Exercice 1 $f(x,y) = \frac{1}{3}x^3 - y^3 - x^2y + 3y$

Points critiques : $\nabla f(x,y) = \begin{pmatrix} x^2 - 2xy \\ -3y^2 - x^2 + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x(x-2y) = 0 & \begin{array}{l} x=0 \\ x=2y \end{array} \\ x^2 + 3y^2 = 3 \end{cases}$

- si $x=0$ on a $3y^2 = 3 \Leftrightarrow y = \pm 1$, donc deux solutions $(0, \pm 1)$
- si $x = 2y$ on a $4y^2 + 3y^2 = 3 \Leftrightarrow y = \pm\sqrt{3}/\sqrt{7}$ et $x = \pm 2\sqrt{3}/\sqrt{7}$, donc deux solutions $\left(\pm\frac{2\sqrt{3}}{\sqrt{7}}, \pm\frac{\sqrt{3}}{\sqrt{7}}\right)$

Au total on a quatre points critiques : $(0,1)$, $(0,-1)$, $\left(\frac{2\sqrt{3}}{\sqrt{7}}, \frac{\sqrt{3}}{\sqrt{7}}\right)$ et $\left(-2\sqrt{3}/\sqrt{7}, -\sqrt{3}/\sqrt{7}\right)$.

Nature des points critiques :

$\text{Hess}\, f(x,y) = \begin{pmatrix} 2x-2y & -2x \\ -2x & -6y \end{pmatrix} \Rightarrow \det \text{Hess}\, f(x,y) = -12\,y(x-y) - 4x^2 = -4\left[x^2 + 3xy - 3y^2\right]$

- $\det \text{Hess}\, f(0,1) = -4(-3) = 12 > 0$, $\frac{\partial^2 f}{\partial x^2}(0,1) = 2.0 - 2 = -2 < 0 \Rightarrow (0,1)$ est un pt. max. loc.
- $\det \text{Hess}\, f(0,-1) = 12 > 0$, $\frac{\partial^2 f}{\partial x^2}(0,-1) = 2.0 - 2(-1) = 2 > 0 \Rightarrow (0,-1)$ est un pt. min. loc.
- $\det \text{Hess}\, f\left(\pm\frac{2\sqrt{3}}{\sqrt{7}}, \pm\frac{\sqrt{3}}{\sqrt{7}}\right) = -4\left[4\cdot\frac{3}{7} + 3\cdot 2\cdot\frac{3}{7} - 3\cdot\frac{3}{7}\right] = -4\cdot\frac{3}{\cancel{7}}\cdot\cancel{7} = -12 < 0 \Rightarrow \left(\pm\frac{2\sqrt{3}}{\sqrt{7}}, \pm\frac{\sqrt{3}}{\sqrt{7}}\right)$ sont pts col.

Exercice 2 $\vec{V}(x,y,z) = \begin{pmatrix} 2x \\ -y \\ -z \end{pmatrix}$ défini sur tout $\mathbb{R}^3$.

1. $\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z) = 2 - 1 - 1 = 0$.
   Puisque $\mathbb{R}^3$ est simplement connexe, $\operatorname{div}\vec{V} = 0 \Longrightarrow \vec{V} = \vec{\text{rot}}\,\vec{U}$ pour un champ $\vec{U}$.

2. $\vec{\text{rot}}\,\vec{V} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -y & -z \end{pmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$.
   Puisque $\mathbb{R}^3$ est contractile, $\vec{\text{rot}}\,\vec{V} = \vec{0} \Longrightarrow \vec{V} = \vec{\text{grad}}\, f$ pour une fonction f.

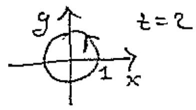
   Calculons f :

$$\begin{cases} \frac{\partial f}{\partial x} = 2x \\ \frac{\partial f}{\partial y} = -y \\ \frac{\partial f}{\partial z} = -z \end{cases} \Rightarrow f(x,y,z) = \int 2x\,dx + g(y,z) = x^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{\partial x^2}{\partial y} + \frac{\partial g}{\partial y} = \frac{\partial g}{\partial y} = -y \Rightarrow g(y,z) = \int -y\,dy + h(z) = -\frac{1}{2}y^2 + h(z)$$

$$\Rightarrow f(x,y,z) = x^2 - \frac{1}{2}y^2 + h(z)$$

$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} = -z \Rightarrow h(z) = -\frac{1}{2}z^2 + c \Longrightarrow f(x,y,z) = x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + c$$

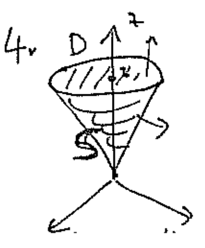


3. $\gamma = \{(x,y,z) \mid x^2 + y^2 = 1,\ z = 2\} = \{\gamma(t) = (\cos t, \sin t, 2) \mid t \in [0, 2\pi]\}$

$$\oint_\gamma \vec{V}\cdot\vec{d\ell} = \int_\gamma \begin{pmatrix} 2x \\ -y \\ -z \end{pmatrix} \cdot \gamma'(t)\,dt = \int_0^{2\pi} \begin{pmatrix} 2\cos t \\ -\sin t \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} dt = \int_0^{2\pi} \left(-2\cos t \sin t - \sin t \cos t + 0\right) dt$$

$$= -3\int_0^{2\pi} \sin t \cos t\,dt = -3\int_0^{2\pi} \frac{\sin(2t)}{2}\,dt = -\frac{3}{4}\int_0^{2\pi} \sin(2t)\,d(2t) = -\frac{3}{4}\left[-\cos(2t)\right]_0^{2\pi} = \frac{3}{4}\left[\cos(4\pi) - \cos 0\right] = 0$$

Autre méthode : $\oint_\gamma \vec{V}\cdot\vec{d\ell} = \oint_\gamma \vec{\nabla} f \cdot \vec{d\ell} = 0$ car la circulation d'un gradient le long d'une courbe fermée est nulle.

