$f : \mathbb{R}^2 \to \mathbb{R}$, $\quad f(x,y) = (x-1)\arctan y + 1 \quad \forall (x,y) \in \mathbb{R}^2$.

Question 1 :

$$\vec{\nabla} f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \arctan y \\ \frac{x-1}{1+y^2} \end{pmatrix}.$$

Question 2 :

$$df_{(0,1)} = \frac{\partial f}{\partial x}(0,1)\,dx + \frac{\partial f}{\partial y}(0,1)\,dy = \arctan 1\,dx + \frac{0-1}{1+1}\,dy = \frac{\pi}{4}dx - \frac{1}{2}dy.$$

Question 3 :

$$\mathrm{Hess}f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{1+y^2} \\ \frac{1}{1+y^2} & \frac{2(x-1)y}{(1+y^2)^2} \end{pmatrix}.$$

Question 4 :

(x,y) est un point critique $\Leftrightarrow \vec{\nabla} f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} \arctan y = 0 \\ \frac{x-1}{1+y^2} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 1 \end{cases}$

$\Rightarrow$ il y a un seul point critique, $(1,0)$.

$\det \mathrm{Hess}f(1,0) = 0 \cdot \frac{2\cdot 0 \cdot 0}{(1+0)^2} - \left(\frac{1}{1+0}\right)^2 = -1 < 0 \Rightarrow (1,0)$ est un point col.

Question 5 :

$$f(x,y) \underset{(x,y)\to(0,0)}{\sim} f(0,0) + \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial y}(0,0)\cdot y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)\,x^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)\cdot xy + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)\cdot y^2.$$

$$(x-1)\arctan y + 1 \underset{(x,y)\to(0,0)}{\sim} 1 + \underbrace{\arctan 0}_{=0}\cdot x + \frac{0-1}{1+0}\cdot y + \frac{1}{2}\,0\cdot x^2 + \frac{1}{1+0}\,xy + \frac{2\cdot(-1)\cdot 0}{(1+0)^2}\cdot y^2$$

$$\sim 1 - y + xy.$$