Statistics of real eigenvalues of asymmetric random matrices and their products

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Contents

Ginibre ensembles

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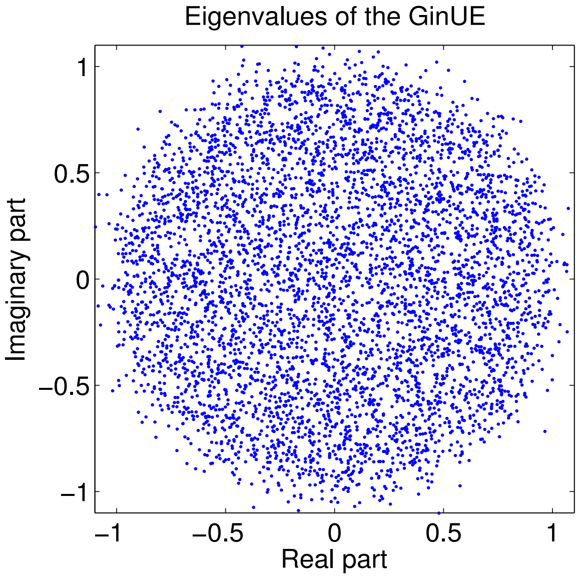
- Known as the *Ginibre orthogonal ensemble (GinOE)*.
- The Ginibre *unitary* ensemble (GinUE) instead consists of complex variables whose real and imaginary parts are $\mathcal{N}(0, 1/2)$.

Eigenvalues of $G/\sqrt{N}$ where $G \sim \mathrm{GinUE}$



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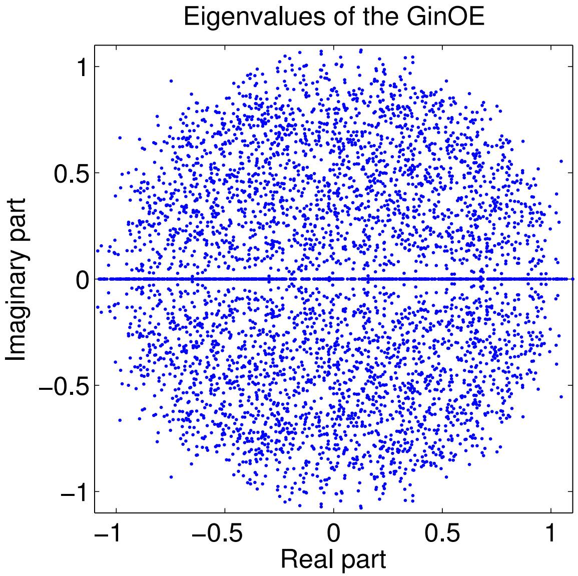

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- Jean Ginibre's 1965 paper introduced a total of three ensembles: GinUE, GinSE and GinOE, in increasing order of difficulty.
- Eigenvalue distribution: Ginibre completely solved the GinUE, partially solved GinSE and left GinOE unsolved.
- Lehmann and Sommers 1991. Joint PDF of GinOE complex eigenvalues $x_j + iy_j$ and $N_{\mathbb{R}}$ real eigenvalues λ_j:

$$\frac{1}{c_N}|\Delta| \exp\left(\sum_{j=1}^{N-N_{\mathbb{R}}} (y_j^2 - x_j^2) - \sum_{j=1}^{N_{\mathbb{R}}} \lambda_j^2/2\right) \prod_{j=1}^{N-N_{\mathbb{R}}} \mathrm{erf}(y_j\sqrt{2})$$

  where Δ is the product of differences over *all* eigenvalues. The factor Δ correlates all real and complex eigenvalues in a non-trivial way.

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- The correlation functions of real eigenvalues converge to those of annihilating Brownian motions (Tribe, Zaboronski '11, T,Z and Siu Kwan Yip '12).
- Recently developed for edge statistics by Poplavskyi, Tribe, Zaboronski (2016).
- Forrester (2013) used this idea to compute the nearest neighbour spacing distribution:

$$p_{\mathrm{GinOE}}(s) \approx s e^{-c_1 s}, \qquad c_1 = \frac{1}{\sqrt{2\pi}}\zeta(3/2)$$

which appears in intermediate spectral statistics ('mermaid statistics' - Beenakker et al. 2013). ($p_{\mathrm{GOE}}(s) \approx s e^{-c_2 s^2}$).

