1, \alpha_2 = 3\ell^2 + \ell + 3$ and $\alpha_3 = 4\ell^2 + 3\ell + 4$.

$$
\begin{array}{llll}
B_1^2 = &]0 \, ; \, \ell^2 + 1[& \overset{\tilde{f}}{\mapsto} \;]3\ell^2 + 3\ell + 3 \, ; \, 4\ell^2 + 3\ell + 4[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]2\ell^2 + \ell + 3 \, ; \, 3\ell^2 + \ell + 3[\\
B_2 = &]\ell^2 + 1 \, ; \, \ell^2 + \ell + 1[& \overset{\tilde{f}}{\mapsto} \;]3\ell^2 + \ell + 3 \, ; \, 3\ell^2 + 2\ell + 3[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]2\ell^2 + \ell + 1 \, ; \, 2\ell^2 + \ell + 2[\\
A_1^1 = &]\ell^2 + \ell + 1 \, ; \, 2\ell^2 + \ell + 2[& \overset{\tilde{f}}{\mapsto} \;]\ell^2 \, ; \, 2\ell^2 + 1[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]3\ell^2 + 3\ell + 3 \, ; \, 4\ell^2 + 3\ell + 3[\\
A_1^2 = &]2\ell^2 + \ell + 2 \, ; \, 2\ell^2 + 2\ell + 2[& \overset{\tilde{f}}{\mapsto} \;]2\ell^2 + 1 \, ; \, 2\ell^2 + \ell + 1[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]4\ell^2 + 3\ell + 3 \, ; \, 4\ell^2 + 3\ell + 4[\\
A_2 = &]2\ell^2 + 2\ell + 2 \, ; \, 3\ell^2 + 2\ell + 2[& \overset{\tilde{f}}{\mapsto} \;]4\ell^2 + 3\ell + 4 \, ; \, 5\ell^2 + 3\ell + 4[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]3\ell^2 + \ell + 3 \, ; \, 3\ell^2 + 2\ell + 3[\\
C_1^1 = &]3\ell^2 + 2\ell + 2 \, ; \, 3\ell^2 + 2\ell + 3[& \overset{\tilde{f}}{\mapsto} \;]2\ell^2 + \ell + 2 \, ; \, 2\ell^2 + \ell + 3[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]\ell^2 + \ell + 1 \, ; \, 2\ell^2 + 1[\\
C_1^2 = &]3\ell^2 + 2\ell + 3 \, ; \, 4\ell^2 + 2\ell + 3[& \overset{\tilde{f}}{\mapsto} \;]2\ell^2 + \ell + 3 \, ; \, 3\ell^2 + \ell + 3[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]2\ell^2 + 1 \, ; \, 2\ell^2 + \ell + 1[\\
C_2 = &]4\ell^2 + 2\ell + 3 \, ; \, 4\ell^2 + 2\ell + 4[& \overset{\tilde{f}}{\mapsto} \;]2\ell^2 + \ell + 1 \, ; \, 2\ell^2 + \ell + 2[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]4\ell^2 + 3\ell + 4 \, ; \, 5\ell^2 + 2\ell + 4[\\
A_3 = &]4\ell^2 + 2\ell + 4 \, ; \, 5\ell^2 + 2\ell + 4[& \overset{\tilde{f}}{\mapsto} \;]0 \, ; \, \ell^2[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]3\ell^2 + 2\ell + 3 \, ; \, 3\ell^2 + 3\ell + 3[\\
B_1^1 = &]5\ell^2 + 2\ell + 4 \, ; \, 5\ell^2 + 3\ell + 4[& \overset{\tilde{f}}{\mapsto} \;]3\ell^2 + 2\ell + 3 \, ; \, 3\ell^2 + 3\ell + 3[& \overset{\tilde{\boldsymbol{\alpha}}}{\mapsto} \;]2\ell^2 + \ell + 2 \, ; \, 2\ell^2 + \ell + 3[
\end{array}
$$

