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Extreme behaviour for bivariate elliptical distributions

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Abstract: The authors examine the asymptotic behaviour of conditional threshold exceedance probabilities for an elliptically distributed pair (X, Y) of random variables. More precisely, they investigate the limiting behaviour of the conditional distribution of Y given that X becomes extreme. They show that this behaviour differs between regularly and rapidly varying tails.

Le comportement extrême des lois elliptiques bivariées

Résumé : Les auteurs s'intéressent au comportement asymptotique de probabilités conditionnelles de dépassement d'un seuil pour une paire (X, Y) de variables aléatoires de loi elliptique. Plus précisément, ils étudient le comportement limite de la loi conditionnelle de Y sachant que X devient extrême. Ils montrent que ce comportement diffère suivant que les queues de la loi sont à variations régulières ou rapides.

1. INTRODUCTION

Multivariate extreme events are common in finance, insurance, hydrology and climatology, among other fields. Multivariate extreme value theory is generally regarded as the standard tool for modeling such events, despite the fact that several authors have recently pointed out their possible limitations; see, e.g., Bruun & Tawn (1998), Heffernan & Tawn (2004), Maulik & Resnick (2004), or Abdous, Fougères & Ghoudi (2004a).

In practice, extreme-value models perform well provided that the data satisfy the so-called asymptotic tail dependence property (see, e.g., Sibuya 1960; Joe 1993). Specifically, a bivariate random vector (X, Y) with marginal distributions F_X and F_Y is said to be tail dependent whenever

$$\lambda = \lim_{u \to 1} \mathrm{P}\big\{F_Y(Y) > u \,|\, F_X(X) > u\big\} > 0,$$

when this limit exists, in which case $\lambda \in (0, 1)$ is called the (upper) tail dependence coefficient.

In recent years, tail dependence has been heavily investigated, particularly in financial contexts, where its applications are numerous; see, e.g., Embrechts, McNeil & Straumann (2002), Schmidt (2002) or Frahm, Junker & Szimayer (2003). Nevertheless, the notion is somewhat limiting, as both components in the definition of λ are required to become extreme at the same rate.

A more general point of view on tail dependence has recently been proposed in Heffernan & Tawn (2004) and Abdous, Fougères & Ghoudi (2004a). Given an arbitrary random pair (X, Y) on $\mathbb{R}^2$, Heffernan and Tawn focus on the limiting form of $\mathrm{P}(Y \le y \,|\, X = x)$ as $x \to \infty$.

In this paper, however, we will follow Abdous, Fougères & Ghoudi (2004a) in considering

$$\theta(x, y) = \mathrm{P}(Y \le y \,|\, X > x)$$

as the main measure of interest. More precisely, we will focus on the limiting behaviour of the probability $\theta(x, y)$, when x becomes extreme, while y is either fixed or becomes extreme at various possible rates, so that the limit of $\theta(x, y)$ when $x \to \infty$ is nondegenerate.

One argument for working with $\theta(x, y)$ instead of $\mathrm{P}(Y \le y \,|\, X = x)$ is that it is a natural generalization of the tail dependence coefficient. Moreover, in finance and risk management, this function is related to contagion and stress testing concepts. Contagion formalizes the fact that $1 - \theta(x, y) > \mathrm{P}(Y > y)$, where X and Y denote two positively dependent market returns. As for stress tests, they measure the influence of large movements in financial markets on portfolio values. Both notions are thus related to the conditional distribution of a portfolio value given unexpected activities in the financial market.

Proposition 3.2 of Abdous, Fougères & Ghoudi (2004a) shows that a simple relation links $\theta(x, y)$ and $\mathrm{P}(Y \le y \,|\, X = x)$ when $y \to \infty$, so that $y = \xi(x) \to \infty$ as $x \to \infty$, with ξ differentiable. In the regular cases where X and Y have densities f_X and f_Y respectively, one obtains

$$\lim_{x\to\infty} \theta(x, y) = \lim_{x\to\infty} \left\{ \mathrm{P}(Y \le y \,|\, X = x) - \frac{f_Y(y)}{f_X(x)} \xi'(x) \mathrm{P}(X > x \,|\, Y = y) \right\}.$$

The aim of this paper is to investigate the possible nondegenerate limiting behaviours of $\theta(x, y)$ as $x \to \infty$, in the specific case where (X, Y) follows a bivariate elliptical distribution. This is of special interest, given that elliptical distributions provide a wide range of tail behaviours and are commonly used in financial and risk models, where the Value-at-Risk (VaR) is a coherent risk measure as used by Artzner, Delbaen, Eber & Heath (1999). Important special cases of elliptical distributions include the multivariate Student distributions and the regular centered Gaussian mixtures. See Abdous, Genest & Rémillard (2004) for additional properties of this class of dependence models.

Background material is reviewed in Section 2. Explicit formulas for $\theta(x, y)$ as $x \to \infty$ are then given in Section 3, both when y becomes extreme at a suitable rate, as well as when y is fixed. These formulas are then illustrated in Section 4, where they are used to provide estimates of $\theta(x, y)$ for given values of x and y. Conclusions and a discussion are presented in Section 5. Proofs are relegated to the Appendix.

2. PRELIMINARIES

The following background information is excerpted from Fang, Kotz & Ng (1990); see also Schmidt (2002). Let X and Y be random variables with cumulative distribution functions F_X and F_Y, means μ_X and μ_Y, and standard deviations σ_X and σ_Y, respectively. When its joint distribution is elliptical, the pair (X, Y) can then be represented in the form

$$(X, Y) = (\mu_X, \mu_Y) + R\big(\sigma_X D\, U_1, \sigma_Y \big\{\rho D\, U_1 + \sqrt{1-\rho^2}\,\sqrt{1-D^2}\, U_2\big\}\big).$$

Here, ρ is Pearson's correlation between X and Y; furthermore, U_1, U_2, D and R are mutually independent random variables, R and D are positive, $D^2 \sim \mathrm{Beta}(1/2, 1/2)$, and

$$\mathrm{P}(U_i = 1) = \mathrm{P}(U_i = -1) = 1/2, \quad i = 1, 2.$$

Consequently, the marginal distribution functions can then be seen to satisfy the relation $\sigma_X F_Y^{-1}(v) = \sigma_Y F_X^{-1}(v)$ for all $v \in (0, 1)$. Furthermore,

$$\mathrm{P}(Y \le y \,|\, X > x) = \mathrm{P}\big\{\widetilde{Y} \le (y - \mu_Y)/\sigma_Y \,|\, \widetilde{X} > (x - \mu_X)/\sigma_X\big\},$$

where $\widetilde{X}$ and $\widetilde{Y}$ are the standardized versions of X and Y. It may thus be assumed without loss of generality that $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$, as will be done in the sequel.

In the next section it will be seen that in essence, the behaviour of $\theta(x, y)$ is ruled by the tail behaviour of the radius component R. In fact, this behaviour will be shown to differ between the classes of regularly and rapidly varying tails (Resnick 1987), whose definitions are recalled next.

