The large-N limit of transition kernels

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04/03/2017

Introduction

X: **real Gaussian variable** with law $\gamma(\mathrm{d}x) = e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}}$.

Heat kernel operator

For $f \in L^2(\mathbb{R})$ and $z \in \mathbb{R}$, we have

$$e^{\frac{1}{2}\Delta} f(z) = \mathbb{E}[f(X+z)] = \langle f | \psi_z \rangle_{L^2(\mathbb{R},\gamma)}.$$

$$\int_{\mathbb{R}} f(x) e^{-\frac{(x-z)^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbb{R}} f(x+z) e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbb{R}} f(x)\psi_z(x) e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}}$$

where $\psi_z(x) = e^{-\frac{z^2}{2} + zx}$.

Questions: In large-N limit? In free probability? q-deformation?

Results about the q-deformation in collaboration with Ching-Wei Ho

Random matrices

X_N: **Gaussian matrix** in the space $\mathbb{H}_N$ of Hermitian matrices, normalized by the following covariance for the entries:

$$\mathbb{E}[X_N(i,j)X_N(k,l)] = \frac{\delta_{ij}\delta_{kl}}{N}\delta_{ij}\delta_{kl}.$$

Y_N: **deterministic matrix** in $\mathbb{H}_N$.

Theorem (Wigner, 1958)

The empirical spectral distribution of X_N converges to the semicircular measure

$$\sigma(\mathrm{d}x) = \frac{1}{4\pi}\sqrt{4-x^2}\mathbb{1}_{[-2,+2]}\mathrm{d}x.$$

Free probability ($N = \infty$)

The basic construct: a noncommutative algebra $\mathcal{A}$ of "random variables" equipped with an "expectation functional" $\tau : \mathcal{A} \to \mathbb{C}$.

Theorem (Voiculescu, 1991)

If the empirical spectral distribution of Y_N converges to a compactly supported measure, there exists x and y elements of $(\mathcal{A}, \tau)$ such that, for any noncommmutative polynomial P,

$$\frac{1}{N}\operatorname{Tr}(P(X_N, Y_N)) \underset{N\to\infty}{\longrightarrow} \tau(P(x, y)).$$

The variables x and y are **freely independent**: the quantity $\tau(P(x, y))$ can be deduced from the semi-circular law of x and the law of y.

Transition operator for $N = \infty$

Theorem (Biane 1998)

If x has a semicircular law and y is ***freely independent*** *from x in $(\mathcal{A}, \tau)$, there is a kernel K_y such that, for all bounded function f,*

$$\tau[f(x+y)|y] = (K_y f)(y).$$

Compare to the classical case: if X is Gaussian and Y independent from X,

$$\mathbb{E}[f(X+Y)|Y] = e^{\frac{1}{2}\Delta} f(Y).$$

Transition kernel

X_N: **Hermitian Gaussian matrix**. Let $f \in L^2(\mathbb{H}_N)$. The function

$$Y_N \in \mathbb{H}_N \mapsto \mathbb{E}\Big[f(X_N + Y_N)\Big]$$

is given by $e^{\frac{1}{2}\Delta_N} f$, the heat-kernel semigroup at time 1.

This transition operator extends to matrix-valued functions, by applying it **entrywise**. If $f : \mathbb{H}_N \to \mathbb{H}_N$, then $e^{\frac{1}{2}\Delta_N} f : \mathbb{H}_N \to \mathbb{H}_N$ is such that

$$e^{\frac{1}{2}\Delta_N} f(Y_N) = \mathbb{E}\Big[f(X_N + Y_N)\Big].$$

Limit of $e^{\frac{1}{2}\Delta_N}$ when $N \to \infty$?

First step: Take a function $f : \mathbb{H}_N \to \mathbb{H}_N$ which is defined from a function $f : \mathbb{R} \to \mathbb{R}$ by functional calculus.

Example: $f(Y_N) = Y_N^3$.

Second step: Apply $e^{\frac{1}{2}\Delta_N}$ to f.