It is easy to check that the following equalities hold (up to singularities):

$$\tilde{\boldsymbol{\alpha}}(B_1^2) = C_1^2 \quad \tilde{\boldsymbol{\alpha}}(A_1^2 \cup A_2 \cup C_1^1) = A_1^1 \qquad \tilde{\boldsymbol{\alpha}}(C_1^2) = A_1^2$$

$$\tilde{\boldsymbol{\alpha}}(B_2) = C_2 \qquad\qquad\qquad\qquad\qquad\qquad \tilde{\boldsymbol{\alpha}}(C_2 \cup A_3) = A_2$$

$$\tilde{\boldsymbol{\alpha}}(A_1^1) = A_3 \qquad\qquad\qquad\qquad\qquad\qquad \tilde{\boldsymbol{\alpha}}(C_1^1) = B_1^1$$

In addition, simply looking at the lengths of the intervals, we get

$$\tilde{f} \circ \tilde{\boldsymbol{\alpha}}(A_1^2) \subset B_1^2 \quad \text{and their right end points match,}$$

$$\tilde{f} \circ \tilde{\boldsymbol{\alpha}}(A_2) = B_2,$$

$$\tilde{f} \circ \tilde{\boldsymbol{\alpha}}(C_1^1) \subset A_1^1 \quad \text{and their left end points match,}$$

$$\tilde{f} \circ \tilde{\boldsymbol{\alpha}}(C_2) \subset A_3 \quad \text{and their right end points match,}$$

$$\tilde{f} \circ \tilde{\boldsymbol{\alpha}}(A_3) = B_1^1.$$

We can finally obtain the images of each interval by $\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}$:

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(B_1^2) = \tilde{f} \circ \tilde{\boldsymbol{\alpha}}(B_1^2) =]2\ell^2 + \ell + 3; 3\ell^2 + \ell + 3[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(B_2) = \tilde{f} \circ \tilde{\boldsymbol{\alpha}}(B_2) =]2\ell^2 + \ell + 1; \ell^2 + \ell + 2[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(A_1^1) = \tilde{f}^2 \circ \tilde{\boldsymbol{\alpha}}(A_1^1) =]3\ell^2 + 3\ell + 3; 4\ell^2 + 3\ell + 3[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(A_1^2) = \tilde{f}^2 \circ \tilde{\boldsymbol{\alpha}}(A_1^2) =]4\ell^2 + 3\ell + 3; 4\ell^2 + 3\ell + 4[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(A_2) = \tilde{f}^2 \circ \tilde{\boldsymbol{\alpha}}(A_2) =]3\ell^2 + \ell + 3; 3\ell^2 + 2\ell + 3[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(C_1^1) = \tilde{f} \circ \tilde{\boldsymbol{\alpha}}(C_1^1) =]\ell^2 + \ell + 1; 2\ell^2 + 1[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(C_1^2) = \tilde{f} \circ \tilde{\boldsymbol{\alpha}}(C_1^2) =]2\ell^2 + 1; 2\ell^2 + \ell + 1[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(C_2) = \tilde{f} \circ \tilde{\boldsymbol{\alpha}}(C_2) =]4\ell^2 + 3\ell + 4; 5\ell^2 + 2\ell + 4[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(A_3) = \tilde{f}^2 \circ \tilde{\boldsymbol{\alpha}}(A_3) =]3\ell^2 + 2\ell + 3; 3\ell^2 + 3\ell + 3[$$

$$\tilde{f}_1 \circ \tilde{\boldsymbol{\alpha}}(B_1^1) = \tilde{f} \circ \tilde{\boldsymbol{\alpha}}(B_1^1) =]2\ell^2 + \ell + 2; 2\ell^2 + \ell + 3[$$