We say that R has a regularly varying tail with index $-\alpha < 0$ whenever

$$\mathcal{H}_\alpha\colon \quad \frac{\mathrm{P}(R > \lambda x)}{\mathrm{P}(R > x)} \longrightarrow \lambda^{-\alpha}$$

as $x \to \infty$. Furthermore, we say that R has a rapidly varying tail whenever

$$\mathcal{H}_\infty\colon \quad \frac{\mathrm{P}(R > \lambda x)}{\mathrm{P}(R > x)} \longrightarrow \begin{cases} \infty & \text{if } 0 < \lambda < 1, \\ 1 & \text{if } \lambda = 1, \\ 0 & \text{if } \lambda > 1, \end{cases}$$

as $x \to \infty$. We will assume a slightly stronger hypothesis than $\mathcal{H}_\infty$, which consists in assuming the existence of an auxiliary function ψ such that

$$\mathcal{H}_\infty^+\colon \quad \lim_{x\to\infty} \frac{\mathrm{P}\{R > x + t\psi(x)\}}{\mathrm{P}(R > x)} = e^{-t}.$$

See Resnick (1987, p. 26) for further details. This function is positive, continuous and satisfies

$$\lim_{t\to\infty} \frac{\psi\{t + x\psi(t)\}}{\psi(t)} = 1, \quad \lim_{t\to\infty} \psi'(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} \frac{\psi(t)}{t} = 0.$$

The class of distribution functions satisfying $\mathcal{H}_\infty^+$ was introduced by de Haan (1970) as the $\Gamma-$ varying class. Note finally that the auxiliary function is unique up to asymptotic equivalence, and that it can be written as $\psi(t) = \{1 - H(t)\}/H'(t)$, where H denotes the distribution function of R; see, e.g., Resnick (1987, p. 40). Some insight into the reasons for looking at the classes $\mathcal{H}_\alpha$ and $\mathcal{H}_\infty^+$ of tails is given in the next section, after Theorem 1.

3. TAIL BEHAVIOUR FOR ELLIPTICAL DISTRIBUTIONS

This section describes how the conditional probability $\theta(x, y) = \mathrm{P}(Y \leq y \,|\, X > x)$ behaves when x becomes extreme while y is also extreme or fixed. Theorem 1 tackles the situation where both x and y become extreme, whereas Theorem 2 deals with the case where y is fixed. The function $t \to \mathrm{sign}(t)$ is defined as

$$\mathrm{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0. \end{cases}$$

THEOREM 1. *Let (X, Y) be a bivariate standardized elliptical random variable. A non-degenerate limit exists for $\mathrm{P}(Y \leq y \,|\, X > x)$ as both $x, y \to \infty$ in either one of the following circumstances:*

(i) *R satisfies $\mathcal{H}_\alpha$, and $y \sim \rho x + zx$ when $x \to \infty$, for $z \in \mathbb{R}$. Then*

$$\lim_{x,y\to\infty} \mathrm{P}(Y \leq y \,|\, X > x) =$$
$$T_{\alpha+1}\left(\frac{z\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\right) - \frac{\mathrm{sign}(\rho+z)}{|\rho+z|^\alpha}\,\overline{T}_{\alpha+1}\left\{\mathrm{sign}(\rho+z)\frac{\sqrt{\alpha+1}}{\sqrt{1-\rho^2}}\left(\frac{1}{\rho+z} - \rho\right)\right\},$$

where $T_\nu(x)$ is the cumulative distribution function of a univariate Student random variable with ν degrees of freedom and $\overline{T}_\nu(x) = 1 - T_\nu(x)$.

(ii) *R satisfies $\mathcal{H}_\infty^+$, and as $x \to \infty$, $y \sim \rho x + z\sqrt{x\psi(x)}$ for some $z \in \mathbb{R}$. Then one has*

$$\lim_{x\to\infty} \mathrm{P}(Y \leq y \,|\, X > x) = \Phi\left(\frac{z}{\sqrt{1-\rho^2}}\right),$$

where Φ is the standard normal distribution function.

Remark. Observe that by setting $z = 1 - \rho$ in Part (i) of Theorem 1, one can recover the formula for the tail dependence coefficient λ given by Embrechts, McNeil & Straumann (2002) for the Student distribution and by Schmidt (2002) in the more general case of elliptical distributions. The same substitution further exhibits the well-known fact that the regularly varying case corresponds to asymptotic dependence.

In view of Theorem 1, the extreme-value rate $y \sim \rho x + zx$ produces a nondegenerate limit (Abdous, Fougères & Ghoudi 2004a). In cases of asymptotic independence, typically occurring under $\mathcal{H}_\infty^+$, this rate turns out to be too fast, however. For, it leads to a degenerate limit for $\theta(x,y)$ as $x \to \infty$. Part (ii) of Theorem 1 precisely exhibits the proper rate to choose, namely $y \sim \rho x + z\sqrt{x\psi(x)}$. This rate is slower than the extreme-value rate, since $\psi(x)/x \to 0$ as $x \to \infty$.

The next theorem states the limiting behaviour of $\theta(x,y)$ as $x \to \infty$ and y remains fixed.

THEOREM 2. *Let (X,Y) be a bivariate standardized elliptical random variable. For any fixed $y \in \mathbb{R}$, one has*

(i) *Under $\mathcal{H}_\alpha$,* $\lim_{x\to\infty} \mathrm{P}(Y \le y \mid X > x) = T_{\alpha+1}\left(-\rho\sqrt{\frac{\alpha+1}{1-\rho^2}}\right)$.

(ii) *Under $\mathcal{H}_\infty^+$,* $\lim_{x\to\infty} \mathrm{P}(Y \le y \mid X > x) = \mathbb{1}(\rho < 0)$ for $\rho \neq 0$ *and for $\rho = 0$, one has*

$$\lim_{x\to\infty} \mathrm{P}(Y \le y \mid X > x) = \begin{cases} 1/2 & \text{if } \lim_{x\to\infty} x\psi(x) = \infty, \\ \Phi\big(y/\sqrt{\lambda}\big) & \text{if } \lim_{x\to\infty} x\psi(x) = \lambda, \\ \{\mathrm{sign}(y)+1\}/2 & \text{if } \lim_{x\to\infty} x\psi(x) = 0. \end{cases}$$

In other words, Theorem 2 states that the only case where a nondegenerate limit occurs for fixed y corresponds to a rapidly varying distribution with $\rho = 0$ and $x\psi(x) \to \lambda$. In this specific situation, the two rates arising from Theorem 1 and 2 coincide. This happens in particular for the normal and the logistic distributions, as shown in Table 1. The table summarizes, for some classical bivariate elliptical distributions, the tail properties required for the application of Theorems 1 and 2.

TABLE 1: Index of regular variation α or auxiliary function ψ for some examples of bivariate elliptical distributions.

Bivariate distribution	Generator $g(u)^\dagger$	α	$\psi(x)$	$\lim_{x\to\infty} x\psi(x)$
Student	$(1+u/\nu)^{-(\nu+2)/2}$	ν	–	–
Logistic	$e^{-u}/(1+e^{-u})^2$	∞	$1/(2x)$	1/2
Kotz, $s<1$	$u^{s-1}\exp(-u^s)$	∞	x^{1-2s}	∞
Normal (Kotz, $s=1$)	$e^{-u/2}$	∞	$1/x$	1
Kotz, $s>1$	$u^{s-1}\exp(-u^s)$	∞	x^{1-2s}	0
Symmetric generalized hyperbolic ‡	$K_{s-1}\{\sqrt{a(b+u)}\}/(b+u)^{(1-s)/2}$	∞	$1/\sqrt{a}$	∞

† Recall that the standardized elliptical density function can be written as $f(x,y) = Cg\{(x^2-2\rho xy+y^2)/(1-\rho^2)\}$, in terms of a generator g and a positive constant C. The density of R is given by $h(r) = Krg(r^2)$, where K is a normalizing constant (see, e.g., Fang, Kotz & Ng 1990).

‡ K_ν is the modified Bessel function of the third kind (or MacDonald's function), s is a real number and a, b are positive real numbers.

Figure 1 illustrates the various limiting behaviours obtained in Theorem 1. One can see from panels (a) and (b) that a nondegenerate behaviour is obtained for $\lim_{x\to\infty} \theta(x, y)$ when $y \sim (\rho + z)x$ in the Student case. One can also see from panels (c) and (d) that $y \sim \rho x + z\sqrt{x\psi(x)}$ in the normal case. Note also that as the degree of freedom of the Student distribution increases, one "gets closer to the $\mathcal{H}_\infty^+$-case" described in Theorem 1(ii), and therefore closer to the discontinuous function $\mathbb{1}_{\mathbb{R}^+}$ when looking at the rate $y \sim (\rho + z)x$.