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Theorem (Edelman, Kostlan and Shub '94)

For an $N \times N$ real Ginibre matrix G, one has

$$\mathbb{E}(N_{\mathbb{R}}) = \sqrt{2N/\pi} + O(1) \qquad N \to \infty$$

and the convergence to the uniform law

$$\frac{1}{\mathbb{E}(N_{\mathbb{R}})} \mathbb{E}\left[\sum_{j=1}^{N} \delta(\lambda_j - x)\right] \to \begin{cases} \frac{1}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

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Fluctuations: Is there a central limit theorem for $N_{\mathbb{R}} - \mathbb{E}(N_{\mathbb{R}})$?

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as $N \to \infty$.

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Compare to known density for the complex eigenvalues (Burda et al., Götze and Tikhomirov, O'Rourke and Soshnikov 2010): $p(z) = \frac{1}{m\pi}|z|^{2/m-2} 1_{|z|<1}$.

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Idea of the proof is to compute moments and show that

$$\lim_{N\to\infty} \frac{1}{\mathbb{E}(N_{\mathbb{R}}^{(m)})} \mathbb{E}\left[\sum_{j=1}^{N_{\mathbb{R}}^{(m)}} \lambda_j^k\right] = \begin{cases} \frac{1}{1+mk}, & k \quad \text{even} \\ 0, & k \quad \text{odd} \end{cases}$$

Scalar case and Meijer G-functions

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$$w_m(x) := \int_{\mathbb{R}^m} \prod_{j=1}^{m} dx_j\, e^{-x_j^2/2} \delta(x - x_1 x_2 \ldots x_m) = G^{m,0}_{0,m}\left(\begin{matrix} \text{—} \\ 0,\ldots,0 \end{matrix} \,\middle|\, \frac{x^2}{2^m} \right).$$

where the *Meijer G-function* is

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix} \,\middle|\, z \right) = \frac{1}{2\pi i} \int_\gamma \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s \, ds$$

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The contour γ connects $-i\infty$ to $+i\infty$ such that *all poles* of $\Gamma(b_j - s)$ on right and $\Gamma(1 - a_k + s)$ on left.

What about the real eigenvalues of X_m $N > 1$?

Products form a Pfaffian point process

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Theorem (Forrester and Ipsen 2016)

The real eigenvalues of the matrix product $G_1 \dots G_m$ form a Pfaffian point process with correlation kernel given by

$$\mathbb{K}(x,y) = \begin{pmatrix} D(x,y) & S(x,y) \\ -S(y,x) & I(x,y) \end{pmatrix}$$

where

$$S(x,y) = \sum_{j=0}^{N-2} \frac{w_m(x)x^j}{(2\sqrt{2\pi}j!)^m}(xA_j(y) - A_{j+1}(y))$$

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In particular, the desired moments are just

$$M_{k,N} := \mathbb{E}\left[\sum_{j=1}^{N_{\mathbb{R}}^{(m)}} \lambda_j^k\right] = \int_{\mathbb{R}} x^k S(x,x)\,dx$$

Moment formula

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The last integral splits in two pieces
$M_{2k,N}(m) = M^{(1)}_{2k,N}(m) - M^{(2)}_{2k,N}(m)$ where

$$M^{(1)}_{2k,N}(m) = N^{-mk} \sum_{j=0}^{(N-2)/2} \frac{2^{(2j+k)m}}{(\sqrt{\pi}(2j)!)^m} (a_{j+1,j+k+1} + a_{j+k+1,j+1})$$

$$M^{(2)}_{2k,N}(m) = N^{-mk} \sum_{j=0}^{N/2-2} \frac{2^{(2j+1+k)m}}{(\sqrt{\pi}(2j+1)!)^m} (a_{j+k+2,j+1} + a_{j+2,j+k+1})$$

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Here $a_{j,k}$ is a particular case of the Meijer-G function

$$\begin{aligned} a_{j,k} &= G^{m+1,m}_{m+1,m+1}\left(\begin{matrix} 3/2-j,\ldots,3/2-j,1 \\ 0,k,\ldots,k \end{matrix} \,\middle|\, 1 \right) \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{(\Gamma(k-s)\Gamma(-1/2+j+s))^m}{-s}\, ds \end{aligned}$$

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The formula

$$\left(\frac{\Gamma(k-s)\Gamma(-1/2+j+s)}{\Gamma(j+k-1/2)}\right) = \int_0^\infty \frac{t^{k-s-1}}{(1+t)^{k+j+1/2}}\,dt$$

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implies that $a_{j+1,j+k+1}$ can be written

$$\Gamma(2j+k+3/2)^m \int_1^\infty \frac{dx_m}{x_m} \prod_{l=1}^{m-1} \left[\int_0^\infty \frac{dx_l}{x_l} \frac{(x_l/x_{l+1})^{j+1/2}}{(1+x_l/x_{l+1})^{2j+k+3/2}}\right] \frac{x_1^{j+k+1}}{(1+x_1)^{2j+k+3/2}}$$

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Because of cancellations (coming from $\mathrm{sgn}(x-y)$) one has to go to sub-leading order.

