Exo1 $f(x,y) = \ln(x^2y^3) \Rightarrow df_{(x,y)} = \frac{2xy^3}{x^2y^3}\,dx + \frac{3x^2y^2}{x^2y^3}\,dy = \frac{2}{x}\,dx + \frac{3}{y}\,dy$

Exo2 $f(x,y) = x^3y - 6x^2 - 2y^2$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} 3x^2y - 12x \\ x^3 - 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 3x(xy-4) = 0 \\ x^3 - 4y = 0 \end{cases} \quad \begin{matrix} x = 0 & ① \\ xy = 4 & ② \end{matrix}$$

① $\begin{cases} x = 0 \\ -4y = 0 \end{cases} \Rightarrow (0,0)$ est sol.

② $\begin{cases} y = \frac{4}{x} \text{ si } x \neq 0 \\ x^3 - \frac{16}{x} = 0 \iff x^4 - 16 = 0 \end{cases}$ $\quad x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow y = \frac{4}{\pm 2} = \pm 2$; $x^2 = -4$ imp $\rightsquigarrow (\pm 2, \pm 2)$ sol.

Donc il y a trois points extrêmes : $(0,0)$, $(2,2)$ et $(-2,-2)$.

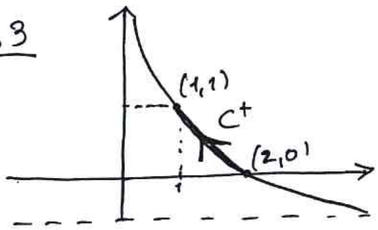
$$\det \mathrm{Hess} f(x,y) = \det \begin{pmatrix} 6xy - 12 & 3x^2 \\ 3x^2 & -4 \end{pmatrix} = -4(6xy-12) - 9x^4 = 3(16 - 8xy - 3x^4)$$

$\det \mathrm{Hess} f(0,0) = 3 \cdot 16 > 0 \Rightarrow (0,0)$ est extremum loc.
$\frac{\partial^2 f}{\partial y^2}(0,0) = -4 < 0$ $\Bigg\} \Rightarrow (0,0)$ est un max. loc.

$\det \mathrm{Hess} f(\pm 2, \pm 2) = 3(16 - 8(\pm 2)(\pm 2) - 3(\pm 2)^4) = 3(16 - 32 - 3 \cdot 16) = 3(-2 \cdot 16 - 32) = -3 \cdot 64 < 0$

$\Longrightarrow (\pm 2, \pm 2)$ sont deux points col.

Exo3



$C^+ = \{ (x,y) \in \mathbb{R}^2 \mid y = \frac{2}{x} - 1 \,,\ x \in [1,2] \}$

orientée $x : 2 \to 1$

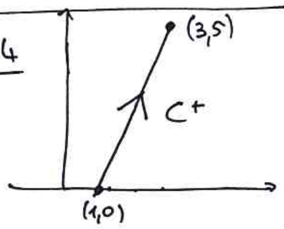
$\vec{V}(x,y) = xy\,\vec{i} + (x^2+1)\,\vec{j}$

$$\int_{C^+} \vec{V} \cdot \vec{dl} = \int_{C^+} xy\,dx + (x^2+1)\,dy = \int_2^1 \left[x\left(\frac{2}{x} - 1\right) + (x^2+1)\left(-\frac{2}{x^2}\right) \right] dx = \int_2^1 \left(\cancel{2} - x - \cancel{2} - \frac{2}{x^2} \right) dx$$

$y = \frac{2}{x} - 1 \Rightarrow dy = -\frac{2}{x^2}\,dx$

$$= \int_1^2 \left(x + \frac{2}{x^2} \right) dx = \left[\frac{1}{2}x^2 - \frac{2}{x} \right]_1^2 = \left(\frac{4}{2} - 1\right) - \left(\frac{1}{2} - 2\right) = 2 - 1 - \frac{1}{2} + 2 = 3 - \frac{1}{2} = \frac{5}{2}.$$

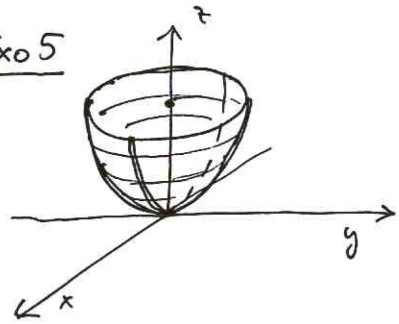
Exo4



$\vec{V} = \overrightarrow{\mathrm{grad}} f \qquad f(x,y) = x^2 - y$

$$\int_{C^+} \overrightarrow{\mathrm{grad}}\, f \cdot \vec{dl} = f(3,5) - f(1,0) = 9 - 5 - 1 = 3.$$