Example: $e^{\frac{1}{2}\Delta_N} f(Y_N) = Y_N^3 + Y_N + \frac{1}{2N}\operatorname{Tr}(Y_N)$.

However, $e^{\frac{1}{2}\Delta_N} f : \mathbb{H}_N \to \mathbb{H}_N$ is not anymore a function given by functional calculus.

Is it possible to make sense of $\lim_N e^{\frac{1}{2}\Delta_N} f : \mathbb{H}_N \to \mathbb{H}_N$ as $N \to \infty$?

Limit of $e^{\frac{1}{2}\Delta_N}$ when $N \to \infty$?

Is it possible to make sense of $\lim_N e^{\frac{1}{2}\Delta_N} f : \mathbb{H}_N \to \mathbb{H}_N$?

Yes, if we enlarge the space of functional calculus.

- Consider normalized trace $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$.
- Consider **trace polynomials**, i.e. polynomials in Y_N and normalized traces of power of Y_N.

Example : $f(Y_N) = Y_N^3 + Y_N \mathrm{tr}(Y_N) + \mathrm{tr}(Y_N^2)\mathrm{tr}(Y_N^3)$.

- **The trace polynomials** form an invariant space for $e^{\frac{1}{2}\Delta_N}$.

Example : If $f(Y_N) = Y_N^3$,

$$\text{then } e^{\frac{1}{2}\Delta_N} f(Y_N) = Y_N^3 + Y_N + \frac{1}{2N}\mathrm{tr}(Y_N).$$

Limit of $e^{\frac{1}{2}\Delta_N}$ when $N \to \infty$?

$\mathbb{C}\{X\}$: space of trace polynomials.

Fact

The action of Δ_N on $\mathbb{C}\{X\}$ decomposes as

$$\Delta_N = \Delta_\infty + \frac{1}{N^2}L,$$

for operators Δ_∞ and L whose actions are ***independent of*** N.

As a consequence, $\lim_{N\to\infty} e^{\frac{1}{2}\Delta_N} = e^{\frac{1}{2}\Delta_\infty} : \mathbb{C}\{X\} \to \mathbb{C}\{X\}$.

Convergence for Gaussian Hermitian matrices (Wigner)

- Convergence of the mean

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}(P(X_N))\right] = e^{\frac{1}{2}\Delta_N}(\operatorname{tr}(P))(0) = e^{\frac{1}{2}(\Delta_\infty + \frac{1}{N^2}L)}(\operatorname{tr}(P))(0)$$
$$\underset{N\to\infty}{\longrightarrow} e^{\frac{1}{2}\Delta_\infty}(\operatorname{tr}(P))(0) + O(1/N^2).$$

- Concentration around the mean

$$\mathbb{V}ar\left[\frac{1}{N}\operatorname{Tr}(P(X_N))\right] = \left[e^{\frac{1}{2}\Delta_N}\left(\operatorname{tr}(P) - (e^{\frac{1}{2}\Delta_N}\operatorname{tr}(P))(0)\right)^2\right](0)$$
$$\underset{N\to\infty}{\longrightarrow} \left[e^{\frac{1}{2}\Delta_\infty}\left(\operatorname{tr}(P) - (e^{\frac{1}{2}\Delta_\infty}\operatorname{tr}(P))(0)\right)^2\right](0) + O(1/N^2).$$

Check that $e^{\frac{1}{2}\Delta_\infty}\left(\operatorname{tr}(P) - (e^{\frac{1}{2}\Delta_\infty}\operatorname{tr}(P))(0)\right)^2$ is 0.

Brownian motions on Lie group

A **Brownian motion** $(g_t)_{t\geq 0}$ on a matrix Lie group G is a Markov process starting at 1_g **whose generator is the Laplacian** $\frac{1}{2}\Delta_G$ for a certain metric.

In particular, the expectation can be computed by the action of the semigroup of generator Δ_G:

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(P(g_t))\right] = \left(e^{\frac{t}{2}\Delta_G}(\mathrm{tr}(P))\right)(1_g).$$

We will use exactly the same proof.