All that is left to do is to define the coding automorphism $\tilde{\alpha}$ (see Sec. 9):

$$\tilde{\alpha} : A_1^1 \rightarrow A_3 B_1^2$$

$$A_1^2 \rightarrow A_1^1 B_1^2$$

$$A_2 \rightarrow A_1^1 B_2$$

$$A_3 \rightarrow A_2 B_1^1$$

$$B_1^1 \rightarrow C_1^1$$

$$B_1^2 \rightarrow C_1^2$$

$$B_2 \rightarrow C_2$$

$$C_1^1 \rightarrow A_1^1$$

$$C_1^2 \rightarrow A_1^2$$

$$C_1 \rightarrow A_2$$

The initial substitution α can be obtained from $\tilde{\alpha}$ by using the natural projection:

$$A_i^{(j)} \mapsto a$$

$$B_i^{(j)} \mapsto b$$

$$C_i^{(j)} \mapsto c$$

for any i and any j when necessary. This ends the proof of Theorem 1.1.

12.2. *Computation of the lengths using the tree substitution*

Following the work of [25], the tree substitution defined by Fig. 15 yields a construction of the limit set T associated to α if we set $T_0^s = \tau^2(X_2)$ (see Fig. 16). To
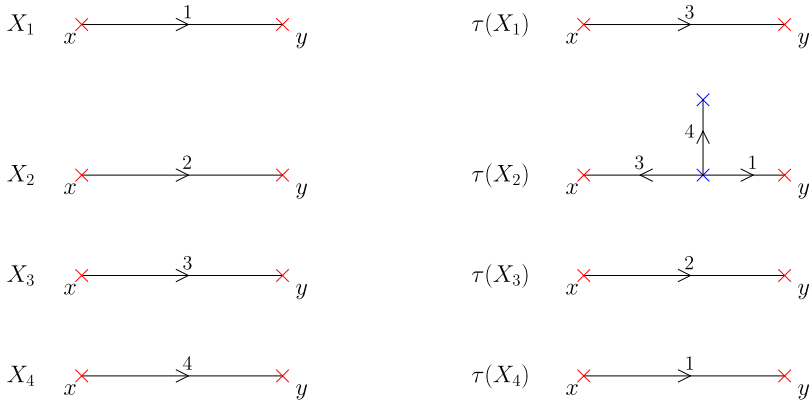
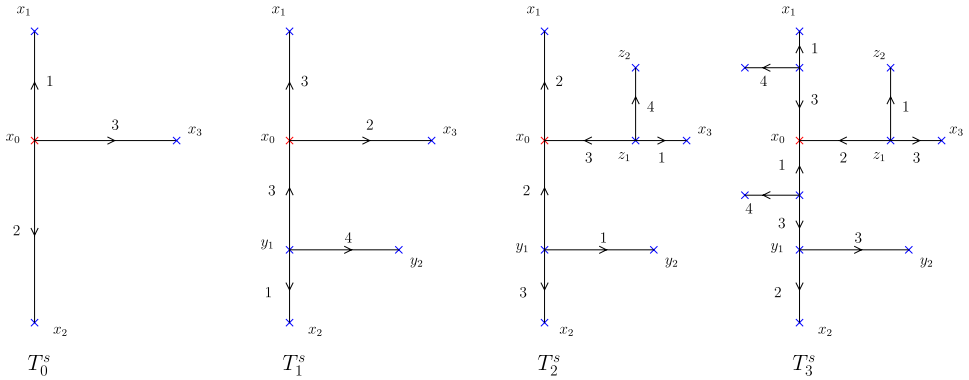Fig. 15.   Tree substitution associated to σ.