Figure 2 shows some of the possible asymptotic behaviours described in Theorem 2. There is only one case which leads to a nondegenerate behaviour of $\theta(x, y)$ when y is fixed. This is illustrated by curve (c) and it occurs under the assumption $\mathcal{H}_\infty^+$ for $\rho = 0$ and $\lim_{x\to\infty} x\psi(x) = c$, where $0 < c < \infty$.

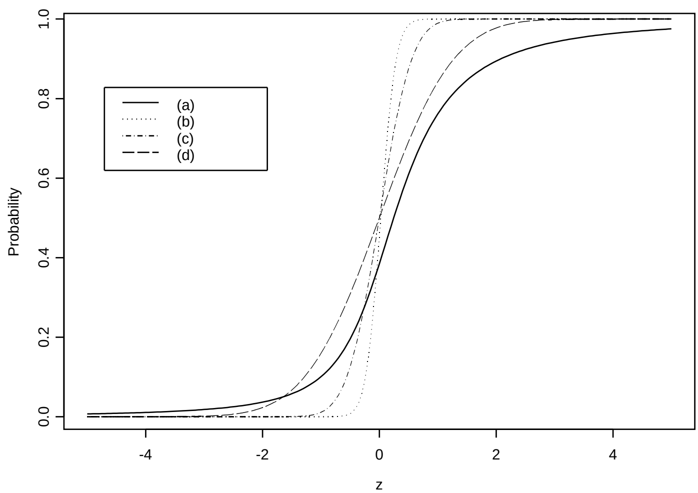

FIGURE 1: Plots of $\lim_{x\to\infty} \mathrm{P}\{Y \le (\rho + z)x \,|\, X > x\}$ in terms of z for (a) a bivariate Student distribution with $\rho = 0.5$ and $\nu = 2$; (b) a bivariate Student distribution with $\rho = 0.5$ and $\nu = 20$ and plots of $\lim_{x\to\infty} \mathrm{P}\{Y \le \rho x + z\sqrt{x\psi(x)} \,|\, X > x\}$ in terms of z for (c) a bivariate normal distribution with $\rho = 0.9$ and (d) a bivariate normal distribution with $\rho = 0$.

4. APPLICATIONS

In practical situations where bivariate random vectors are observed, one often has to evaluate the probability that one of the components belongs to a specific domain, given that the other is greater than a fixed value. Whenever the domain of interest does not contain any of the observed vectors, the classical empirical estimates of this probability either cannot be evaluated or take degenerate values of 0 or 1. To overcome this deficiency, one can rely on the estimation of the limits given in Theorems 1 and 2, whenever the distribution can be assumed to be elliptical. Evaluation of these limits only requires an estimate of the correlation coefficient ρ, and the tail index α, or the auxiliary function ψ.

The first two parameters have been widely studied in the literature. As for the estimation of the auxiliary function ψ, it is considered in a forthcoming paper; see Abdous, Fougères & Ghoudi (2004b). Practical aspects of Theorem 1 and Theorem 2 when dealing with real data are investigated therein as well. As an illustration, we will focus hereafter on the regularly varying case, which requires the estimation of ρ and α only.

Estimation of ρ is classic. One can merely use ρ_n, the empirical version of ρ. However, in order to avoid the instability of ρ_n in heavy-tailed cases, it is advisable to make use of an estimate based on the relationship $\rho = \sin(\pi\tau/2)$ that exists between Kendall's tau and Pearson's correlation coefficient for elliptical distributions. See Lindskog (2000) or Hult & Lindskog (2002) for a discussion of this approach.

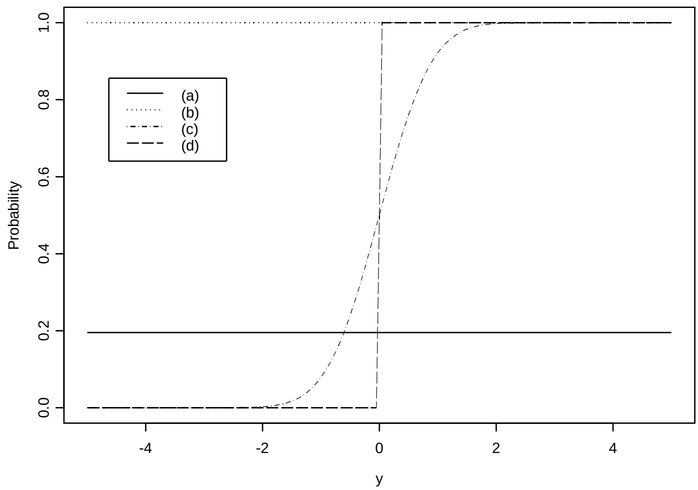

FIGURE 2: Plots of $\lim_{x\to\infty} \mathrm{P}(Y \le y \,|\, X > x)$ in terms of y for (a) a bivariate Student distribution with $\rho = 0.5$ and $\nu = 2$; (b) a bivariate normal distribution with $\rho = -0.9$; (c) a bivariate logistic distribution with $\rho = 0$; and (d) a bivariate Kotz distribution with $s > 1$ and $\rho = 0$.

Several options exist for the estimation of the tail index α, but none of them dominates its competitors universally. The most classic estimator is that of Hill (1975). Its performance strongly depends on the sample fraction used for the estimation, and many strategies have been proposed to optimize this choice. We refer for example to the papers of Dekkers, Einmahl & de Haan (1989), Drees & Kaufmann (1998), Gomes & Oliveira (2001), or Matthys & Beirlant (2002), among others. Some graphical methods such as Hill plots have also been proposed; see, e.g., Resnick & Stărică (1997) and Drees, de Haan & Resnick (2000). As reported in the above references, the bias term remains large in case of light tails, even if the sample fraction is selected optimally. In this paper, we adopt the approach of Huisman, Koedijk, Kool & Palm (2001). These authors proposed a specific bias correction technique, which presents some improvement over the previous estimators.

In what follows, we rely on a small simulation study to illustrate a way of using Theorem 1 (i) in practice. Bivariate Student distributions with various degrees of freedom ν have been simulated. Without loss of information, we provide in Table 2 below results for two models only, $\nu = 2$ and 20. In both cases, we simulated 1000 samples of size 500. For each sample and various values of x and y, we evaluated:

- an estimation of the probability $\theta(x, y)$ using Theorem 1(i);
- an estimation of the probability $\theta(x, y)$ using the empirical distribution;
- the exact value of $\theta(x, y)$, computed by numerical integration;
- the exact value of the limit stated in Theorem 1(i), also computed by numerical integration.

The values of x and y for which the probability is estimated correspond to the 0.975, 0.99, 0.999, 0.9999 and 0.99999 marginal quantiles. Table 2 contains the mean and standard deviation obtained from the 1000 samples of bivariate Student distributions with 2 and 20 degrees of freedom. Different values of ρ have also been examined and are not reported here, since they lead to quite similar performances.

First note that the empirical estimator fails for large quantiles. In particular, when x is the quantile of order 99.99%, it was only possible to compute it for 4.5% of the simulations. This proportion drops to around 0.2 % for x is the quantile of order 99.999%. The results also show that the estimates obtained via Theorem 1 provide an excellent alternative even for extreme quantiles. As outlined in Table 2, the accuracy of these estimates increases with the heaviness of the tails. A careful look into the simulation details shows that for light tails the loss of accuracy is due to the poor behaviour of the tail index estimator. This has already been discussed by Huisman, Koedijk, Kool & Palm (2001).

5. CONCLUSION AND DISCUSSION

In this paper, the extreme behaviour of the conditional distribution associated with a random pair of elliptical random variables (X, Y) has been considered. The asymptotic behaviour of $\theta(x, y) = \mathrm{P}(Y \leq y \mid X > x)$ as $x \to \infty$ has been studied, and expressions have been determined for y, as a function of x, which lead to nondegenerate limit. Such functions $y = \varphi(x)$ differ, depending on the tail behaviour of the radial component associated with (X, Y). The estimation problem connected with the theoretical results obtained has been treated in the regularly varying case. The rapidly varying case involves the estimation of a less classical tool—namely, the auxiliary function defined in Section 2—and will be discussed in a subsequent paper; see Abdous, Fougères & Ghoudi (2004b). The simulation results highlight the relevance of Theorem 1 as an alternative way to estimate the probability of extreme events, as required in the financial context of contagion or stress testing, for example.

