Corrigé du CC3 – Math2 – Jeudi 24 avril 2014 – A. Frabetti

Exo 1 : $f(x,y) = \ln(x+y^2+1)$ $\qquad f(0,0) = \ln 1 = 0$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} \frac{1}{x+y^2+1} \\ \frac{2y}{x+y^2+1} \end{pmatrix} \qquad \vec{\nabla} f(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$H_f(x,y) = \begin{pmatrix} -\frac{1}{(x+y^2+1)^2} & \frac{-2y}{(x+y^2+1)^2} \\ \frac{-2y}{(x+y^2+1)^2} & \frac{2(x+y^2+1)-2y\cdot 2y}{(x+y^2+1)^2} \end{pmatrix} \qquad H_f(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\frac{2(x+y^2+1)-2y\cdot 2y}{(x+y^2+1)^2} = \frac{2(x-y^2+1)}{(x+y^2+1)^2}$$

Taylor en $(0,0)$: $\ln(x+y^2+1) \sim 0 + x + 0\cdot y - \frac{1}{2}x^2 + 0\cdot xy + \frac{2}{2}y^2 = x - \frac{1}{2}x^2 + y^2$.

Exo 2 : $f(x,y) = x^2(y^2-1) - 2y$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} 2x(y^2-1) \\ 2x^2y - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} x(y^2-1) = 0 \\ x^2y = 1 \end{cases}$$

- $x = 0 \Rightarrow 0\cdot y = 1$ impossible
- $y = \pm 1 \Rightarrow$
  - $y = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ $\quad (\pm 1, 1)$
  - $y = -1 \Rightarrow x^2 = -1$ imp.

Il y a donc deux points critiques : $(\pm 1, 1)$

$$\det H_f(x,y) = \det \begin{pmatrix} 2(y^2-1) & 4xy \\ 4xy & 2x^2 \end{pmatrix} = 4x^2(y^2-1) - 16x^2y^2 = 4x^2(y^2-1-4y^2) = -4x^2(3y^2+1)$$

$\det H_f(\pm 1, 1) = -4\cdot 1\cdot(3+1) = -16 < 0 \implies$ deux points col.

Exo 3 : $\vec{V}(\rho,\theta,z) = z\rho\, \vec{e}_\rho + \rho^2\theta\, \vec{e}_\theta + z\theta\, \vec{k}$

$$\operatorname{div}\vec{V}(\rho,\theta,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(z\rho^2) + \frac{1}{\rho}\frac{\partial}{\partial\theta}(\rho^2\theta) + \frac{\partial}{\partial z}(z\theta) = \frac{1}{\rho}\cdot 2\rho z + \frac{1}{\rho}\cdot\rho^2 + \theta = 2z + \rho + \theta.$$

Exo 4 :
$$\vec{U} = \begin{pmatrix} 2xz \\ z^3 \\ x^2 - 2yz \end{pmatrix} \Rightarrow \vec{\operatorname{rot}}\, \vec{U} = \begin{pmatrix} -2z - 3z^2 \\ -(2x-2x) \\ 0-0 \end{pmatrix} = \begin{pmatrix} -2z-3z^2 \\ 0 \\ 0 \end{pmatrix} \neq \vec{0} \Rightarrow \vec{U} \text{ n'a pas de potentiel.}$$

$$\vec{V} = \begin{pmatrix} 2xz \\ -z^2 \\ x^2 - 2yz \end{pmatrix} \Rightarrow \vec{\operatorname{rot}}\, \vec{V} = \begin{pmatrix} -2z-(-2z) \\ -(2x-2x) \\ 0-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V} \text{ a un potentiel scalaire.}$$

Cherchons f t.q. $\vec{V} = \vec{\operatorname{grad}}\, f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$:

$$\begin{cases} \frac{\partial f}{\partial x} = 2xz \\ \frac{\partial f}{\partial y} = -z^2 \\ \frac{\partial f}{\partial z} = x^2 - 2yz \end{cases}$$

$\frac{\partial f}{\partial x} = 2xz \implies f(x,y,z) = \int 2xz\, dx + g(y,z) = x^2z + g(y,z)$

$\frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y} \overset{!}{=} -z^2 \Rightarrow g(y,z) = -\int z^2 dy + h(z) = -yz^2 + h(z)$

$\Rightarrow f(x,y,z) = x^2z - yz^2 + h(z)$

$\frac{\partial f}{\partial z} = x^2 - 2yz + h'(z) \overset{!}{=} x^2 - 2yz \Rightarrow h'(z) = 0 \Rightarrow h(z) = \text{cost.}$

$\implies f(x,y,z) = x^2 - 2yz + C$.