Theorem (Biane, Rains, Xu 1997)

Convergence of the Brownian motion $(U_N(t))_{t\geq 0}$ on the unitary group $\mathbb{U}_N$ (unitary matrices of size $N \times N$): for all $t \geq 0$, and polynomial P,

$$\frac{1}{N}\operatorname{Tr}(P(U_N(t))) \text{ converges almost surely as } N \to \infty.$$

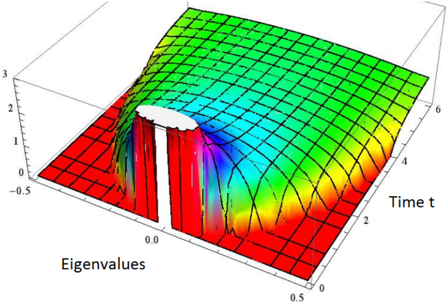

Figure: The limiting distribution of the eigenangles of $U_N(t)$

Demonstration

The action of $\Delta_{\mathbb{U}_N}$ on $\mathbb{C}\{X\}$ decomposes as $\Delta_{\mathbb{U}_N} = \Delta_{\mathbb{U}} + \frac{1}{N^2} D_{\mathbb{U}}$, from which we can deduce that

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}(P(X_N))\right] = e^{\frac{1}{2}\Delta_{\mathbb{U}_N}}(\operatorname{tr}(P))(1) \underset{N\to\infty}{\longrightarrow} e^{\frac{1}{2}\Delta_{\mathbb{U}}}(\operatorname{tr}(P))(1) + O(1/N^2)$$

and

$$\mathbb{V}ar\left[\frac{1}{N}\operatorname{Tr}(P(X_N))\right] \underset{N\to\infty}{\longrightarrow} O(1/N^2).$$

The same proof gives similar results for Brownian motions on others Lie groups.

- **Lévy (2011)**: for the orthogonal group $\mathbb{O}_N$ (and for the symplectic group $\mathbb{S}p(N)$)

$$\Delta_{\mathbb{O}_N} = \Delta_{\mathbb{O}} + \frac{1}{N} C_{\mathbb{O}} + \frac{1}{N^2} D_{\mathbb{O}}.$$

- **C. (2013)**: for the general linear group GL_N
- **Kemp (2015)**: for a two-parameter family of Brownian motions on GL_N which interpolates between $\mathbb{U}_N$ and GL_N

$$\Delta_{GL_N} = \Delta_{GL} + \frac{1}{N^2} D_{GL}.$$

It is also possible to consider more general situations.

- **Ulrich (2015)**: for the M^2 blocks of size $N \times N$ of a Brownian motion on $\mathbb{U}_{NM}$ (when $N \to \infty$)

$$\Delta_{\mathbb{U}_{NM}} = \Delta_{\mathbb{U},M} + \frac{1}{N^2} L_{\mathbb{U},M}.$$

$\mathbb{C}\{X\}$ must be replaced by the space $\mathbb{C}\{X_{ij} : 1 \leq i,j \leq M\}$ of trace polynomials in the M^2 blocks.

- **Gabriel (2015)**: for a random walk $(S_N(t))_{t\geq 0}$ on $\mathfrak{S}_N$ with generator

$$\mathcal{L}_{\mathfrak{S}_N} = \mathcal{L} + O(1/N)$$

$\mathbb{C}\{X\}$ must be replaced by a particular space of functions given by **traffic operations** (in the sense of Male), and we have

$$\mathbb{V}ar\left[\frac{1}{N}\mathrm{Tr}(P(S_N(t)))\right] \underset{N\to\infty}{\longrightarrow} \left[e^{\mathcal{L}}\Big(\mathrm{tr}(P) - (e^{\mathcal{L}}\mathrm{tr}(P))(0)\Big)^2\right](1) \neq 0$$