Interestingly, Theorem 1 also settles in part a conjecture formulated by Eddy & Gale (1981) while they were studying the convex hull of a spherically symmetric random vector (X, Y). They noticed that the random measure associated with this convex hull is related to the extreme behaviour of the underlying distribution. In particular, in their Theorem 4.1, they identified the random measure for a special class $\mathcal{C}$ of spherical distributions with exponential tails, namely for survival functions $\overline{H}(r) = \mathrm{P}(R > r)$ of the form $r^{\alpha} \exp(-r^{\beta})$ for arbitrary real α and for $\beta > 1$.

The key point in their proof consisted of showing that for $z \in \mathbb{R}$ and $t > 0$,

$$\lim_{n\to\infty} \frac{\mathrm{P}(X > b_n + a_n^{-1}t, Y < a_n^{-1}c_n^{-1}z)}{\mathrm{P}(X > b_n)} = \Phi(z)\exp(-t),$$

where b_n is the solution of $\overline{H}(b_n) = 1/n$, $a_n = \beta b_n^{\beta-1}$ and $c_n = \{\beta \log(n)\}^{-1/2}$. The above limit can be rewritten as

$$\lim_{n\to\infty} \frac{\mathrm{P}(X > b_n + a_n^{-1}t, Y < a_n^{-1}c_n^{-1}z)}{\mathrm{P}(X > b_n + a_n^{-1}t)} \frac{\mathrm{P}(X > b_n + a_n^{-1}t)}{\mathrm{P}(X > b_n)}.$$

Observe that for the class $\mathcal{C}$, the auxiliary function ψ is given by $\psi(x) = x^{1-\beta}/\beta$ and the normalizing sequences satisfy $a_n^{-1} = \psi(b_n)$ and $a_n^{-1}c_n^{-1} = \sqrt{b_n\psi(b_n)}$. By definition of ψ, one sees that

$$\lim_{n\to\infty} \frac{\mathrm{P}(X > b_n + a_n^{-1}t)}{\mathrm{P}(X > b_n)} = e^{-t}.$$

Setting $x = b_n + a_n^{-1}t$ and exploiting properties of ψ given in Lemmas 1.2 and 1.3 of Resnick (1987, pp. 40–41), one obtains

$$\lim_{n\to\infty} \frac{\mathrm{P}(X > b_n + a_n^{-1}t, Y < a_n^{-1}c_n^{-1}z)}{\mathrm{P}(X > b_n + a_n^{-1}t)} = \lim_{x\to\infty} \frac{\mathrm{P}\{X > x, Y < z\sqrt{b_n\psi(b_n)}\}}{\mathrm{P}(X > x)},$$

which converges to $\Phi(z)$ by Theorem 1(ii).

TABLE 2: Simulation results for two bivariate Student distributions with $(\nu, \rho) = (2, 0.5)$ (top) and $(\nu, \rho) = (20, 0.5)$ (bottom). In each cell, line 1 provides the average (standard deviation) of the estimation of $\theta(x, y)$ based on Theorem 1(i) with $z = y/x - \rho$. Line 2 gives the same information for the empirical estimate of $\theta(x, y)$. Line 3 provides the theoretical value of $\theta(x, y)$ (left) and its associated limit given by Theorem 1(i) (right). Asterisks are used when there are no available results. Values of x and y are chosen as the marginal quantiles with probability p, where p labels rows and columns.

	Probabilities associated with x									
Probabilities associated with y	0.975		0.99		0.999		0.9999		0.99999	
	0.617	(0.047)	0.444	(0.037)	0.255	(0.031)	0.211	(0.031)	0.198	(0.030)
0.975	0.599	(0.150)	0.447	(0.260)	0.266	(0.416)	0.240	(0.431)	$\star$	
	0.598	0.609	0.438	0.440	0.257	0.257	0.213	0.213	0.201	0.201
	0.794	(0.050)	0.617	(0.047)	0.302	(0.032)	0.223	(0.031)	0.202	(0.030)
0.99	0.778	(0.125)	0.607	(0.258)	0.305	(0.434)	0.240	(0.431)	$\star$	
	0.775	0.786	0.605	0.609	0.302	0.302	0.225	0.225	0.204	0.204
	0.971	(0.018)	0.933	(0.031)	0.617	(0.047)	0.304	(0.032)	0.223	(0.031)
0.999	0.970	(0.050)	0.931	(0.135)	0.618	(0.444)	0.350	(0.476)	0.250	(0.500)
	0.970	0.972	0.930	0.932	0.609	0.609	0.304	0.304	0.225	0.225
	0.996	(0.004)	0.991	(0.008)	0.932	(0.031)	0.617	(0.047)	0.304	(0.032)
0.9999	0.997	(0.015)	0.993	(0.048)	0.943	(0.215)	0.680	(0.471)	0.500	(0.577)
	0.997	0.997	0.992	0.992	0.930	0.931	0.609	0.609	0.304	0.304
	0.999	(0.001)	0.999	(0.002)	0.991	(0.008)	0.932	(0.031)	0. 617	(0.047)
0.99999	1.000	(0.003)	1.000	(0.006)	0.996	(0.056)	0.980	(0.141)	$\star$	
	1.000	1.000	0.999	0.999	0.992	0.992	0.930	0.930	0.609	0.609

	Probabilities associated with x									
Probabilities associated with y	0.975		0.99		0.999		0.9999		0.99999	
	0.870	(0.040)	0.760	(0.048)	0.524	(0.050)	0.380	(0.049)	0.294	(0.047)
0.975	0.788	(0.121)	0.722	(0.226)	0.529	(0.461)	0.440	(0.501)	0.400	(0.548)
	0.788	0.985	0.718	0.924	0.537	0.621	0.381	0.373	0.266	0.232
	0.940	(0.027)	0.870	(0.040)	0.658	(0.050)	0.490	(0.050)	0.377	(0.049)
0.99	0.887	(0.094)	0.837	(0.188)	0.671	(0.434)	0.560	(0.501)	0.400	(0.548)
	0.887	0.998	0.837	0.985	0.681	0.818	0.516	0.563	0.375	0.367
	0.989	(0.009)	0.970	(0.017)	0.870	(0.040)	0.724	(0.049)	0.583	(0.051)
0.999	0.981	(0.041)	0.965	(0.092)	0.908	(0.278)	0.820	(0.388)	0.600	(0.548)
	0.981	1.000	0.968	1.000	0.904	0.985	0.794	0.892	0.651	0.715
	0.997	(0.003)	0.992	(0.007)	0.953	(0.023)	0.870	(0.040)	0.755	(0.048)
0.9999	0.998	(0.014)	0.996	(0.031)	0.978	(0.134)	0.940	(0.240)	$\star$	
	0.998	1.000	0.995	1.000	0.979	0.999	0.936	0.985	0.853	0.919
	0.999	(0.001)	0.997	(0.003)	0.983	(0.012)	0.942	(0.026)	0.870	(0.040)
0.99999	1.000	(0.004)	1.000	(0.004)	0.999	(0.017)	$\star$		$\star$	
	1.000	1.000	0.999	1.000	0.997	1.000	0.985	0.999	0.953	0.985

Eddy & Gale (1981) conjectured that their theorem remains true under the more general Condition 4.2 in their paper. Theorem 1(ii) shows that this is indeed true, provided that the distribution satisfies the hypothesis $\mathcal{H}_\infty^+$. Note that under this setting, b_n is still the solution of

$$\overline{H}\,(b_n) = 1/n, \quad a_n = \psi(b_n)^{-1} \quad \text{and} \quad c_n = \sqrt{\psi(b_n)/b_n}\,.$$