Fig. 16.   Initial trees.

define an order, we simply specify the order at the (only) branching point of $\tau(X_2)$ (there are only two possibilities that will yield the same combinatorial result). The order chosen is specified in Figs. 15 and 16; its combinatorial interpretation (see Sec. 10) is the following.

$$\tilde{\tau} : 1^+ \rightarrow 3^+$$
$$1^- \rightarrow 3^-$$
$$2^+ \rightarrow 3^- 4^+ 4^- 1^+$$
$$2^- \rightarrow 3^+ 1^-$$
$$3^+ \rightarrow 2^+$$
$$3^- \rightarrow 2^-$$
$$4^+ \rightarrow 1^+$$
$$4^- \rightarrow 1^-$$

The matrix associated to this substitution is given below.

$$
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

We now explain how the lengths of these intervals can be obtained, thanks to the tree substitution. We denote $\lambda_{\boldsymbol{\alpha}}$ the factor of the homothety $\boldsymbol{\alpha}$; we still denote ℓ the real number solution of $X^3 - X^2 - 1 = 0$. Let us stress that the homothety $\boldsymbol{\alpha}$ acts differently on the distance and the measure of the tree. While distances are contracted by a factor $\lambda_{\boldsymbol{\alpha}}$, measures are contracted by a factor ℓ^{-1} (see Theorem 6.2): for any subset E of T,

$$
\mathcal{H}^{\delta}(\boldsymbol{\alpha}(E)) = \ell^{-1}\mathcal{H}^{\delta}(E) \tag{12.1}
$$

(where δ is the Hausdorff dimension of T and $\mathcal{H}^{\delta}$ is its associated Hausdorff measure).

In this specific case, all we need to compute the lengths of the intervals are the values of $\mathcal{H}^{\delta}(]]\mathcal{P}, c\mathcal{P}[[^{\epsilon})$ and $\mathcal{H}^{\delta}(]]\mathcal{P}, \mathcal{S}_0[[^{\epsilon})$ (see Fig. 13) for $\epsilon \in \{+, -\}$. Since $\mathcal{P}$ is the fixed point of $\boldsymbol{\alpha}$ (see Sec. 12.3), we deduce the other lengths using the equalities:

$$
\begin{aligned}
\boldsymbol{\alpha}(]]\mathcal{P}, b\mathcal{P}[[^{\epsilon}) &=]]\mathcal{P}, c\mathcal{P}[[^{\epsilon}, \\
\boldsymbol{\alpha}(]]\mathcal{P}, c\mathcal{P}[[^{\epsilon}) &=]]\mathcal{P}, a\mathcal{P}[[^{\epsilon}, \\
\boldsymbol{\alpha}(]]\mathcal{P}, \mathcal{S}_{i+1}[[^{\epsilon}) &=]]\mathcal{P}, \mathcal{S}_i[[^{\epsilon}, \quad \text{for all } 0 \leq i \leq 3.
\end{aligned} \tag{12.2}
$$

The vector $\mathbf{V}$ defined by

$$
\frac{\ell+1}{\ell^2}(\ell^2 \quad \ell \quad \ell^4 \quad \ell^3 \quad \ell^3 \quad \ell^2 \quad \ell \quad 1)
$$

is a left eigenvector of the incidence matrix of the oriented tree substitution (the normalization constant is chosen for convenience). We immediately obtain

$$
\begin{aligned}
\mathcal{H}^{\delta}(]]\mathcal{P}, c\mathcal{P}[[^{+}) &= \ell(\ell+1) \\
\mathcal{H}^{\delta}(]]\mathcal{P}, c\mathcal{P}[[^{-}) &= \ell + 1.
\end{aligned} \tag{12.3}
$$