Central limit theorems

The Taylor expansion in $\frac{1}{N^2}$ can be used to prove central limit theorems:

$$e^{\frac{t}{2}\Delta_{\mathbb{U}_N}} = e^{\frac{t}{2}(\Delta_{\mathbb{U}}+\frac{1}{N^2}D_{\mathbb{U}})} = e^{\frac{t}{2}\Delta_{\mathbb{U}}} + \frac{1}{2N^2}\int_0^t e^{\frac{s}{2}\Delta_{\mathbb{U}}} D_{\mathbb{U}} e^{\frac{t-s}{2}\Delta_{\mathbb{U}}} + o(1/N^2)$$

- **Lévy-Maïda (2010)** : CLT for the Brownian motion on $\mathbb{U}_N$

$$N\left[\frac{1}{N}\mathrm{Tr}(P(U_N(t))) - \mathbb{E}\frac{1}{N}\mathrm{Tr}(P(U_N(t)))\right] \text{ is asymptotically Gaussian}$$

- **Dahlqvist (2014)** : CLT for the Brownian motion on $\mathbb{O}_N$ and $\mathbb{S}p_N$ + estimates of the Laplace transform
- **C.-Kemp (2014)** : CLT for the Brownian motion on GL_N

Back to $N = 1$

X: **real Gaussian variable** with law $\gamma(\mathrm{d}x) = e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}}$.

Heat kernel operator

For $f \in L^2(\mathbb{R})$ and $z \in \mathbb{R}$, we have

$$e^{\frac{1}{2}\Delta} f(z) = \mathbb{E}[f(X+z)] = \langle f | \psi_z \rangle_{L^2(\mathbb{R},\gamma)}$$

$$\int_{\mathbb{R}} f(x) e^{-\frac{(x-z)^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbb{R}} f(x+z) e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbb{R}} f(x) \overline{\psi_z(x)} e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}}$$

where $\psi_z(x) = e^{-\frac{z^2}{2} + zx}$.

$\langle f|\psi_z\rangle_{L^2(\mathbb{R},\gamma)}$ is the Segal-Bargmann transform

Set $\psi_z(x) = e^{-\frac{\bar{z}^2}{2}+\bar{z}x}$ and $\gamma^{\mathbb{C}}(\mathrm{d}z)$ the complex Gaussian variable of complex variance 1.

Theorem (Bargmann, Segal - 1958)

We have the resolution of the identity

$$Id_{L^2(\mathbb{R},\gamma)} = \int_{\mathbb{C}} |\psi_z\rangle\langle\psi_z| \,\mathrm{d}\gamma^{\mathbb{C}}(z).$$

In the sense that

$$\langle f|f\rangle_{L^2(\mathbb{R},\gamma)} = \int_{\mathbb{C}} \langle f|\psi_z\rangle_{L^2(\mathbb{R},\gamma)}\langle\psi_z|f\rangle_{L^2(\mathbb{R},\gamma)} \,\mathrm{d}\gamma^{\mathbb{C}}(z).$$

Equivalently, the functional which maps f to $z \mapsto \langle f|\psi_z\rangle_{L^2(\mathbb{R},\gamma)}$ is an isometry of Hilbert space.

q-Gaussian variables (Bozejko and Speicher, 1991)

Interpolation between the Gaussian and the semicircular law

- $q = 1$: Gaussian distribution $\mathrm{d}\gamma_1$
- $0 < q < 1$:

$$\mathrm{d}\gamma_q(x) = 1_{|x| \le 2/\sqrt{1-q}} \frac{1}{\pi} \sqrt{1-q} \sin\theta \prod_{n=1}^{\infty} (1-q^n)|1-q^n e^{2i\theta}|^2 \mathrm{d}x$$

  where $\theta \in [0, \pi]$ is such that $x = 2\cos(\theta)/\sqrt{1-q}$
- $q = 0$: semicircular distribution $\mathrm{d}\gamma_0$

q-deformation of the Segal-Bargmann transform

For $0 \leq q < 1$, and $|z| < 1/\sqrt{1-q}$, set

$$\psi_z^q(x) = \prod_{k=0}^{\infty} \frac{1}{1-(1-q)q^k \bar{z} x + (1-q) q^{2k} \bar{z}^2},$$

and $\gamma_q^{\mathbb{C}}$ a particular measure on $\mathbb{C}$ concentrated on a family of concentric circles.

