QCM - c b d a a

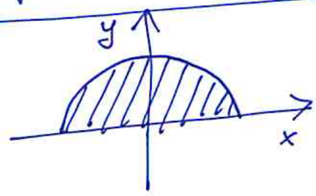
Exo 1 $\quad f(x,y) = \dfrac{e^{3x}}{2y+1} \qquad f(0,0) = \dfrac{e^0}{0+1} = 1$

$$\left[\begin{array}{ll} \dfrac{\partial f}{\partial x}(x,y) = \dfrac{3e^{3x}}{2y+1} & \dfrac{\partial f}{\partial x}(0,0) = \dfrac{3e^0}{0+1} = 3 \\ \dfrac{\partial f}{\partial y}(x,y) = -\dfrac{2e^{3x}}{(2y+1)^2} & \dfrac{\partial f}{\partial y}(0,0) = \dfrac{-2e^0}{(0+1)^2} = -2 \end{array}\right.$$

$$\left[\begin{array}{ll} \dfrac{\partial^2 f}{\partial x^2}(x,y) = \dfrac{9e^{3x}}{2y+1} & \dfrac{\partial^2 f}{\partial x^2}(0,0) = 9 \\ \dfrac{\partial^2 f}{\partial y^2}(x,y) = +\dfrac{2e^{3x}}{(2y+1)^{\cancel{4}\,3}}\cdot 2\cancel{(2y+1)}\cdot 2 = \dfrac{8e^{3x}\cancel{(2y+1)}}{(2y+1)^3} & \dfrac{\partial^2 f}{\partial y^2}(0,0) = 8 \\ \dfrac{\partial^2 f}{\partial x\partial y}(x,y) = -\dfrac{2\cdot 3e^{3x}}{(2y+1)^2} & \dfrac{\partial^2 f}{\partial x\partial y}(0,0) = -6 \end{array}\right.$$

$$\Longrightarrow \quad \frac{e^{3x}}{2y+1} = 1 + 3x - 2y + \frac{9}{2}x^2 - 6xy + 4y^2 + o(x^2+y^2) \text{ pour } (x,y) \to (0,0)$$

Exo 2 a)



$D^+ = \{(\rho,\varphi) \mid 0 \le \rho \le 1,\ 0 \le \varphi \le \pi\}$

$\hat{\mu}(\rho,\varphi) = \rho \sin\varphi$

b) $M = \iint_{D^+} \mu(x,y)\,dx\,dy = \iint_{D^+} \hat{\mu}(\rho,\varphi)\,\rho\,d\rho\,d\varphi = \iint_{D^+} \rho^2 \sin\varphi\,d\rho\,d\varphi$

$= \int_0^1 \rho^2\,d\rho \int_0^\pi \sin\varphi\,d\varphi = \left[\frac{1}{3}\rho^3\right]_0^1 \left[-\cos\varphi\right]_0^\pi$

$= \frac{1}{3}\cdot\left(-\cos\pi + \cos 0\right) = \frac{1}{3}(1+1) = \frac{2}{3}.$

c) $x_G = \frac{1}{M}\iint_{D^+} x\,\mu(x,y)\,dx\,dy = \frac{1}{M}\iint_{D^+} \rho\cos\varphi\cdot\rho\sin\varphi\,\rho\,d\rho\,d\varphi$

$= \frac{3}{2}\int_0^1 \rho^3\,d\rho \int_0^\pi \cos\varphi\sin\varphi\,d\varphi = \frac{3}{2}\cdot\left[\frac{1}{4}\rho^4\right]_0^1 \left[\frac{1}{2}\sin^2\varphi\right]_0^\pi = \frac{3}{2}\cdot\frac{1}{4}\cdot 0 = 0$

$y_G = \frac{1}{M}\iint_{D^+} y\,\mu(x,y)\,dx\,dy = \frac{1}{M}\iint_{D^+} \rho\sin\varphi\ \rho\sin\varphi\ \rho\,d\rho\,d\varphi$

$= \frac{3}{2}\int_0^1 \rho^3\,d\rho \int_0^\pi \sin^2\varphi\,d\varphi = \frac{3}{2}\left[\frac{1}{4}\rho^4\right]_0^1 \left[\frac{1}{2}\varphi - \sin\varphi\cos\varphi\right]_0^\pi$