It is explained in Sec. 12.3 that the singularity $\mathcal{S}_0$ is a point of $[\mathcal{P}, c\mathcal{P}]$; recall that the edge $(x_0, x_3, 3)$ of the initial tree T_0^s (Fig. 16) is mapped to the interval $[\mathcal{P}, c\mathcal{P}]$ of T. Applying τ^4 to the tree $(\{x_0, x_3\}, \{(x_0, x_3, 3)\})$ yields the tree on Fig. 17. Note that Fig. 17 respects the cyclic order. Suppose the edge $(z_3, z_1, 3)$ (Fig. 13) is mapped to the interval $[u\mathcal{P}, v\mathcal{P}]$ in T (u and v are elements of $F(\mathcal{A})$: further details can be found in Sec. 12.3). Our point is that the homothety with center $\mathcal{S}_0$
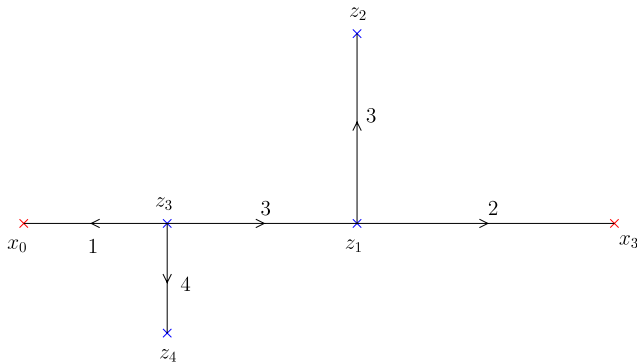
Fig. 17.   Image of the tree $(\{x_0, x_3\}, \{(x_0, x_3, 3)\})$ by τ^4.

and factor λ_α^4 maps the interval $[\mathcal{P}, c\mathcal{P}]$ to the interval $[u\mathcal{P}, v\mathcal{P}]$. We obtain from Proposition 10.1 and Eq. (12.3) that

$$\mathcal{H}^\delta(]]\mathcal{P}, u\mathcal{P}[[^+) = \left(\frac{1}{\ell^4}\right)\left(\frac{\ell+1}{\ell}\right),$$

$$\mathcal{H}^\delta(]]\mathcal{P}, u\mathcal{P}[[^-) = \left(\frac{1}{\ell^4}\right)(\ell+1)$$

(recall the point z_3 of Fig. 17 is mapped to $u\mathcal{P} \in T$) and deduce

$$\mathcal{H}^\delta(]]\mathcal{P}, \mathcal{S}_0[[^+) = \left(\frac{\ell+1}{\ell^5}\right)\sum_{k\geq 0}\left(\frac{1}{\ell^{4k}}\right) = \ell^{-2}$$

$$\mathcal{H}^\delta(]]\mathcal{P}, \mathcal{S}_0[[^-) = \left[\left(\frac{\ell+1}{\ell^4}\right) + \left(\frac{1}{\ell^4}\right)\left(\frac{\ell+1}{\ell^2}\right)(\ell+1)\right]\sum_{k\geq 0}\left(\frac{1}{\ell^{4k}}\right) = \ell.$$

We finally obtain the lengths of the intervals by using Eq. (12.2):

$$\begin{pmatrix} B_1^2 & B_2 & A_1^1 & A_1^2 & A_2 & C_1^1 & C_1^2 & C_2 & A_3 & B_1^1 \\ \ell^3 & \ell & \ell^3 & \ell & \ell^2 & 1 & \ell^2 & 1 & \ell^2 & \ell \end{pmatrix}.$$

12.3. *Singularities in the limit set*

We explain here how the tree substitution allows us to place the singularities that appear in the definition of the interval exchange transformation on the contour of T. There are two steps.

First, we explain how the combinatorial properties of the tree substitution give insight on the orbits of points of T. Specifically, we associate a word of Σ_α^+ (see Sec. 2.2) to any branching point of any T_n^s (recall that for any $n \in \mathbb{N}$, we have defined $T_n^s = \tau^n(T_0^s)$ and that T_0^s is pictured in Fig. 16).