Theorem (van Leeuwen and Maassen, 1995)

We have the resolution of the identity

$$Id_{L^2(\mathbb{R},\gamma_q)} = \int_{\mathbb{C}} |\psi_z^q\rangle\langle\psi_z^q| \, \mathrm{d}\gamma_q^{\mathbb{C}}(z).$$

q-deformation of the Segal-Bargmann transform

Between the Gaussian ($q = 1$) and the semicircular ($q = 0$):

Theorem (C.-Ho, 2017)

For any polynomial P, we have

- $q = 1$: $$e^{\frac{1}{2}\Delta} f(z) = \mathbb{E}[f(X+z)|z] = \langle f|\psi_z\rangle_{L^2(\mathrm{d}\gamma)}$$
- $0 < q < 1$: $$??? = \tau[P(x+z)|z] = \langle P|\psi_z^q\rangle_{L^2(\mathrm{d}\gamma_q)}$$ *for $x \sim \mathrm{d}\gamma_q$ and $z \sim \mathrm{d}\gamma_q^{\mathbb{C}}$ which are "q-independent"*
- $q = 0$: $$e^{\frac{1}{2}\Delta_\infty} P(z) = \tau[P(x+z)|z] = \langle P|\psi_z^0\rangle_{L^2(\mathrm{d}\gamma_0)}$$ *for $x \sim \mathrm{d}\gamma_0$ and $z \sim \mathrm{d}\gamma_0^{\mathbb{C}}$ which are freely independent*

Random matrices and q-deformation

In 2001, Śniady defines a random matrix model for the measure $\mathrm{d}\gamma_q$:

- γ_N is a measure on $\mathbb{H}_N$ such that, if $X_N \sim \gamma_N$, then X_N converges in noncommutative distribution to γ_q.
- $\gamma_N^{\mathbb{C}}$ is a measure on $\mathbb{M}_N$ such that, if $Z_N \sim \gamma_N$, then Z_N converges in noncommutative distribution to $\gamma_q^{\mathbb{C}}$.

Because $\tau[P(x+z)|z] = \langle P|\psi_z^q\rangle_{L^2(\mathrm{d}\gamma_q)}$ for $x \sim \mathrm{d}\gamma_q$ and $z \sim \mathrm{d}\gamma_q^{\mathbb{C}}$ which are "q-independent", we have the following result.

Theorem (q=0 by Biane in 1997, 0<q<1 by C.-Ho in 2017)

The following ***classical Segal-Bargmann transform*** *of a polynomial P*

$$M \mapsto \langle P|\psi_M\rangle_{L^2(\mathbb{H}_N,\gamma_N)}$$

*converges to the q-****deformed Segal-Bargmann transform*** *of the same polynomial*

$$z \mapsto \langle P|\psi_z\rangle_{L^2(\mathbb{R},\gamma_q)}$$

in the following sense: if Z_N is a random matrix of law $\gamma_N^{\mathbb{C}}$,

$$\mathbb{E}\left[\left\|\langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{H}_N,\gamma_N)} - \langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{R},\gamma_q)}\right\|^2\right] \underset{N\to\infty}{\longrightarrow} 0.$$

Sketch of proof: If X_N is a random matrix with law X_N, we have $\langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{H}_N,\gamma_N)} = \mathbb{E}[P(X_N + Z_N)|Z_N]$, and we can prove that

$$\mathbb{E}\left[\left\|\mathbb{E}[P(X_N + Z_N)|Z_N] - \langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{R},\gamma_q)}\right\|^2\right]$$

converges to

$$\left\|\tau[P(x+z)|z] - \langle P|\psi_z\rangle_{L^2(\mathbb{R},\gamma_q)}\right\|^2 = 0$$

where $x \sim \mathrm{d}\gamma_q$ and $z \sim \mathrm{d}\gamma_q^{\mathbb{C}}$ are "q-independent".

Thank you!