Math 2 - 2017 - Corrigé de l'examen final - 1ère session - 11 mai 2017 (1

QCM : 1) c 2) a 3) c 4) d 5) b

Exercice 1 $f(x,y) = 5x^2 + 6xy + 2y^2 + 2x + 2y + 1$ $D_f = \mathbb{R}^2$

pts critiques : $\vec{\nabla} f(x,y) = \begin{pmatrix} 10x + 6y + 2 \\ 6x + 4y + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 5x + 3y + 1 = 0 \\ 3x + 2y + 1 = 0 \end{cases} \Leftrightarrow$

$\begin{cases} 6y = -10x - 2 \\ 9x + (-10x-2) + 3 = 0 \end{cases}$ $\begin{cases} y = -(5x+1)/3 \\ -x + 1 = 0 \end{cases}$ $\begin{cases} x = 1 \\ y = -6/3 = -2 \end{cases} \Leftrightarrow$ un seul pt. critique $(1,-2)$

nature du pt critique $(1,-2)$:

$H_f(x,y) = \begin{pmatrix} 10 & 6 \\ 6 & 4 \end{pmatrix} \Rightarrow \det H_f(1,-2) = 10 \times 4 - 6^2 = 40 - 36 = 4 > 0 \Rightarrow$ extremum local

$\frac{\partial^2 f}{\partial x^2}(1,-2) = 10 > 0 \Rightarrow$ $\boxed{\text{minimum local}}$

Exercice 2 $D = [-1,1] \times [0,1]$ $\mu(x,y) = (x^2+1)e^y$

a) $M = \iint_D \mu(x,y)\,dx\,dy = \int_{-1}^{1} dx \int_0^1 dy\,(x^2+1)e^y = \int_{-1}^{1}(x^2+1)\,dx \int_0^1 e^y\,dy$

$= \underbrace{\left[\frac{1}{3}x^3 + x\right]_{-1}^{1}}_{\text{fct. impair}} \cdot \left[e^y\right]_0^1 = 2\left[\frac{1}{3}x^3 + x\right]_0^1 \cdot \left[e^y\right]_0^1 = 2\left(\frac{1}{3}+1\right)\cdot(e-1) = \boxed{\frac{8}{3}(e-1)}$

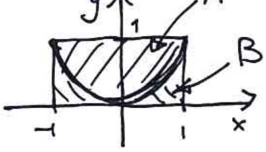
b) $G(x_G, y_G)$ avec

$x_G = \frac{1}{M}\iint_D x\,\mu(x,y)\,dx\,dy = \frac{3}{8(e-1)}\int_{-1}^{1} \overset{x^3+x}{x(x^2+1)}\,dx \int_0^1 e^y\,dy = \frac{3}{8(e-1)} \underbrace{\left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_{-1}^{1}}_{=0 \text{ car fct. pair}} \left[e^y\right]_0^1 = 0$

$y_G = \frac{1}{M}\iint_D y\,\mu(x,y)\,dx\,dy = \frac{3}{8(e-1)}\int_{-1}^{1}(x^2+1)\,dx \int_0^1 y e^y\,dy$ où $\int y e^y\,dy = ye^y - \int e^y\,dy = ye^y - e^y = (y-1)e^y$

$= \frac{3}{8(e-1)}\, 2\cdot\left[\frac{1}{3}x^3 + x\right]_0^1 \left[(y-1)e^y\right]_0^1$

$= \frac{3}{8(e-1)}\, 2\left(\frac{1}{3}+1\right)\left(0 - (-1)e^0\right) = \frac{\cancel{3}}{\cancel{8}(e-1)} \cdot \frac{\cancel{8}}{\cancel{3}} \cdot 1 = \frac{1}{e-1} \Rightarrow \boxed{G\left(0, \frac{1}{e-1}\right)}$

c) 

$D = A \cup B$

Aire $B = \int_{-1}^{1} f(x)\,dx$ où $y = f(x) = x^2$, donc

$= \int_{-1}^{1} x^2\,dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = 2\left[\frac{1}{3}x^3\right]_0^1 = \boxed{\frac{2}{3}}$

$\Rightarrow$ Aire A = Aire D − Aire B $= 2\cdot 1 - \frac{2}{3} = \frac{6-2}{3} = \boxed{\frac{4}{3}}$

Alternative : $A = \{(x,y)\ |\ -1 \le x \le 1,\ x^2 \le y \le 1\}$

$\Rightarrow$ Aire $A = \iint_A dx\,dy = \int_{-1}^{1} dx \int_{x^2}^{1} dy = \int_{-1}^{1} dx\,[y]_{x^2}^{1} = \int_{-1}^{1}(1-x^2)\,dx$

$= \left[x - \frac{1}{3}x^3\right]_{-1}^{1} = 2\left[x - \frac{1}{3}x^3\right]_0^1 = 2\left(1 - \frac{1}{3}\right) = 2\cdot\frac{3-1}{3} = \boxed{\frac{4}{3}}$ ok.