For any $n \in \mathbb{N}$, define $\mathcal{B}_n^s$ as the set of branching points of T_n^s. Suppose x and y are two points of $\mathcal{B}_n^s$ such that (x, y, i) is an edge of T_n^s; define $\chi_n(x, y) = \mathcal{A}(i)$

(where $\mathcal{A}(i)$ is the ith letter of the alphabet) and $\chi_n(y, x) = \mathcal{A}(i)^{-1}$. Now, for any two points x and y of $\mathcal{B}_n^s$, if $(x, b_1, \ldots, b_k, y)$ is a path in T_n^s, then define $\chi_n(x, y) = \chi_n(x_0, b_1) \cdots \chi_n(b_k, y)$. Note that, since x and y are branching points, $\chi_n(x, y)$ is an element of $F(\mathcal{A})$. A vertex of T_n^s is also a vertex of T_{n+1}^s by definition of tree substitutions; it is important to note that $\chi_n(x, y) = \alpha(\chi_{n+1}(x, y))$ for any branching points x, y of T_n^s.

Recall that x_0 is the central vertex of T_0^s (see Fig. 16) and define the map f_0 by:

$$f_0 : \bigcup_{n \in \mathbb{N}} \to F(\mathcal{A})$$

$$x \mapsto \chi_n(x_0, x)$$

where n is any integer such that x is a vertex of T_n^s.

Finally, define $\omega = \lim_{n \to +\infty} \alpha^n(a)$. The important point of our first step is the following.

Proposition 12.2. ([25]) *If x is a vertex of any tree T_n^s, with image z in T, then the coding of the (there is only one for these points) positive orbit of z under the system of partial isometries yields exactly the word $f_0(x)\omega$ of Σ_α^+. In other words, $Q(f_0(x)\omega) = z$.*

The reader is referred to [25] for a proof of this result. It is also proven that there is a bijection between the set of branching points of T and the set $\{S^n(\omega), n \in \mathbb{N}\}$.

This brings us to Step 2. We are going to give the Q-preimages of a singularity, then approach one of them with a sequence of words of $\{S^n(\omega), n \in \mathbb{N}\}$ and use Proposition 5.1 to place it in the tree.

As mentioned before, it is important to see that the partial isometries are defined on closed set, meaning the intersection between two adjacent domains is non-empty. It follows that $\mathcal{S}_0$ and $\mathcal{S}_1$ (see Fig. 13) have more than one positive orbit under the action of the system of partial isometries. Using [25], it is however possible to prove that they each have a unique negative orbit.

From a symbolical point of view, we are going to find pairs $(X, Y), (X, Y')$ of points of Σ_α agreeing on the first coordinate. An algorithm for finding these pairs is given in [26]; it uses the prefix–suffix automata.

We refer to [16] for definitions and results regarding the prefix–suffix automata and prefix–suffix developments and to [26] for a slight adaption of these definitions. For almost (with respect to the unique invariant probability measure of (Σ_α, S)) every point of Σ_α, the prefix–suffix development of a point (X, Y) is a sequence $\rho(X, Y) = (p_i, a_i, s_i)_{n \in \mathbb{N}}$ of elements of $\mathcal{A}^* \times \mathcal{A} \times \mathcal{A}^*$ (where $\mathcal{A}^*$ is the free monoïd with basis $\mathcal{A}$) such that:

- for all $i \geq 0$, $\alpha(a_{i+1}) = p_i a_i s_i$,
- $X = p_0^{-1} \alpha(p_1^{-1}) \cdots \alpha^n(p_n^{-1}) \cdots$,
- $Y = a_0 s_0 \alpha(s_1) \cdots \alpha^n(s_n) \cdots$.

For the automorphism we are considering, the algorithm of [26] provides two pairs of points agreeing on the first coordinate. We use prefix–suffix developments to define the pairs $(X_0, Y_0), (X_0, Y_0')$ and $(X_1, Y_1), (X_1, Y_1')$. Brackets followed by the symbol $*$ are repeated indefinitely; the empty word is denoted ϵ.

