# Ultracontractive bounds on Hamilton-Jacobi solutions


By

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Abstract.- Following the equivalence between logarithmic Sobolev inequality, hypercontractivity of the heat semigroup showed by Gross and hypercontractivity of Hamilton-Jacobi equations, we prove, like the Varopoulos theorem, the equivalence between Euclidean-type Sobolev inequality and an ultracontractive control of the Hamilton-Jacobi equations. We obtain also ultracontractive estimations under general Sobolev inequality which imply in the particular case of a probability measure, transportation inequalities.



Keywords : Hamilton-Jacobi equation; Sobolev inequality; Ultracontractivity; Transportation inequality.


## 1 Introduction

The main results of the following paper are for the border of three theorems, which are respectively the theorem of hypercontractivity of Gross, the theorem of hypercontractivity of Bobkov-Gentil-Ledoux and the theorem of ultracontractivity of Varopoulos. Let us describe these results.

The fundamental work by Gross [Gro75] put forward the equivalence between logarithmic Sobolev inequalities and hypercontractivity of the associated heat semigroup. Let us consider for example a probability measure $\mu$ on the Borel sets of $\mathbb{R}^n$ satisfying the logarithmic Sobolev inequality

$$\rho\,\mathbf{Ent}_\mu(f^2) \leqslant 2\int |\nabla f|^2 d\mu \tag{1}$$


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for some $\rho > 0$ and all smooth enough functions $f$ on $\mathbb{R}^n$ where

$$\mathbf{Ent}_\mu(f^2) = \int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu$$

and where $|\nabla f|$ is the Euclidean length of the gradient $\nabla f$ of $f$. The canonical Gaussian measure with density $(2\pi)^{-n/2} \mathrm{e}^{-|x|^2/2}$ with respect to the LEBESGUE measure on $\mathbb{R}^n$ is the basic example of measure $\mu$ satisfying (1) with $\rho = 1$.

For simplicity, assume furthermore that $\mu$ has a strictly positive smooth density which may be written $\mathrm{e}^{-U}$ for some smooth function $U$ on $\mathbb{R}^n$. Denote by $\mathbf{L}$ the second order diffusion operator $\mathbf{L} = \Delta - \langle \nabla U, \nabla \rangle$ with invariant measure $\mu$. Integration by parts for $\mathbf{L}$ is described by

$$\int f(-\mathbf{L}g) d\mu = \int \langle \nabla f, \nabla g \rangle d\mu$$

for every smooth functions $f, g$. Under mild growth conditions on $U$ one may consider the time reversible (with respect to $\mu$) semigroup $(\mathbf{P_t})_{t \geqslant 0}$ with generator $\mathbf{L}$. Given a function $f$ (in the domain of $\mathbf{L}$), $u = u(x,t) = \mathbf{P_t} f(x)$ is the fundamental solution of the initial value problem (heat equation with respect to $\mathbf{L}$)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathbf{L}u &= 0 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\ \qquad\quad u &= f \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

One of the main results of the contribution [Gro75] by GROSS is that the logarithmic SOBOLEV inequality (1) for $\mu$ holds if and only if the associated heat semigroup $(\mathbf{P_t})_{t \geqslant 0}$ is hypercontractive in the sense that, for every (or some) $1 < p < q < \infty$, and every $f$ (in $\mathbf{L}^p$),

$$\|\mathbf{P_t} f\|_q \leqslant \|f\|_p \tag{2}$$

for every $t > 0$ large enough so that

$$\mathrm{e}^{2\rho t} \geqslant \frac{q-1}{p-1}\,. \tag{3}$$

In (2), the $\mathbf{L}^p$-norms are understood with respect to the measure $\mu$. The key idea of the proof is to consider a function $q(t) = 1 + (p-1)\,\mathrm{e}^{2\rho t}$ of $t \geqslant 0$ such that $q(0) = p$ and to take the derivative in time of $F(t) = \|\mathbf{P_t} f\|_{q(t)}$ (for a non-negative smooth function $f$ on $\mathbb{R}^n$).

Following GROSS's idea, the main result of [BGL00] is to establish a similar relationship for the solutions of HAMILTON-JACOBI partial differential equations. Consider the HAMILTON-JACOBI initial value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{1}{2} |\nabla v|^2 & = 0 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\ v & = f \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{4}$$

Solutions of (4) are described by the HOPF-LAX representation formula as infimum-convolutions. Namely, given a (Lipschitz) function $f$ on $\mathbb{R}^n$, define the infimum-convolution of $f$ with quadratic cost as

$$\mathbf{Q_t} f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2t} |x - y|^2 \right\}, \quad t > 0,\, x \in \mathbb{R}^n. \tag{5}$$

The family $(\mathbf{Q_t})_{t \geqslant 0}$ defines a semigroup with infinitesimal (non-linear) generator $-\frac{1}{2} |\nabla f|^2$. That is, $v = v(x,t) = \mathbf{Q_t} f(x)$ is a solution of the HAMILTON-JACOBI initial value problem (4) (at least almost everywhere). Actually, if in addition $f$ is bounded, the HOPF-LAX formula $\mathbf{Q_t} f$ is the pertinent mathematical solution of (4), that is its unique viscosity solution (cf. e.g. [Bar94], [Eva98]).