APPENDIX

A. Proof of Theorem 1.

Note that if (X, Y) has an elliptic distribution with correlation ρ, then $(X, -Y)$ is also elliptically distributed, but with correlation $-\rho$. Therefore, one just needs to establish the proof for positive ρ. From now on, assume that $\rho \geq 0$. Using notation given in Section 2, one gets

$$\begin{aligned}
&\mathrm{P}(Y > y \,|\, X > x) \\
&\quad = \frac{1}{\mathrm{P}(RDU_1 > x)}\, \mathrm{P}\Big[RDU_1 > x;\, RD\big\{\rho U_1 + \sqrt{1-\rho^2}\,\sqrt{(1-D^2)/D^2}\,U_2\big\} > y\Big] \\
&\quad = \frac{1}{\mathrm{P}(RD > x)}\, \mathrm{P}\Big[RD > x;\, RD\big\{\rho + \sqrt{1-\rho^2}\,\sqrt{(1-D^2)/D^2}\,U_2\big\} > y\Big].
\end{aligned}$$

In the last equality, we conditioned on U_1 and used the fact that U_1 and U_2 are two independent Bernoulli random variables. Conditioning on U_2 and using the definition of D yields

$$\begin{aligned}
\mathrm{P}(Y > y \,|\, X > x) \;=\; & \frac{1}{2\Delta(x)}\Big[\int_0^1 \mathrm{P}\big\{R > x/u, Rh_+(u,\rho) > y/u\big\} f(u)\,du \\
& + \int_0^1 \mathrm{P}\big\{R > x/u, Rh_-(u,\rho) > y/u\big\} f(u)\,du\Big],
\end{aligned}$$

where $f(u) = 1/\sqrt{1-u^2}$, $u \in (0,1)$, is the probability density function of D divided by $2/\pi$,

$$h_\pm(u,\rho) = \rho \pm \sqrt{1-\rho^2}\,\sqrt{1/u^2 - 1} \quad \text{and} \quad \Delta(x) = \int_0^1 \mathrm{P}(R > x/u) f(u)\,du.$$

Note that h_+ is decreasing from ∞ to ρ and that $h_-(u,\rho) = 2\rho - h_+(u,\rho)$. It follows that when $\rho \geq 0$, one has $h_+(u,\rho) \geq 0$ for all $u \in (0,1)$, and $h_-(u,\rho) \geq 0$ for all $u \geq \sqrt{1-\rho^2}$.

To avoid unnecessary repetition of arguments which are common to the proofs of (i) and (ii), let $\ell(\,\cdot\,)$ be a positive function. Algebraic manipulations show that

$$\mathrm{P}\big\{Y > \rho x + z\ell(x) \,|\, X > x\big\} = A_1(x,z,\rho) + A_2(x,z,\rho), \tag{1}$$

where

$$\begin{aligned}
A_1(x,z,\rho) &= \frac{1}{2\Delta(x)} \int_0^1 \mathrm{P}\left[R > \max\left\{\frac{x}{u}, \frac{\rho x + z\ell(x)}{uh_+(u,\rho)}\right\}\right] f(u)\,du, \\
A_2(x,z,\rho) &= \frac{1}{2\Delta(x)} \int_{\sqrt{1-\rho^2}}^1 \mathrm{P}\left[R > \max\left\{\frac{x}{u}, \frac{\rho x + z\ell(x)}{uh_-(u,\rho)}\right\}\right] f(u)\,du \\
&\quad + \frac{1}{2\Delta(x)} \int_0^{\sqrt{1-\rho^2}} \mathrm{P}\left\{\frac{x}{u} < R < \frac{\rho x + z\ell(x)}{uh_-(u,\rho)}\right\} f(u)\,du.
\end{aligned}$$

Now set

$$u_\star = 1/\sqrt{1 + z^2\ell^2(x)/\{x^2(1-\rho^2)\}} \quad \text{and} \quad \overline{H}\,(x) = 1 - H(x) = \mathrm{P}(R > x).$$

Long but straightforward computations yield

$$A_1(x,z,\rho) = \frac{1}{2} - \frac{1}{2\Delta(x)}\left\{B_1 - C_1(x,z,\rho)\right\}\mathbb{1}(z \geq 0), \tag{2}$$

$$\begin{aligned} A_2(x,z,\rho) &= \frac{D_2(x,z,\rho)}{2\Delta(x)}\,\mathbb{1}(z\geq 0) + \frac{D_1(x,z,\rho)}{2\Delta(x)}\,\mathbb{1}\left\{-\rho x/\ell(x) \leq z < 0\right\} \\ &\quad + \frac{B_1}{2\Delta(x)}\mathbb{1}(z<0) - \frac{D_3(x,z,\rho)}{2\Delta(x)}\mathbb{1}\left\{z < -\rho x/\ell(x)\right\}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} B_1 &= \int_{u_\star}^1 \overline{H}\,(x/u) f(u)\,du, \\ C_1(x,z,\rho) &= \int_{u_\star}^1 \overline{H}\left\{\frac{\rho x + z\ell(x)}{uh_+(u,\rho)}\right\} f(u)\,du, \\ D_1(x,z,\rho) &= \int_{\sqrt{1-\rho^2}}^{u_\star} \overline{H}\left\{\frac{\rho x + z\ell(x)}{uh_-(u,\rho)}\right\} f(u)\,du, \\ D_2(x,z,\rho) &= \int_{\sqrt{1-\rho^2}}^{1} \overline{H}\left\{\frac{\rho x + z\ell(x)}{uh_-(u,\rho)}\right\} f(u)\,du, \end{aligned}$$

and

$$D_3(x,z,\rho) = \int_{u_\star}^{\sqrt{1-\rho^2}} \overline{H}\left\{\frac{\rho x + z\ell(x)}{uh_-(u,\rho)}\right\} f(u)\,du.$$

The proof of (i) or (ii) is completed by studying the asymptotic behaviour of each term in decompositions (2) and (3). We will start with the proof of (i). Note that in this case,

$$\ell(x) = x, \quad u_\star = 1/\sqrt{1 + z^2/(1-\rho^2)}$$

and that $\overline{H}$ is regularly varying with index $-\alpha$. Since $\overline{H}\,(x/u)/\overline{H}\,(x) \leq 1$ for any $x > 0$ and $u \in (0,1)$, the Dominated Convergence Theorem shows that

$$\lim_{x\to\infty} \frac{\Delta(x)}{\overline{H}\,(x)} = \int_0^1 u^\alpha f(u)\,du = \frac{\sqrt{\pi}\,\Gamma\{(\alpha+1)/2\}}{2\Gamma\{(\alpha+2)/2\}},$$

where the last equality arises from (1) and (4) of Lemma 1 in Subsection 6.3. The same argument yields

$$\lim_{x\to\infty} B_1/\overline{H}\,(x) = \int_{u_\star}^1 u^\alpha f(u)\,du.$$

As a consequence, it follows from Lemma 1 that

$$\lim_{x\to\infty} \frac{B_1}{2\Delta(x)} = T_{\alpha+1}\left(|z|\sqrt{\frac{\alpha+1}{1-\rho^2}}\right) - \frac{1}{2}.$$

The asymptotic behaviour of the term C_1 is only needed for $z > 0$. By the Dominated Convergence Theorem, we obtain in this case

$$\lim_{x\to\infty} \frac{C_1(x,z,\rho)}{\overline{H}\,(x)} = \int_{u_\star}^1 \left\{\frac{uh_+(u,\rho)}{\rho+z}\right\}^\alpha f(u)\,du.$$

