Corrigé du CC2 - Math 2 - Jeudi 25 novembre 2010 - Ale Frabetti

Fonction gaussienne $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^{-(x^2+y^2)}$.

<u>Question 1</u> : $r = \sqrt{x^2+y^2} > 0 \Rightarrow f(x,y) = e^{-(x^2+y^2)} = e^{-r^2} \xrightarrow[r^2 \to +\infty]{} e^{-\infty} = 0^+$.

<u>Question 2</u> : $f(0,0) = e^{-0} = 1$.

<u>Question 3</u> : $\begin{cases} \frac{\partial f}{\partial x}(x,y) = -2x\, e^{-(x^2+y^2)} \\ \frac{\partial f}{\partial y}(x,y) = -2y\, e^{-(x^2+y^2)} \end{cases} \Rightarrow \nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} -2x\, e^{-(x^2+y^2)} \\ -2y\, e^{-(x^2+y^2)} \end{pmatrix} = -2e^{-(x^2+y^2)} \begin{pmatrix} x \\ y \end{pmatrix}$.

<u>Question 4</u> : $df_{(0,1)} = \frac{\partial f}{\partial x}(0,1)\, dx + \frac{\partial f}{\partial y}(0,1)\, dy = (-2 \cdot 0\, e^{-1})\, dx + (-2 \cdot 1 \cdot e^{-1})\, dy = -\frac{2}{e}\, dy$.

<u>Question 5</u> : $\begin{cases} \frac{\partial^2 f}{\partial x^2}(x,y) = -2\, e^{-(x^2+y^2)} + (-2x)(-2x)\, e^{-(x^2+y^2)} = (-2+4x^2)\, e^{-(x^2+y^2)} = 2(2x^2-1)\, e^{-(x^2+y^2)} \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) = (-2x)(-2y)\, e^{-(x^2+y^2)} = 4xy\, e^{-(x^2+y^2)} \\ \frac{\partial^2 f}{\partial y^2}(x,y) = -2\, e^{-(x^2+y^2)} + (-2y)(-2y)\, e^{-(x^2+y^2)} = 2(2y^2-1)\, e^{-(x^2+y^2)} \end{cases}$

$$\Rightarrow \text{Hess}\, f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \text{“} & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = 2\, e^{-(x^2+y^2)} \begin{pmatrix} 2x^2-1 & 2xy \\ 2xy & 2y^2-1 \end{pmatrix}.$$

<u>Question 6</u> : (x,y) est un point critique $\Leftrightarrow \nabla f(x,y) = 0 \Leftrightarrow -2\underbrace{e^{-(x^2+y^2)}}_{\neq 0} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow (x,y) = (0,0)$

Donc $(0,0)$ est le seul point critique.

$\det \text{Hess}\, f(0,0) = \det \left(2 \cdot e^0 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) = 4 \cdot (-1)(-1) = 4 > 0$ donc $(0,0)$ est min. ou max. loc.

Puisque $\frac{\partial^2 f}{\partial x^2}(0,0) = 2 \cdot e^0 \cdot (-1) = -2 < 0$, le point $(0,0)$ est un max. local.

<u>Question 7</u> : Taylor à l'ordre 2 autour de $(0,0)$:

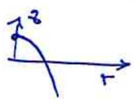
$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\, x + \frac{\partial f}{\partial y}(0,0)\, y + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0,0)\, x^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)\, xy + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(0,0)\, y^2 + o(x^2+y^2)$$

Donc $f(x,y) = 1 + 0 \cdot x + 0 \cdot y + \frac{1}{2} \cdot (-2)\, x^2 + 0 \cdot xy + \frac{1}{2}(-2)\, y^2 + o(x^2+y^2)$

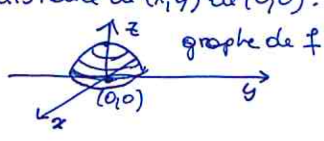
$$= 1 - x^2 - y^2 + o(x^2+y^2) = 1 - (x^2+y^2) + o(x^2+y^2).$$

<u>Question 8</u> : autour de $(0,0)$ on a $f(x,y) \sim 1 - r^2$ où $r = \sqrt{x^2+y^2}$ est la distance de (x,y) de $(0,0)$.

Donc autour de $(0,0)$ f est une fonction "radiale" qui se comporte comme la parabole $z = 1 - r^2$



$\Longrightarrow$



<u>Question 9</u> : Taylor à l'ordre 2 autour de $(1,1)$:

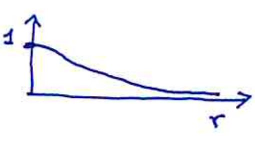
$$f(x,y) = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(1,1)(x-1)^2 + \frac{\partial^2 f}{\partial x \partial y}(1,1)(x-1)(y-1) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(1,1)(y-1)^2 + o\big((x-1)^2+(y-1)^2\big)$$

$$f(x,y) = e^{-2} + (-2)e^{-2}(x-1) + (-2)e^{-2}(y-1) + \frac{1}{\cancel{2}}\cancel{2}\, e^{-2}(2-1)(x-1)^2 + 2e^{-2} \cdot 2(x-1)(y-1) + \frac{1}{\cancel{2}}\cancel{2}\, e^{-2}(2-1)(y-1)^2 + o\big((x-1)^2+(y-1)^2\big)$$

$$= \frac{1}{e^2}\left[1 - 2(x-1) - 2(y-1) + (x-1)^2 + 4(x-1)(y-1) + (y-1)^2\right] + o\big((x-1)^2+(y-1)^2\big).$$

<u>Question 10</u> : $f(x,y) = e^{-r^2}$ dépend seulement de $r = \sqrt{x^2+y^2} > 0$, a un seul max. loc en $r=0$, et tend à 0 pour $r \to \infty$,

i.e.



donc :