An other way to introduce the HAMILTON-JACOBI solutions is to use the vanishing viscosity. Let $\mathbf{L}$ an infinitesimal diffusion generator, like Laplacian, and $(\mathbf{P}_t)_{t \geqslant 0}$ the associated heat semigroup. Given a smooth function $f$, and $\varepsilon > 0$, denote namely by $v^\varepsilon = v^\varepsilon(x,t)$ the solution of the initial value partial differential equation

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} + \dfrac{1}{2} |\nabla v^\varepsilon|^2 - \varepsilon \mathbf{L} v^\varepsilon & = 0 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\ v^\varepsilon & = f \quad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

As $\varepsilon \to 0$, it is expected that $v^\varepsilon$ approaches in a reasonable sense the solution $v$ of (4). It is easy to check that $u^\varepsilon = \mathrm{e}^{-v^\varepsilon / 2\varepsilon}$ is a solution of the heat equation $\frac{\partial u^\varepsilon}{\partial t} = \varepsilon \mathbf{L} u^\varepsilon$ (with initial value $\mathrm{e}^{-f/2\varepsilon}$). Therefore,

$$u^\varepsilon = \mathbf{P_{\varepsilon t}} \big( \mathrm{e}^{-f/2\varepsilon} \big).$$

It must be emphasized that the perturbation argument by a small noise has a clear picture in the probabilistic language of large deviations. Namely, the asymptotic of

$$v^\varepsilon = -2\varepsilon \log \mathbf{P_{\varepsilon t}} \big( \mathrm{e}^{-f/2\varepsilon} \big) \tag{6}$$

as $\varepsilon \to 0$ is a LAPLACE-VARADHAN asymptotic with rate described precisely by the infimum convolution of $f$ with the quadratic large deviation rate function for the heat semigroup (see [Bar94] or [BGL00]).

The main results in [BGL00] is that if the logarithmic Sobolev inequality (1) holds, then for each $t \geqslant 0$, $a \in \mathbb{R}$ and each (say Lipschitz bounded) function $f$,

$$\left\| e^{\mathbf{Q_t} f} \right\|_{a+\rho t} \leqslant \left\| e^{f} \right\|_{a}. \tag{7}$$

Conversely, if (7) holds for every $t \geqslant 0$ and some $a \neq 0$, then the logarithmic Sobolev inequality (1) holds. Compared respect to classical hypercontractivity, it is worthwhile noting that $(\mathbf{Q_t})_{t\geqslant 0}$ is defined independently of the underlying measure $\mu$.

Can one obtain more than hypercontractivity? Varopoulos answers that under a stronger constraint we obtain an ultracontractive control of the semigroup. Let us recall the Varopoulos's theorem.

Let us consider a measure $\mu$ (not necessary of probability) on a smooth Riemanniann manifold $M$, satisfying a Sobolev inequality in dimension $n$, with $n > 2$:

$$\|f\|^2_{2n/(n-2)} \leqslant a\|\nabla f\|_2^2 + b\|f\|_2^2, \tag{8}$$

for some $a, b \geqslant 0$ and any smooth enough function $f$ with compact support. In (8) the $\mathbf{L}^p$-norms are understood with respect to the measure $\mu$. The fundamental example is the Lebesgue's measure in $\mathbb{R}^n$ which satisfies a Sobolev inequality of dimention $n$ with $b = 0$.

Denote by $\mathbf{L}$ a diffusion generator and $(\mathbf{P_t})_{t\geqslant 0}$ the heat semigroup associated. Assume that the measure $\mu$ is reversible whith respect to the operator $\mathbf{L}$ and $-\int f \mathbf{L} f d\mu = \|\nabla f\|_2^2$.

One of the main results of Varopoulos (see [Var84], [Var85] or [Var91]) is that the Sobolev inequality (8) for $\mu$ holds if and only if the semigroup $(\mathbf{P_t})_{t\geqslant 0}$ is ultracontractive in the sense that there is a constant $k > 0$ such that for each $t \in ]0, 1]$ and each function $f$ (in $\mathbf{L}^1$), we have

$$\|\mathbf{P_t} f\|_\infty \leqslant \|f\|_1 \frac{k}{t^{n/2}}.$$

At the light of the three theorems we study, like in [BGL00] for the logarithmic Sobolev inequality, the implication of Sobolev inequality (8) to the Hamilton-Jacobi semigroup $(\mathbf{Q_t})_{t\geqslant 0}$.

The next section deals with the $\mathbb{R}^n$ case and the Lebesgue measure. We prove, by 3 methods, an optimal ultracontractive estimate for the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$ in $\mathbb{R}^n$. In particular, we use the vanishing viscosity (inequality (6)) and the Brunn-Minskowski inequality.