$= \frac{3}{2}\cdot\frac{1}{4}\cdot\left(\frac{1}{2}\pi - \underbrace{\sin\pi}_{=0}\cos\pi - 0 + \underbrace{\sin 0}_{=0}\cos 0\right) = \frac{3}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\pi = \frac{3\pi}{16}$

$\Longrightarrow \quad G\left(0, \frac{3\pi}{16}\right).$

2)

Exo 3 $\vec{E}(x,y) = 2x\sin y\ \vec{\imath} + x^2\cos y\ \vec{\jmath}$

a) $\vec{\mathrm{rot}}\,\vec{E} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2x\sin y & x^2\cos y & 0 \end{vmatrix} = 0.\vec{\imath} - 0\,\vec{\jmath} + (2x\cos y - 2x\cos y)\cdot\vec{k} = \vec{0}$

Par le lemme de Poincaré, $\vec{E}$ est conservatif sur tout ensemble simplement connexe de $\mathbb{R}^2$. Puisque $D_{\vec{E}} = \mathbb{R}^2$ est simplement connexe, le champ $\vec{E}$ est conservatif sur $\mathbb{R}^2$.

b) Cherchons $f : \mathbb{R}^2 \to \mathbb{R}$ t.q. $\vec{\mathrm{grad}}\, f = \vec{E}$.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x\sin y \implies f(x,y) = \displaystyle\int 2x\sin y\, dx + g(y) = x^2\sin y + g(y) \\[2ex] \dfrac{\partial f}{\partial y} = x^2\cos y \qquad \dfrac{\partial f}{\partial y} = x^2\cos y + g'(y) = x^2\cos y \iff g'(y) = 0 \iff g(y) = c \end{cases}$$

$$\implies f(x,y) = x^2\sin y + c, \quad \forall c \in \mathbb{R}.$$

c) $\displaystyle\int_\gamma \vec{E}\cdot\vec{d\ell} = \int_A^B \vec{\mathrm{grad}}\, f\cdot\vec{d\ell} = f(B) - f(A) = f(5, \pi/2) - f(1, 0)$

$= 25\sin(\pi/2) + \cancel{c} - \sin 0 - \cancel{c} = 25.$

Exo 4 $f(u,v) = (u, v, u\sin v)$ $u \in [0,1]$ $v \in [0, \pi/2]$

$\vec{V}(x,y,z) = -x\,\vec{\jmath} + y^2\,\vec{k}$

$\vec{V}(f(u,v)) = -u\,\vec{\jmath} + v^2\,\vec{k} = \begin{pmatrix} 0 \\ -u \\ v^2 \end{pmatrix}$

$\dfrac{\partial f}{\partial u} \wedge \dfrac{\partial f}{\partial v} = \begin{pmatrix} 1 \\ 0 \\ \sin v \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ u\cos v \end{pmatrix} = \begin{pmatrix} -\sin v \\ -(u\cos v) \\ 1 \end{pmatrix}$

$$\iint_S \vec{V}\cdot\vec{dS} = \iint_S \begin{pmatrix} 0 \\ -u \\ v^2 \end{pmatrix}\cdot\begin{pmatrix} -\sin v \\ -u\cos v \\ 1 \end{pmatrix} du\,dv = \iint_S (u^2\cos v + v^2)\, du\,dv$$

$$= \int_0^1 du \int_0^{\pi/2} dv\,(u^2\cos v + v^2) = \int_0^1 du\,\left[u^2\sin v + \frac{1}{3}v^3\right]_0^{\pi/2}$$

$$= \int_0^1 \left(u^2\sin\frac{\pi}{2} + \frac{1}{3}\frac{\pi^3}{2^3} - u^2\sin 0 - 0\right) du$$

$$= \int_0^1 \left(u^2 + \frac{\pi^3}{24}\right) du$$

$$= \left[\frac{1}{3}u^3 + \frac{\pi^3}{24}u\right]_0^1 = \frac{1}{3} + \frac{\pi^3}{24} = \frac{8+\pi^3}{24}.$$