Using Lemma 1, we see that

$$\lim_{x\to\infty}\frac{C_1(x,z,\rho)}{2\Delta(x)}=\frac{T_{\alpha+1}\left(\sqrt{\alpha+1}\,\frac{\sqrt{1-\rho^2}}{\rho}\right)-T_{\alpha+1}\left\{\sqrt{\frac{\alpha+1}{1-\rho^2}}\left(\frac{1}{\rho+z}-\rho\right)\right\}}{|\rho+z|^\alpha}.$$

Concerning the terms D_1, D_2 and D_3, note that D_1 and D_2 are only needed for $\rho+z>0$, while D_3 is only used for $\rho+z<0$. Similar arguments to those used for C_1 prove that

$$\begin{aligned}
\lim_{x\to\infty}\frac{D_1(x,z,\rho)}{\overline{H}(x)} &= \int_{\sqrt{1-\rho^2}}^{u_\star}\left\{\frac{uh_-(u,\rho)}{\rho+z}\right\}^\alpha f(u)\,du,\\
\lim_{x\to\infty}\frac{D_1(x,z,\rho)}{2\Delta(x)} &= \frac{1}{|\rho+z|^\alpha}\,\overline{T}_{\alpha+1}\left\{\sqrt{\frac{\alpha+1}{1-\rho^2}}\left(\frac{1}{\rho+z}-\rho\right)\right\},\\
\lim_{x\to\infty}\frac{D_2(x,z,\rho)}{\overline{H}(x)} &= \int_{\sqrt{1-\rho^2}}^{1}\left\{\frac{uh_-(u,\rho)}{\rho+z}\right\}^\alpha f(u)\,du,\\
\lim_{x\to\infty}\frac{D_2(x,z,\rho)}{2\Delta(x)} &= \frac{1}{|\rho+z|^\alpha}\,\overline{T}_{\alpha+1}\left(\sqrt{\alpha+1}\,\frac{\sqrt{1-\rho^2}}{\rho}\right),\\
\lim_{x\to\infty}\frac{D_3(x,z,\rho)}{\overline{H}(x)} &= \int_{u_\star}^{\sqrt{1-\rho^2}}\left\{\frac{uh_-(u,\rho)}{\rho+z}\right\}^\alpha f(u)\,du
\end{aligned}$$

and

$$\lim_{x\to\infty}\frac{D_3(x,z,\rho)}{2\Delta(x)} = \frac{1}{|\rho+z|^\alpha}\,\overline{T}_{\alpha+1}\left\{\operatorname{sign}(\rho+z)\sqrt{\frac{\alpha+1}{1-\rho^2}}\left(\frac{1}{\rho+z}-\rho\right)\right\}.$$

Collecting the above terms completes the proof of (i).

For (ii), the function $\ell(x)$ corresponds to $\sqrt{x\psi(x)}$ and $u_\star$ is taken to be equal to

$$1/\sqrt{1+z^2\psi(x)/\{x(1-\rho^2)\}}\,,$$

which goes to one as $x\to\infty$. Let

$$t_\star = x(1/u_\star-1)/\psi(x)$$

and note that it goes to $z^2/(2-2\rho^2)$ as $x\to\infty$.

As in the proof of (i), we will examine the asymptotics of each term involved in decompositions (2) and (3). First, we establish that

$$\lim_{x\to\infty}\frac{\Delta(x)}{\overline{H}(x)\sqrt{\psi(x)/x}} = \sqrt{\pi/2}\,.$$

A change of variable $1/u = 1+t\psi(x)/x$ in the definition of Δ yields

$$\frac{\Delta(x)}{\overline{H}(x)\sqrt{\psi(x)/x}} = \int_0^\infty \frac{\overline{H}\{x+t\psi(x)\}/\overline{H}(x)}{\{1+t\psi(x)/x\}\sqrt{2t}\,\sqrt{1+t\psi(x)/(2x)}}\,dt. \tag{4}$$

Next, recall that $\lim_{x\to\infty}\psi'(x)=0$, so for any $\varepsilon>0$, there exists x_0 such that for $x>x_0$ one has $|\psi'(x)|<\varepsilon$. Since we are interested in the limit as $x\to\infty$, it is assumed from now on that $x>x_0$. Therefore, one has

$$|\psi\{x+u\psi(x)\}-\psi(x)| = \left|\int_x^{x+u\psi(x)}\psi'(s)\,ds\right| \le \left|\int_x^{x+u\psi(x)}\varepsilon\,ds\right| \le \varepsilon u\psi(x),$$

so that for all $x > x_0$,

$$1/(1+\varepsilon u) \le \frac{\psi(x)}{\psi\{x+u\psi(x)\}}.$$

Proposition 1.4 of Resnick (1987) states that for x greater than some x_0^*,

$$\overline{H}\,(x) = c(x)\exp\left\{-\int_{x_0^*}^{x}\frac{1}{\psi(u)}\,du\right\},$$

where $\lim_{x\to\infty} c(x) = c > 0$. For $x > \tilde{x}_0 = \max(x_0, x_0^*)$, and for $t > 0$,

$$\frac{c(x)\overline{H}\,\{x+t\psi(x)\}}{c\{x+t\psi(x)\}\overline{H}\,(x)} = \exp\left[-\int_0^t \frac{\psi(x)}{\psi\{x+s\psi(x)\}}\,ds\right] \le \exp\left(-\int_0^t \frac{ds}{1+\varepsilon s}\right) = \frac{1}{(1+\varepsilon t)^{1/\varepsilon}}.$$

Since $c(x)$ converges to some constant $c > 0$, there exists $x_3 > 0$ such that for $x > x_3$ one has $c\{x+t\psi(x)\}/c(x) \le 2$. Choosing $\varepsilon = 1$ leads to

$$\frac{\overline{H}\,\{x+t\psi(x)\}}{\overline{H}\,(x)} \le \frac{2}{1+t},$$

for $t > 0$ and $x > x_* = \max(x_3, \tilde{x}_0)$. Therefore, the integrand in (4) is bounded by the integrable function $\sqrt{2}\,/\{\sqrt{t}\,(1+t)\}$. The Dominated Convergence Theorem and $\mathcal{H}_\infty^+$ thus give

$$\lim_{x\to\infty}\frac{\Delta(x)}{\overline{H}\,(x)\sqrt{\psi(x)/x}} \quad = \quad \int_0^\infty \frac{e^{-t}}{\sqrt{2t}}\,dt = \sqrt{\frac{\pi}{2}}. \tag{5}$$

Similar arguments lead to

$$\lim_{x\to\infty}\frac{B_1}{\overline{H}\,(x)\sqrt{\psi(x)/x}} \quad = \quad \sqrt{2}\int_0^{|z|/\sqrt{2(1-\rho^2)}} \exp(-u^2)\,du,$$

and combining this limit with (5) then yields

$$\lim_{x\to\infty}\frac{B_1}{2\Delta(x)} = \frac{1}{\sqrt{\pi}}\int_0^{|z|/\sqrt{2(1-\rho^2)}} \exp(-u^2)\,du = \Phi\left(\frac{|z|}{\sqrt{1-\rho^2}}\right) - \frac{1}{2}.$$

The term C_1 is only needed for $z > 0$ and it will be shown that

$$\lim_{x\to\infty}\frac{C_1(x,z,\rho)}{\overline{H}\,(x)\sqrt{\psi(x)/x}} = 0. \tag{6}$$

Indeed, observe that for any $u \ge u_\star$, one has

$$\frac{\rho x + z\sqrt{x\psi(x)}}{uh_+(u,\rho)} > \frac{x}{u}.$$

Therefore, the integrand in $C_1(x,z,\rho)/\left\{\overline{H}\,(x)\sqrt{\psi(x)/x}\,\right\}$ is bounded by the integrand in $\Delta(x)/\left\{\overline{H}\,(x)\sqrt{\psi(x)/x}\,\right\}$. Since the latter was shown to be bounded by an integrable function, one can apply the Dominated Convergence Theorem to obtain (6). Using the same change of variables as in (4), it suffices to verify that for any $t \in \left(0, z/\sqrt{2-2\rho^2}\,\right)$,

$$\lim_{x\to\infty}\frac{1}{\overline{H}\,(x)}\overline{H}\left[\frac{\left\{\rho x + z\sqrt{x\psi(x)}\,\right\}\{1+t\psi(x)/x\}}{\rho + \sqrt{(1-\rho^2)\{2t\psi(x)/x + t^2\psi^2(x)/x^2\}}}\right] = 0.$$

