Corrigé du CC3 - Math 2 - Vendredi 25 avril 2014 - A. Frabetti

Exo 1: $f(x,y) = \ln(x^2+2y+1)$ $\qquad f(0,0) = \ln 1 = 0$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} \frac{2x}{x^2+2y+1} \\ \frac{2}{x^2+2y+1} \end{pmatrix} \qquad \vec{\nabla} f(0,0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$H_f(x,y) = \begin{pmatrix} \frac{2(x^2+2y+1) - 2x\cdot 2x}{(x^2+2y+1)^2} & \frac{-4x}{(x^2+2y+1)^2} \\ \frac{-4x}{(x^2+2y+1)^2} & \frac{-4}{(x^2+2y+1)^2} \end{pmatrix} = \begin{pmatrix} \frac{2(-x^2+2y+1)}{(x^2+2y+1)^2} & \text{idem} \\ \text{idem} & \text{idem} \end{pmatrix} \qquad H_f(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

Taylor en $(0,0)$: $\ln(x^2+2y+1) \sim 0 + 0\cdot x + 2\cdot y + \frac{2}{2}x^2 + 0\cdot xy - \frac{4}{2}y^2 = 2y + x^2 - 2y^2$,

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Exo 2: $f(x,y) = (x^2-4)(y^2-1) - 4x$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} 2x(y^2-1) - 4 \\ 2(x^2-4)y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} x(y^2-1) = 2 \\ (x^2-4)y = 0 \end{cases}$$

- $x = \pm 2$:
  - $x=2 \Rightarrow y^2-1 = 1 \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2} \rightsquigarrow (2, \pm\sqrt{2})$
  - $x=-2 \Rightarrow y^2-1 = -1 \Rightarrow y^2 = 0 \Rightarrow y = 0$
- $y = 0 \Rightarrow x\cdot(-1) = 2 \Rightarrow x = -2 \rightsquigarrow (-2,0)$

Il y a donc trois points critiques: $(-2,0)$ et $(2, \pm\sqrt{2})$.

$$\det H_f(x,y) = \det \begin{pmatrix} 2(y^2-1) & 4xy \\ 4xy & 2(x^2-4) \end{pmatrix} = 4(x^2-4)(y^2-1) - 16x^2y^2 = 4\left[(x^2-4)(y^2-1) - 4x^2y^2\right]$$

$\det H_f(-2,0) = 4\cdot\left[\underbrace{(4-4)}_{0}\cdot(-1) - 0\right] = 0 \Rightarrow$ on ne peut rien dire sur $(-2,0)$

$\det H_f(2,\pm\sqrt{2}) = 4\left[\underbrace{(4-4)}_{0}\cdot(2-1) - 4\cdot 4\cdot 2\right] = -16\cdot 8 < 0 \Rightarrow (2, \pm\sqrt{2})$ sont deux points col.

Exo 3 $\vec{V}(r,\theta,\varphi) = r\theta\, \vec{e}_r + r\theta\, \vec{e}_\theta + r\cos\varphi\, \vec{e}_\varphi$

$$\operatorname{div}\vec{V}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\theta) + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\theta}(r\theta) + \frac{1}{r\cos\varphi}\frac{\partial}{\partial\varphi}(r\cos^2\varphi)$$

$$= \frac{1}{\cancel{r^2}}\cdot 3\cancel{r^2}\theta + \frac{1}{\cancel{r\cos\varphi}}\cdot\cancel{r} + \frac{1}{\cancel{r\cos\varphi}}\cdot(-2\,\cancel{r\cos\varphi}\sin\varphi) = 3\theta + \frac{1}{\cos\varphi} - 2\sin\varphi.$$

Exo 4: $\vec{V} = \begin{pmatrix} yz^2 \\ xz \\ 2xyz - x \end{pmatrix} \Rightarrow \overrightarrow{\text{rot}}\,\vec{V} = \begin{pmatrix} 2xz - x \\ -(2yz - 1 - 2yz) \\ z - z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ z - z^2 \end{pmatrix} \neq \vec{0} \Rightarrow \vec{V}$ n'a pas de potentiel.

$\vec{U} = \begin{pmatrix} yz^2 - z \\ xz^2 \\ 2xyz - x \end{pmatrix} \Rightarrow \overrightarrow{\text{rot}}\,\vec{U} = \begin{pmatrix} 2xz - 2xz \\ -(2yz - 1 - 2yz + 1) \\ z^2 - z^2 \end{pmatrix} = \vec{0} \Rightarrow \vec{U}$ a un pot. scalaire.

Cherchons f t.q. $\vec{U} = \overrightarrow{\text{grad}}\, f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$:

$$\begin{cases} \frac{\partial f}{\partial x} = yz^2 - z \\ \frac{\partial f}{\partial y} = xz^2 \\ \frac{\partial f}{\partial z} = 2xyz - x \end{cases}$$

$\Rightarrow f(x,y,z) = \int (yz^2 - z)\,dx + g(y,z) = xyz^2 - xz + g(y,z)$

$\frac{\partial f}{\partial y} = xz^2 + \frac{\partial g}{\partial y} \overset{!}{=} xz^2 \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = h(z)$

$\frac{\partial f}{\partial z} = 2xyz - x + h'(z) \overset{!}{=} 2xyz - x \Rightarrow h'(z) = 0 \Rightarrow h(z) = \text{const},$

$\Rightarrow f(x,y,z) = xyz^2 - xz + C,$