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Highly universal *bounded variance* central limit theorems.

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Theorem (Rand. Mat. Theor. Appl. 2017)

The variance of the total number of real eigenvalues of a real $(2n \times 2n)$ Gaussian random matrix is given by the following explicit formula

$$\boxed{\mathrm{Var}(N_{\mathbb{R}}) = \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{k=1}^{n} \frac{\Gamma(2k-3/2)}{\Gamma(2k-1)} - \frac{2}{\pi} \sum_{k_1,k_2} \frac{\Gamma(k_1+k_2-3/2)^2}{\Gamma(2k_1-1)\Gamma(2k_2-1)}}$$

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- One can derive similar exact formulae for any monomial eigenvalue statistics: $\mathrm{Cov}(X_N(\lambda^p), X_N(\lambda^q))$.

Central limit theorem

Theorem (*Rand. Mat. Theor. Appl.* 2017)

Let G be a $2n \times 2n$ matrix of standard i.i.d. Gaussians. Let $\lambda_1, \ldots, \lambda_{N_{\mathbb{R}}}$ be the real eigenvalues and let $P(\lambda)$ be an even polynomial. Then

$$n^{-1/4}\left(\sum_{j=1}^{N_{\mathbb{R}}} P(\lambda_j/\sqrt{2n}) - \mathbb{E}\left(\sum_{j=1}^{N_{\mathbb{R}}} P(\lambda_j/\sqrt{2n})\right)\right) \longrightarrow \mathcal{N}(0, \sigma^2(P))$$

as $n \to \infty$, where

$$\sigma^2(P) = \frac{2-\sqrt{2}}{\sqrt{\pi}} \int_{-1}^{1} P(x)^2 \, dx$$

Central limit theorem

Theorem (*Rand. Mat. Theor. Appl.* 2017)

Let G be a $2n \times 2n$ matrix of standard i.i.d. Gaussians. Let $\lambda_1, \dots, \lambda_{N_{\mathbb{R}}}$ be the real eigenvalues and let $P(\lambda)$ be an even polynomial. Then

$$\boxed{n^{-1/4}\left(\sum_{j=1}^{N_{\mathbb{R}}} P(\lambda_j/\sqrt{2n}) - \mathbb{E}\left(\sum_{j=1}^{N_{\mathbb{R}}} P(\lambda_j/\sqrt{2n})\right)\right) \longrightarrow \mathcal{N}(0, \sigma^2(P))}$$

as $n \to \infty$, where

$$\sigma^2(P) = \frac{2-\sqrt{2}}{\sqrt{\pi}} \int_{-1}^{1} P(x)^2 \, dx$$

See also pre-print by Phil Kopel (2015) - same CLT provided $\mathrm{supp}(f) \subset (-1+\epsilon, 1-\epsilon)$ (so not for $N_{\mathbb{R}}$).

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The moment generating function of any even linear statistic is a determinant:

$$\mathbb{E}e^{s\sum_{j=1}^{N_{\mathbb{R}}} f(\lambda_j)} = \det\left(\delta_{jk} + \frac{A[e^{s(f(x)+f(y))} - 1]_{2j,2k-1}}{\sqrt{2\pi\Gamma(2j-1)\Gamma(2k-1)}}\right)_{j,k=1}^{n}$$

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$$A[\psi]_{jk} = \frac{1}{2}\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\ \psi(x)\psi(y)e^{-x^2/2-y^2/2}P_{j-1}(x)P_{k-1}(y)\mathrm{sign}(y-x)$$

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$$\kappa_p = p! \sum_{m=1}^{p} \frac{(-1)^{q+1}}{q} \sum_{\nu_1+\ldots+\nu_q=p} \frac{\mathrm{Tr}(M_n^{(\nu_1)}[f]\ldots M_n^{(\nu_q)}[f])}{\nu_1!\ldots\nu_q!}$$

where $M_n^{(\nu)}[f]_{j,k} = A[(f(x)+f(y))^{\nu}]_{2j,2k-1}, \; j,k = 1,\ldots,n.$

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Ultimately obtain $\kappa_p = O(\sqrt{n})$ as $n \to \infty$.

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$$\mathbb{E}(N_{\mathbb{R}}(P)) = \sqrt{N}$$

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Note that for real Ginibre

$$c = \lim_{N\to\infty} \frac{\mathrm{Var}(N_{\mathbb{R}})}{\mathbb{E}(N_{\mathbb{R}})} \sim 2 - \sqrt{2} = 0.5857\ldots$$

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Thank you.