CONTRÔLE TERMINAL DE MATH2 - 21/01/09 - CORRIGÉ - Ale Frabetti

Ex. 1 $f(x,y) = x^3 - 3xy + y^2$, $D_f = \mathbb{R}^2$

1\. Points critiques: $\vec{\nabla} f(x,y) = \begin{pmatrix} 3x^2 - 3y \\ -3x + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} y = \frac{3}{2}x \\ 3x(x - \frac{3}{2}) = 0 \end{cases} \Leftrightarrow$ $x=0,\ y=0$; $x = \frac{3}{2},\ y = \frac{9}{4}$

Donc il y a deux pts critiques: $(0,0)$ et $(\frac{3}{2}, \frac{9}{4})$.

Nature des pts critiques : $\mathrm{Hess} f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 2 \end{pmatrix}$, $\det \mathrm{Hess} f(x,y) = 12x - 9$.

En $(0,0)$ on a $\det \mathrm{Hess} f(0,0) = -9 < 0$, donc $(0,0)$ est un point col.

En $(\frac{3}{2}, \frac{9}{4})$ on a $\det \mathrm{Hess} f(\frac{3}{2}, \frac{9}{4}) = 12 \cdot \frac{3}{2} - 9 = 18 - 9 = 9 > 0$, donc $(\frac{3}{2}, \frac{9}{4})$ est un extremum local.

Puisque $\frac{\partial^2 f}{\partial x^2}(\frac{3}{2}, \frac{9}{4}) = 6 \cdot \frac{3}{2} = 9 > 0$, le pt. $(\frac{3}{2}, \frac{9}{4})$ est un minimum local.

2\. Puisque $\mathbb{R}^2$ n'a pas de bord, la fct f a des extrema globaux seulement si les extrema loc. sont globaux. Le seul extremum loc est le minimum $(\frac{3}{2}, \frac{9}{4})$: ce point n'est pas un min. absolu, car la fct f tend à $-\infty$, par ex. pour $y=0$ et $x \to -\infty$.

Ex. 2 $\vec{V}(x,y) = 3x^2y\,\vec{\imath} + (x^3 - 2y)\,\vec{\jmath}$, $D_{\vec{V}} = \mathbb{R}^2$.

1\. Puisque $\mathbb{R}^2$ est simplement connexe, par le Thm Poincaré on a $\vec{V} = \vec{\mathrm{grad}}\, f \Leftrightarrow \vec{\mathrm{rot}}\, \vec{V} = \vec{0}$.

Calculons : $\vec{\mathrm{rot}}\, \vec{V}(x,y) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 - 2y & 0 \end{vmatrix} = \vec{k}\left(\frac{\partial(x^3 - 2y)}{\partial x} - \frac{\partial(3x^2y)}{\partial y} \right) = \vec{k}(3x^2 - 3x^2) = \vec{0}$.

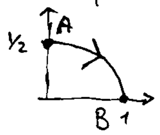
Donc $\vec{V} = \vec{\mathrm{grad}}\, f$, calculons f:

$\begin{cases} \frac{\partial f}{\partial x} = 3x^2y \\ \frac{\partial f}{\partial y} = x^3 - 2y \end{cases}$ $\Leftrightarrow$ $f(x,y) = \int 3x^2y\, dx + g(y) = x^3y + g(y)$

$\frac{\partial f}{\partial y} = x^3 + g'(y) \Leftrightarrow g'(y) = -2y \Leftrightarrow g(y) = -y^2 + c$.

Donc $f(x,y) = x^3y - y^2 + c$.

2\. C^+ = quart d'ellipse

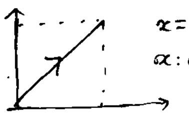


La circulation de $\vec{V}$ le long de C^+ est

$$\int_{C^+} \vec{V} \cdot \vec{d\ell} = \int_{C^+} \vec{\mathrm{grad}}\, f \cdot \vec{d\ell} = f(B) - f(A) = f(1,0) - f(0, \tfrac{1}{2}) = c - (-\tfrac{1}{4} + c) = \tfrac{1}{4}.$$

Ex. 3 $\vec{u}(x,y) = -xy\,\vec{\imath} + y\,\vec{\jmath}$, $D_{\vec{u}} = \mathbb{R}^2$

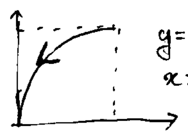
1\. γ_1



$x = y$, $x : 0 \to 1$

$$\int_{\gamma_1} \vec{u} \cdot \vec{d\ell} = \int_{\gamma_1,\ dy = dx} (-xy\, dx + y\, dy) = \int_0^1 (-x \cdot x\, dx + x\, dx) = \int_0^1 (-x^2 + x)\, dx = \left[-\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 \right]_0^1 = -\tfrac{1}{3} + \tfrac{1}{2} = \tfrac{1}{6}$$

2\. γ_2

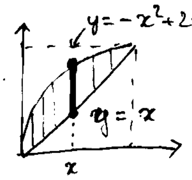


$y = -x^2 + 2x$, $x : 1 \to 0$

$$\int_{\gamma_2} \vec{u} \cdot \vec{d\ell} = \int_{\gamma_2,\ dy = (-2x+2)dx} (-xy\, dx + y\, dy) = \int_1^0 \left(-x(-x^2 + 2x) + (-x^2 + 2x)(-2x + 2) \right) dx = \int_1^0 (3x^3 - 8x^2 + 4x)\, dx$$

$$= \left[\tfrac{3}{4}x^4 - \tfrac{8}{3}x^3 + 2x^2 \right]_1^0 = -\tfrac{3}{4} + \tfrac{8}{3} - 2 = -\tfrac{1}{12}.$$

3\. D



$D = \{(x,y) \in \mathbb{R}^2,\ y \geq x,\ y \leq -x^2 + 2x\}$

$$I = \iint_D x\, dx\, dy = \int_0^1 dx \int_x^{-x^2+2x} x\, dy = \int_0^1 [xy]_{y=x}^{y=-x^2+2x} dx = \int_0^1 (-x^3 + 2x^2 - x^2)\, dx = \int_0^1 (-x^3 + x^2)\, dx$$

$$= \left[-\tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 \right]_0^1 = -\tfrac{1}{4} + \tfrac{1}{3} = \tfrac{1}{12}.$$

4\. $\vec{V} = \vec{\mathrm{rot}}\, \vec{u} = \vec{k} \cdot \left(\frac{\partial y}{\partial x} - \frac{\partial(-xy)}{\partial y} \right) = x\,\vec{k}$, donc $\iint_D \vec{\mathrm{rot}}\, \vec{u} \cdot \vec{dS} = \iint_D x\, dx\, dy = I$.

Green-Riemann: $\iint_D \vec{\mathrm{rot}}\, \vec{u} \cdot \vec{dS} = \oint_{\partial D^+} \vec{u} \cdot \vec{d\ell}$ où $\partial D^+ = \gamma_1 \cup \gamma_2$. Alors on a :

$$I = \iint_D \vec{\mathrm{rot}}\, \vec{u} \cdot \vec{dS} = \oint_{\partial D^+} \vec{u} \cdot \vec{d\ell} = \int_{\gamma_1} \vec{u} \cdot \vec{d\ell} + \int_{\gamma_2} \vec{u} \cdot \vec{d\ell} = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$ ok!