4. 

Soit Ω la partie de $\mathbb{R}^3$ qui se trouve à l'intérieur du cône.
Par le thm. d'Ostrogradski on a

$$\iint_{D^+ \cup S^+} \vec{V}\cdot\vec{dS} = \iiint_\Omega \operatorname{div}\vec{V}\cdot dx\,dy\,dz = 0 \quad \text{car } \operatorname{div}\vec{V} = 0.$$

2) 5. Disque orienté  , cherchons une bonne paramétrisation :

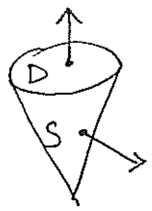
Si $\sigma(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 2)$ avec $\rho \in [0,1]$ et $\theta \in [0,2\pi]$ on a

$$\frac{\partial\sigma}{\partial\rho} = (\cos\theta, \sin\theta, 0), \quad \frac{\partial\sigma}{\partial\theta} = (-\rho\cos\theta, \rho\sin\theta, 0) \Rightarrow \frac{\partial\sigma}{\partial\rho}\wedge\frac{\partial\sigma}{\partial\theta} = (0,0,\rho) \text{ bien orienté.}$$

Donc $D = \{(x,y,z) \mid x^2+y^2 \leq 1, z=2\} = \{\sigma(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 2) \mid \rho\in[0,1], \theta\in[0,2\pi]\}$

$$\iint_D \vec{V}\cdot\vec{dS} = \iint_D \begin{pmatrix}2x\\-y\\-z\end{pmatrix}\cdot\frac{\partial\sigma}{\partial\rho}\wedge\frac{\partial\sigma}{\partial\theta}\,d\rho\,d\theta = \iint_D \begin{pmatrix}2\rho\cos\theta\\-\rho\sin\theta\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\rho\end{pmatrix}\cdot d\rho\,d\theta = \int_0^1\left(\int_0^{2\pi}-2\rho\,d\theta\right)d\rho$$

$$= \int_0^1 -2\rho\,d\rho \cdot \int_0^{2\pi} d\theta = \left[-\rho^2\right]_0^1 \cdot \left[\theta\right]_0^{2\pi} = -1\cdot 2\pi = -2\pi.$$

6. 

$$\underbrace{\iint_{D^+\cup S^+}\vec{V}\cdot\vec{dS}}_{=0} = \underbrace{\iint_{D^+}\vec{V}\cdot\vec{dS}}_{=-2\pi} + \iint_{S^+}\vec{V}\cdot\vec{dS} \Rightarrow \iint_{S^+}\vec{V}\cdot\vec{dS} = 0-(-2\pi) = 2\pi.$$

7. $S = \{\sigma(\theta,\rho) = (\rho\cos\theta, \rho\sin\theta, 2\rho) \mid \theta\in[0,2\pi], \rho\in[0,1]\}$

$$\frac{\partial\sigma}{\partial\theta} = (-\rho\sin\theta, \rho\cos\theta, 0), \quad \frac{\partial\sigma}{\partial\rho} = (\cos\theta, \sin\theta, 2)$$

$$\frac{\partial\sigma}{\partial\theta}\wedge\frac{\partial\sigma}{\partial\rho} = (2\rho\cos\theta, -(-2\rho\sin\theta), -\rho\sin^2\theta-\rho\cos^2\theta) = (2\rho\cos\theta, 2\rho\sin\theta, -\rho)$$

$$\iint_S \vec{V}\cdot\vec{dS} = \iint_S \begin{pmatrix}2x\\-y\\-z\end{pmatrix}\cdot\frac{\partial\sigma}{\partial\theta}\wedge\frac{\partial\sigma}{\partial\rho}\cdot d\theta\,d\rho = \iint_{\substack{\theta\in[0,2\pi]\\ \rho\in[0,1]}} \begin{pmatrix}2\rho\cos\theta\\-\rho\sin\theta\\-2\rho\end{pmatrix}\cdot\begin{pmatrix}2\rho\cos\theta\\2\rho\sin\theta\\-\rho\end{pmatrix} d\theta\,d\rho$$

$$= \int_0^1\left(\int_0^{2\pi}\left(4\rho^2\cos^2\theta - 2\rho^2\underbrace{\sin^2\theta}_{=1-\cos^2\theta} + 2\rho^2\right)d\theta\right)d\rho = \int_0^1\left(\int_0^{2\pi}(6\rho^2\cos^2\theta)\,d\theta\right)d\rho = 6\int_0^1\rho^2\,d\rho\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \left[2\rho^3\right]_0^1\cdot\frac{1}{2}\int_0^{2\pi}(1+\cos(2\theta))\,d\theta = \cancel{2}\cdot\frac{1}{\cancel{2}}\left[\int_0^{2\pi}d\theta + \int_0^{2\pi}\cos(2\theta)\frac{d(2\theta)}{2}\right] = 2\pi + \underbrace{\left[\frac{\sin(2\theta)}{2}\right]_0^{2\pi}}_{=0} = 2\pi.$$

8. $\vec{W}$ tq. $\vec{\mathrm{rot}}\,\vec{W} = 2\vec{V}$.

$$\oint_{\gamma^+}\vec{W}\cdot\vec{d\ell} \underset{\text{Stokes}}{=} \iint_{D^+}\vec{\mathrm{rot}}\,\vec{W}\cdot\vec{dS} = 2\iint_{D^+}\vec{V}\cdot\vec{dS} = -4\pi.$$

où $\partial D^+ = \gamma^+$