Because of H_∞^+, this limit holds as soon as the quantity between square brackets in the above equation, say $Q(x)$, is such that $\{Q(x)-x\}/\psi(x) \to \infty$ as $x \to \infty$. Straightforward simplifications lead to

$$\frac{Q(x)-x}{\psi(x)} = \frac{\sqrt{x/\psi(x)}\left[z\{1+t\psi(x)/x\} - \sqrt{1-\rho^2}\sqrt{2t+t^2\psi(x)/x}\,\right]+\rho t}{\rho+\sqrt{(1-\rho^2)\{2t\psi(x)/x+t^2\psi^2(x)/x^2\}}}.$$

The expression between square brackets is positive for each $t \in \left(0, z/\sqrt{2-2\rho^2}\,\right)$, and as $\psi(x)/x \to 0$ when $x \to \infty$, the proof of (6) is complete. As for the term D_2, observe that for $z > 0$

$$\begin{aligned}\frac{D_2(x,z,\rho)}{2\Delta(x)} &\le \frac{1}{2\Delta(x)}\int_{\sqrt{1-\rho^2}}^1 \overline{H}\left\{\frac{\rho x + z\sqrt{x\psi(x)}}{\rho u}\right\} f(u)\,du \\ &\le \frac{\overline{H}\,(x)\sqrt{\psi(x)/x}}{2\Delta(x)}\int_0^\infty \frac{\overline{H}\left\{x+t\psi(x)+z\sqrt{x\psi(x)}\,\right\}}{\overline{H}\,(x)\{1+t\psi(x)/x\}\sqrt{2t}\,\sqrt{1+t\psi(x)/(2x)}}\,dt.\end{aligned}$$

As in (5), the integrand is bounded by the integrable function $\sqrt{2}\,/\left\{\sqrt{t}\,(1+t)\right\}$. Thus, the Dominated Convergence Theorem yields

$$\lim_{x\to\infty}\frac{D_2(x,z,\rho)}{2\Delta(x)} = 0$$

for $z > 0$. Consider now D_1 which is only needed for $z < 0$. Similar arguments to those used in (6) show that $D_1(x,z,\rho)/\{2\Delta(x)\} \to 0$ as $x \to \infty$. To complete the proof, note that if $\rho > 0$, the set

$$\{z < -\rho x/\ell(x)\} = \left\{z < -\rho\sqrt{x/\psi(x)}\,\right\}$$

is empty for x large enough, as z is fixed and $\lim_{x\to\infty} x/\psi(x) = \infty$. Whereas, if $\rho = 0$, observe that, in the argument used for C_1, the term $\{Q(x)-x\}/x$ still goes to infinity. Therefore,

$$\lim_{x\to\infty} C_1(x,z,0)/\left\{\overline{H}\,(x)\sqrt{\psi(x)/x}\,\right\} = 0.$$

Mimicking the argument, one gets

$$\lim_{x\to\infty} D_3(x,z,0)/\left\{\overline{H}\,(x)\sqrt{\psi(x)/x}\,\right\} = 0.$$

Once again, collecting the limits of these terms completes the proof. □

B. Proof of Theorem 2.

The proof uses decompositions (2) and (3) with z replaced by $-\rho x/\ell(x) + y/\ell(x)$. As in Theorem 1, one needs to study the asymptotic behaviour of each term involved in these decompositions.

The proof of (i) will be considered first. Recall that in this case, $\ell(x) = x$, i.e., $z = -\rho + y/x$. Note that

$$u_\star = \sqrt{1-\rho^2}\,/\sqrt{1-2\rho y/x + y^2/x^2} \to \sqrt{1-\rho^2}$$

as $x \to \infty$. For $\rho > 0$ and $x > y/\rho$, one gets $z < 0$ so that one just needs to consider the terms Δ, B_1, D_1 and D_3. As in the previous theorem, applying the Dominated Convergence Theorem, we prove that

$$\lim_{x\to\infty} B_1/\overline{H}\,(x) = \int_{\sqrt{1-\rho^2}}^1 u^\alpha f(u)\,du.$$

The terms $D_1/\overline{H}\,(x)$ and $D_3/\overline{H}\,(x)$ converge to zero, since their integrands are dominated by an integrable function and $\lim_{x\to\infty} u_\star = \sqrt{1-\rho^2}$. When $\rho = 0$, one has to consider the asymptotics of Δ, B_1, C_1 and D_2 for $y \geq 0$ and the asymptotics of Δ, B_1 and D_3 for $y < 0$. Similar arguments to those used previously show that these quantities converge to 0 as $x \to \infty$. Using Lemma 1 and collecting the terms completes the proof of (i).

To establish (ii), we follow the same procedure. In particular, $u_\star$ remains the same as in (i) and for $\rho > 0$,

$$z = -\rho\sqrt{x/\psi(x)} + y/\sqrt{x\psi(x)} < 0.$$

Therefore one just needs to consider the terms Δ, B_1, D_1 and D_3. The same argument as that used for the term B_1 in the proof of Theorem 1 yields

$$\lim_{x\to\infty} \frac{B_1}{2\Delta(x)} = \frac{1}{2}.$$

The contribution of D_1 and D_3 to the limit is shown to be negligible. As the proofs are quite similar, only the asymptotic behaviour of D_1 is given next.

For $y > 0$ and u between $\sqrt{1-\rho^2}$ and $u_\star$, one has $y/h_-(u,\rho) \geq x$, i.e.,

$$\overline{H}\left\{\frac{y}{uh_-(u,\rho)}\right\} \leq \overline{H}\left(\frac{x}{u}\right),$$

which by the change of variable $1/u = 1 + t\psi(x)/x$ and the arguments of Theorem 1(ii) shows that

$$\frac{D_1(x,z,\rho)}{\overline{H}(x)\sqrt{\psi(x)/x}} \leq \int_{t_*}^{\infty} \frac{1}{(t+1)\sqrt{2t}}\,dt.$$

The latter converges to zero since $t_\star = x(1/u_\star - 1)/\psi(x) \to \infty$.

To complete the proof, it remains to consider the case $\rho = 0$. Recall that, in this case, only the terms B_1, C_1 and D_3 are needed and that $u_\star = 1/\sqrt{1+y^2/x^2}$ converges to 1. Using the change of variable $1/u = 1 + t\psi(x)/x$, one sees that $t_\star = x(1/u_\star - 1)/\psi(x)$ is such that $t_\star x\psi(x) \to y^2/2$ as $x \to \infty$. The rest of the proof is divided in three subcases depending on the limit of $x\psi(x)$.

Subcase 1: $\lim_{x\to\infty} x\psi(x) = \infty$. One sees easily that $t_\star \to 0$. Furthermore, the arguments in the proof of Theorem 1 show that

$$\frac{B_1}{\overline{H}(x)\sqrt{\psi(x)/x}}, \quad \frac{C_1(x,z,0)}{\overline{H}(x)\sqrt{\psi(x)/x}} \quad \text{and} \quad \frac{D_3(x,z,0)}{\overline{H}(x)\sqrt{\psi(x)/x}}$$

are bounded by

$$\int_0^{t_*} \frac{1}{(t+1)\sqrt{2t}}\,dt,$$

which converges to zero and yields $\lim_{x\to\infty} \mathrm{P}(Y \leq y \,|\, X > x) = 1/2$.