2/ Exercice 3 $\vec{E}(x,y) = \cos x \sin y \, \vec{\imath} + \sin x \cos y \, \vec{\jmath}$

a) $\vec{rot}\,\vec{E} = \left[\frac{\partial}{\partial x}(\sin x \cos y) - \frac{\partial}{\partial y}(\cos x \sin y)\right] \vec{k} = (\cos x \cos y - \cos x \cos y)\,\vec{k} = \vec{0}$

$D_{\vec{E}} = \mathbb{R}^2$ simplement connexe.

Par le lemme de Poincaré : $\vec{rot}\,\vec{E} = 0$ et $D_{\vec{E}}$ simpl. connexe $\Rightarrow$ $\vec{E}$ est conservatif sur $D_{\vec{E}}$.

i.e. $\exists f$ déf sur $\mathbb{R}^2$ t.q. $\vec{\nabla} f = \vec{E}$.

b) $\vec{grad}\, f(x,y) = \vec{E}(x,y) \Leftrightarrow \begin{cases} \frac{\partial f}{\partial x} = \cos x \sin y \\ \frac{\partial f}{\partial y} = \sin x \cos y \end{cases} \Rightarrow f(x,y) = \int \cos x \sin y \, dx + g(y)$

$= \sin x \sin y + g(y)$

$\Rightarrow \frac{\partial f}{\partial y} = \sin x \cos y + g'(y) = \sin x \cos y \Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{const. } c$

$\Rightarrow \boxed{f(x,y) = \sin x \sin y + c}$.

c) Si γ joigne $A(0,0)$ à $B(\frac{\pi}{4}, \frac{\pi}{2})$, alors

$\int_\gamma \vec{E} \cdot \vec{dl} = \int_\gamma \vec{grad}\, f \cdot \vec{dl} = f(B) - f(A) = f(\frac{\pi}{4}, \frac{\pi}{2}) - f(0,0)$

$= \sin\frac{\pi}{4} \sin\frac{\pi}{2} - \sin 0 \sin 0 = \frac{\sqrt{2}}{2} \cdot 1 - 0 = \boxed{\frac{\sqrt{2}}{2}}$

Exercice 4 $\vec{B}(x,y,z) = (z-y)\,\vec{\imath} + (x^2-1)\,\vec{\jmath} - x\,\vec{k}$

$\gamma(t) = (\cos t, \sin t, \sin t \cos t)$, $t \in [0, 2\pi]$

a) $\gamma'(t) = (-\sin t, \cos t, \cos^2 t - \sin^2 t) = -\sin t\,\vec{\imath} + \cos t\,\vec{\jmath} + (\cos^2 t - \sin^2 t)\,\vec{k}$

b) $\vec{B}(\gamma(t)) = \vec{B}(\cos t, \sin t, \sin t \cos t)$

$= (\sin t \cos t - \sin t)\,\vec{\imath} + \underbrace{(\cos^2 t - 1)}_{= -\sin^2 t}\,\vec{\jmath} - \cos t\,\vec{k}$

c) $\int_\gamma \vec{B} \cdot \vec{dl} = \int_0^{2\pi} \vec{B}(\gamma(t)) \cdot \gamma'(t)\, dt$

$= \int_0^{2\pi} \left[-(\sin t \cos t - \sin t)\sin t - \sin^2 t \cos t - \cos t(\cos^2 t - \sin^2 t)\right] dt$

$= \int_0^{2\pi} (\cancel{-\sin^2 t \cos t} + \sin^2 t - \sin^2 t \cos t - \cos^3 t + \cancel{\cos t \sin^2 t})\, dt$

$= \int_0^{2\pi} (\sin^2 t - \underbrace{(\sin^2 t + \cos^2 t)}_{=1} \cos t)\, dt$

$= \int_0^{2\pi} (\sin^2 t - \cos t)\, dt$

$= \left[\frac{1}{2}t - \frac{1}{2}\sin t \cos t - \sin t\right]_0^{2\pi}$

$= \left[\frac{1}{2}t\right]_0^{2\pi} = \frac{2\pi}{2} = \boxed{\pi}$.