- $\rho(X_0, Y_0) = (\epsilon, a, b)(\epsilon, a, b)(\epsilon, a, \epsilon)(\epsilon, c, \epsilon)[(a, b, \epsilon)(\epsilon, a, b)(\epsilon, a, \epsilon)(\epsilon, c, \epsilon)]*,$
- $\rho(X_0, Y_0') = (\epsilon, c, \epsilon)(a, b, \epsilon)(\epsilon, a, b)(\epsilon, a, b)[(\epsilon, a, \epsilon)(\epsilon, c, \epsilon)(a, b, \epsilon)(\epsilon, a, b)]*.$
- $\rho(X_1, Y_1) = [(\epsilon, a, b)(\epsilon, a, \epsilon)(\epsilon, c, \epsilon)(a, b, \epsilon)]*,$
- $\rho(X_1, Y_1') = (a, b, \epsilon)(\epsilon, a, b)(\epsilon, a, b)(\epsilon, a, \epsilon)[(\epsilon, c, \epsilon)(a, b, \epsilon)(\epsilon, a, b)(\epsilon, a, \epsilon)]*.$

The point here is that $Q^2(X_0, Y_0) = Q^2(X_0, Y_0') = S_0$ and $Q^2(X_1, Y_1) = Q^2(X_1, Y_1') = S_1$. Note that the first letters of Y_0 and Y_0' (respectively, Y_1 and Y_1') are a and c (respectively, a and b), which is consistent with the fact that S_0 (respectively, S_1) belongs to both T_a and T_c (respectively, T_a and T_b).

Remark 12.3. Let (X, Y) be a point of Σ_α with prefix–suffix development $(p_i, a_i, s_i)_{i \in \mathbb{N}}$. The point $\partial^2 \alpha^{-1}(X, Y)$ is still in Σ_α if and only if $p_0 = \epsilon$. In that case, the prefix–suffix development of $\partial^2 \alpha^{-1}(X, Y)$ is $(p_{i+1}, a_{i+1}, s_{i+1})_{i \in \mathbb{N}}$. It is then obvious that $S_1 = \boldsymbol{\alpha}^{-1}(S_0)$.

Placing S_0 (or any other singularity) is now easy. Pick one of the Q^2-preimages: say (X_0, Y_0). Using its prefix–suffix development, define $u_0^{-1} = \alpha^4(a^{-1})$, and for any $n \in \mathbb{N}^*$, $u_n^{-1} = u_{n-1}^{-1} \alpha^{4n}(a^{-1})$. We have explained how the points $Q(u_n^{-1}\omega)$ can be placed, thanks to the tree substitution, and we use Proposition 5.1 to obtain:

$$S_0 = \lim_{n \to +\infty} Q(u_n^{-1}\omega).$$

The edge $(x_0, x_3, 3)$ (Fig. 16) is mapped on T to the interval $[\mathcal{P}, c\mathcal{P}]$, and the point z_3 (Fig. 17) is mapped on T the point $u\mathcal{P}$ (for some $u \in F_\mathcal{A}$). From Step 1, the positive orbit of the point $u\mathcal{P}$ is the reduction of the word $u_0^{-1}\omega$. We use the following proposition to place S_0 on the interval $[\mathcal{P}, c\mathcal{P}]$.

Proposition 12.4. ([25]) *Let r be the length of the interval $[\mathcal{P}, c\mathcal{P}]$; we define the vector $\mathbf{W} = (r\lambda_\alpha \ \ r\lambda_\alpha^{-1} \ \ r \ \ r\lambda_\alpha^2)$. The embedding of any tree T_n^s in T is such that any edge (x, y, i) with label $i \in \{1, 2, 3, 4\}$ becomes an interval of length $\lambda_\alpha^n \mathbf{W}(i)$.*

The interval $[\mathcal{P}, u\mathcal{P}]$ then has length $r\lambda_\alpha^5$. The periodicity of the prefix–suffix expansion, along with the tree substitution allows us to conclude that the length of the interval $[\mathcal{P}, S_0]$ is

$$r\lambda_\alpha^5 \sum_{k \geq 0} \lambda_\alpha^{4k} = \frac{r\lambda_\alpha^5}{1 - \lambda_\alpha^4}.$$

Acknowledgments

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