Subcase 2: $\lim_{x\to\infty} x\psi(x) = \lambda > 0$. Note that $t_\star \to y^2/2\lambda$. The argument of Theorem 1(ii) shows that

$$\lim_{x\to\infty} \frac{B_1}{2\Delta(x)} = \Phi\left(\frac{|y|}{\sqrt{\lambda}}\right) - \frac{1}{2}.$$

For the terms C_1 and D_3, one sees that

$$\frac{Q(x)-x}{\psi(x)} = \sqrt{\frac{x}{\psi(x)}}\,\frac{|y|\{1+t\psi(x)/x\} - \sqrt{x\psi(x)}\,\sqrt{2t+t^2\psi(x)/x}}{\sqrt{2t+t^2\psi(x)/x}} \to \infty$$

for any $t \in (0, t_\star)$. By the argument of Theorem 1(ii), one gets

$$\frac{C_1(x,z,0)}{\overline{H}(x)\sqrt{\psi(x)/x}} \to 0 \quad \text{and} \quad \frac{D_3(x,z,0)}{\overline{H}(x)\sqrt{\psi(x)/x}} \to 0.$$

Collecting these terms completes the proof for this subcase.

Subcase 3: $\lim_{x\to\infty} x\psi(x) = 0$. In this case, $t_\star \to \infty$. Nevertheless, the limit of $\{Q(x) - x\}/x$ as $x \to \infty$ is still equal to infinity, i.e.,

$$\frac{C_1(x,z,0)}{\overline{\overline{H}}(x)\sqrt{\psi(x)/x}} \to 0 \quad \text{and} \quad \frac{D_3(x,z,0)}{\overline{\overline{H}}(x)\sqrt{\psi(x)/x}} \to 0.$$

Using the technique in the proof of Theorem 1(ii), one verifies easily that

$$\lim_{x\to\infty} \frac{B_1}{2\Delta(x)} = \frac{1}{2}.$$

C. Auxiliary lemma.

This subsection contains the statement and the proof of the auxiliary lemma used in the proofs of Theorems 1 and 2.

LEMMA 1. *Let* $0 \le \rho \le 1$, $\alpha > 0$, $0 < a \le b \le 1$ *and* $-\infty < c \le d < \infty$. *The following equalities are then satisfied:*

(i) $$\int_a^b \frac{u^\alpha}{\sqrt{1-u^2}}\,du = \int_{\sqrt{1/b^2-1}}^{\sqrt{1/a^2-1}} (1+x^2)^{-(\alpha+2)/2}\,dx;$$

(ii) $$\int_a^b \{\rho u + \sqrt{1-\rho^2}\sqrt{1-u^2}\}^\alpha \frac{du}{\sqrt{1-u^2}} = \int_{x_a^+}^{x_b^+} (1+x^2)^{-(\alpha+2)/2}\,dx,$$

where $x_t^+ = \left(t\sqrt{1-t^2} - \rho\sqrt{1-\rho^2}\right)/(1-\rho^2-t^2)$;

(iii) $$\int_{\sqrt{1-\rho^2}}^a \left|\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}\right|^\alpha \frac{du}{\sqrt{1-u^2}} = -\ell(\rho,a)\int_{x_a^-}^\infty (1+x^2)^{-(\alpha+2)/2}\,dx,$$

where $\ell(\rho,a) = \operatorname{sign}(1-\rho^2-a^2)$ *and* $x_a^- = \ell(\rho,a)\dfrac{a\sqrt{1-a^2}+\rho\sqrt{1-\rho^2}}{1-\rho^2-a^2}$;

(iv) $$\frac{\Gamma\{(\alpha+2)/2\}}{\sqrt{\pi}\,\Gamma\{(\alpha+1)/2\}}\int_c^d (1+x^2)^{-(\alpha+2)/2}\,dx = T_{\alpha+1}\left(d\sqrt{\alpha+1}\right) - T_{\alpha+1}\left(c\sqrt{\alpha+1}\right),$$

where $T_\nu(x)$ *is the cumulative distribution function of a univariate Student random variable with* ν *degrees of freedom.*

Proof. Identity (i) is a result of the change of variable $u = 1/\sqrt{1+x^2}$, i.e., $x = \sqrt{1/u^2-1}$.

In order to get (ii), observe that the function $\rho u + \sqrt{1-\rho^2}\sqrt{1-u^2}$ is increasing for $0 \le u \le \rho$ and decreasing for $\rho \le u \le 1$. Note also that

$$0 < \rho u + \sqrt{1-\rho^2}\sqrt{1-u^2} \le 1$$

for $0 \le u \le 1$. Now consider the change of variable

$$\rho u + \sqrt{1-\rho^2}\sqrt{1-u^2} = 1/\sqrt{1+x^2}$$

with $x \ge 0$ if $u \ge \rho$ and $x \le 0$ for $u \le \rho$. This yields

$$u = \left(\rho + x\sqrt{1-\rho^2}\right)/\sqrt{1+x^2}$$

and

$$x = \left(u\sqrt{1-u^2} - \rho\sqrt{1-\rho^2}\right)/(1-\rho^2-u^2).$$

Therefore,

$$\sqrt{1-u^2} = \left(-\rho x + \sqrt{1-\rho^2}\,\right)/\sqrt{1+x^2}$$

and

$$du = \left(-\rho x + \sqrt{1-\rho^2}\,\right)/(1+x^2)^{3/2}.$$

Applying the change of variable completes the proof of (ii).

Now consider the term (iii). Observe that the function $\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}$ is increasing for $0 \le u \le 1$. Note also that

$$\left|\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}\,\right| \le 1$$

for $0 \le u \le 1$ and that

$$\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2} = 0$$

for $u = \sqrt{1-\rho^2}$. Now consider the change of variable

$$\left|\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}\,\right| = 1/\sqrt{1+x^2}$$

with $x \ge 0$ if $u \ge \sqrt{1-\rho^2}$ and $x \le 0$ for $u \le \sqrt{1-\rho^2}$. This gives

$$x = -\left(u\sqrt{1-u^2} + \rho\sqrt{1-\rho^2}\,\right)/(1-\rho^2-u^2).$$

Note that there is an indetermination for $u = \sqrt{1-\rho^2}$. The rest of the proof is divided in two subcases. First subcase $a \ge \sqrt{1-\rho^2}$. In this setting, applying the above change of variable gives

$$\begin{aligned}
&\int_{\sqrt{1-\rho^2}}^{a} \left|\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}\,\right|^{\alpha} \frac{du}{\sqrt{1-u^2}} \\
&\quad = \int_{-(a\sqrt{1-a^2}+\rho\sqrt{1-\rho^2}\,)/(1-\rho^2-a^2)}^{\infty} (1+x^2)^{-(\alpha+2)/2}\,dx.
\end{aligned}$$

For the second subcase for $a < \sqrt{1-\rho^2}$, the same change of variable yields

$$\begin{aligned}
&\int_{\sqrt{1-\rho^2}}^{a} \left|\rho u - \sqrt{1-\rho^2}\sqrt{1-u^2}\,\right|^{\alpha} \frac{du}{\sqrt{1-u^2}} \\
&\quad = -\int_{-\infty}^{-(a\sqrt{1-a^2}+\rho\sqrt{1-\rho^2}\,)/(1-\rho^2-a^2)} (1+x^2)^{-(\alpha+2)/2}\,dx,
\end{aligned}$$

which by symmetry reduces to

$$\begin{aligned}
&-\int_{(a\sqrt{1-a^2}+\rho\sqrt{1-\rho^2}\,)/(1-\rho^2-a^2)}^{\infty} (1+x^2)^{-(\alpha+2)/2}\,dx \\
&= -\mathrm{sign}(1-\rho^2-a^2)\int_{\mathrm{sign}(1-\rho^2-a^2)(a\sqrt{1-a^2}+\rho\sqrt{1-\rho^2}\,)/(1-\rho^2-a^2)}^{\infty} (1+x^2)^{-(\alpha+2)/2}\,dx.
\end{aligned}$$

Finally the proof of item (iv) is straightforward, provided that one uses the change of variable $x = t/\sqrt{\alpha+1}$.

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