2) Exo 5



paraboloïde $= \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = z\}$

$S = \{(x,y,z) = f(r,\theta) = (r\cos\theta, r\sin\theta, r^2) \mid r \in [0,1], \theta \in [0,2\pi]\}$

i.e. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r^2 \end{cases}$

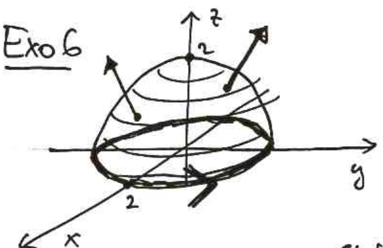
$$\begin{cases} \dfrac{\partial \vec{f}}{\partial r}(r,\theta) = (\cos\theta, \sin\theta, 2r) \\ \dfrac{\partial \vec{f}}{\partial \theta}(r,\theta) = (-r\sin\theta, r\cos\theta, 0) \end{cases} \Longrightarrow \frac{\partial \vec{f}}{\partial r} \wedge \frac{\partial \vec{f}}{\partial \theta}(r,\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$\vec{V}(x,y,z) = y\vec{i} - x\vec{j} + z\vec{k} \Longrightarrow \vec{V}(f(r,\theta)) = r\sin\theta\,\vec{i} - r\cos\theta\,\vec{j} + r^2\vec{k}$$

$$\iint_S \vec{V}\cdot\vec{dS} = \iint_{[0,1]\times[0,2\pi]} \begin{pmatrix} r\sin\theta \\ -r\cos\theta \\ r^2 \end{pmatrix} \cdot \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r \end{pmatrix} dr\,d\theta = \iint_{[0,1]\times[0,2\pi]} \left(\cancel{-2r^3\sin\theta\cos\theta} + \cancel{2r^3\cos\theta\sin\theta} + r^3\right) dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r^3\,dr = [\theta]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^1 = 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}.$$

Exo 6



S^+ = demi-sphère de rayon 2 orienté sortant

∂S^+ = cercle d'éq. $\begin{cases} x^2+y^2 = 4 \\ z = 0 \end{cases}$ orienté dans le sens antihoraire

$\vec{U}(x,y,z) = (x^2-y^2)\vec{i} + 2xy\,\vec{j}$

$$\iint_S \vec{\mathrm{rot}}\,\vec{U}\cdot\vec{dS} \overset{\text{Stokes}}{=\!=} \oint_{\partial S^+} \vec{U}\cdot\vec{dl} = \int_{\text{cercle}} (x^2-y^2)\,dx + 2xy\,dy =$$

coord. polaires sur plan $z=0$

$\begin{cases} x = 2\cos\theta & dx = -2\sin\theta\,d\theta \\ y = 2\sin\theta & dy = 2\cos\theta\,d\theta \end{cases}$ $\theta \in [0,2\pi]$

$$= \int_0^{2\pi} \left[4(\cos^2\theta - \sin^2\theta)(-2\sin\theta) + 2\cdot 4\cos\theta\sin\theta\cdot 2\cos\theta\right] d\theta = \int_0^{2\pi} \left(-8\cos^2\theta\sin\theta + 8\sin^3\theta + 16\cos^2\theta\sin\theta\right) d\theta$$

$$= \int_0^{2\pi} 8\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}\sin\theta\,d\theta = 8\left[-\cos\theta\right]_0^{2\pi} = 0.$$

Exo 7

coquille $= \partial\Omega$ $\quad \mathrm{Vol}(\Omega) = 13$

$$\vec{V}(\rho,\theta,z) = \rho^2\,\vec{e}_\theta + (2z + \cos\theta)\,\vec{k}$$

$$\iint_{\partial\Omega} \vec{V}\cdot\vec{dS} \overset{\text{Gauss}}{=\!=} \iiint_\Omega \mathrm{div}\,\vec{V}\cdot dx\,dy\,dz = 2\iiint_\Omega dx\,dy\,dz = 2\,\mathrm{Vol}(\Omega) = 26,$$

$$\mathrm{div}\,\vec{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot 0) + \frac{1}{\rho}\frac{\partial}{\partial\theta}(\rho^2) + \frac{\partial}{\partial z}(2z + \cos\theta) = 0 + 0 + 2 = 2$$