In section 3, we prove that a measure $\mu$ on a manifold $M$, satisfies an Euclidean-type Sobolev inequality (where $b = 0$ in the inequality (8)) if ond only if the following control of the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$ holds for each $\beta > \alpha > 0$, $t > 0$ and for any bounded function $f$, such that $\left\|e^f\right\|_1 < \infty$,

$$\left\|e^{\mathbf{Q_t}f}\right\|_\beta \leqslant \left\|e^f\right\|_\alpha \left(\frac{k\alpha(\beta-\alpha)}{t\beta}\right)^{\frac{n}{2}\frac{\beta-\alpha}{\beta\alpha}}. \tag{9}$$

When $\beta = \infty$ and $\alpha = 1$ the inequality (9) becomes

$$\left\|e^{\mathbf{Q_t}f}\right\|_\infty \leqslant \left\|e^f\right\|_1 \left(\frac{k}{t}\right)^{\frac{n}{2}}, \tag{10}$$

for every bounded $f$ and $t > 0$.

Section 4 finally states such a control for the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$, when the measure satisfies a Sobolev inequality with constants $a > 0$ and $b > 0$. We prove that under a general Sobolev inequality the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$ satisfies the inequality (10) for every $t \in ]0,1]$. When the manifold is compact, some interesting inequalities are obtained for the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$, which also imply some transportation inequalities for probability measures.

# 2 The $\mathbb{R}^n$ case

## 2.1 Ultracontractive bounds of the Hamilton-Jacobi equations in $\mathbb{R}^n$

Before getting into more complicated cases let us start with the example of the Lebesgue measure on $\mathbb{R}^n$. If $f$ is a bounded Lipschitz function on $\mathbb{R}^n$, define $\mathbf{Q_t}f$ by

$$\mathbf{Q_t}f(x) = \inf_{y\in\mathbb{R}^n}\left\{f(y) + \frac{1}{2t}|x-y|^2\right\}, \quad t > 0,\ x \in \mathbb{R}^n, \tag{11}$$

and $\mathbf{Q_0}f(x) = f(x)$. The function $\mathbf{Q_t}f$ is known as the Hopf-Lax solution of the Hamilton-Jacobi equation

$$\frac{\partial \mathbf{Q_t}f}{\partial t}(x) = -\frac{1}{2}|\nabla \mathbf{Q_t}f(x)|^2, \tag{12}$$

with initial value $f$.

Then, in $\mathbb{R}^n$, considering the Lebesgue measure yields the following theorem.

**Theorem 2.1** *Let $f$ be a bounded Lipschitz function on $\mathbb{R}^n$ and let $\alpha$ and $\beta$ be two constants such that $0 < \alpha \leqslant \beta$. Note $\|\cdot\|_p$ is the $\mathbf{L}^p$-norm of the* LEBESGUE *measure in $\mathbb{R}^n$, then*

$$\left\| e^{\mathbf{Q_t} f} \right\|_\beta \leqslant \left\| e^f \right\|_\alpha \left( \frac{\beta - \alpha}{2\pi t} \right)^{\frac{n}{2}\frac{\beta-\alpha}{\beta\alpha}} \left( \frac{\alpha}{\beta} \right)^{\frac{n}{2}\frac{\alpha+\beta}{\alpha\beta}}, \tag{13}$$

*for any $t > 0$.*

***Proof***

◄ In this following proof we use the method developed by DAVIES and BAKRY in [Dav90] and [Bak94]. To prove the inequality (13) we use the following inequality

$$\mathbf{Ent}_{dx}(e^g) \leqslant \frac{n}{2} \left( \int e^g dx \right) \log \left( \frac{1}{2e\pi n} \frac{\int |\nabla g|^2 e^g dx}{\int e^g dx} \right), \tag{14}$$

for any smooth function $g$ on $\mathbb{R}^n$. This inequality is called *Euclidean logarithmic* SOBOLEV inequality and can be obtained as a consequence of the logarithmic SOBOLEV inequality for the Gaussian measure on $\mathbb{R}^n$ (see for example [Car91] or Chapters 4 or 10 of [ABC$^+$00]). And by the concavity of the logarithmic function, inequality (14) is equivalent to the family of logarithmic SOBOLEV inequalities, for each $x > 0$,

$$\mathbf{Ent}_{dx}(e^g) \leqslant \frac{n}{2x} \int |\nabla g|^2 e^g dx + \frac{n}{2} \log \left( \frac{1}{2\pi e^2 n} x \right) \int e^g dx. \tag{15}$$

Thanks to the property of $\mathbf{Q_t} f$,

$$\lambda \geqslant 0,\ t \geqslant 0,\ \ \mathbf{Q_t}(\lambda f) = \lambda \mathbf{Q}_{\lambda t} f, \tag{16}$$

we just have to prove the inequality (13) when $t = 1$. Let $\alpha$ and $\beta$ be such that $0 < \alpha \leqslant \beta$, and define $F(t) = \left\| e^{\mathbf{Q_t} f} \right\|_{q(t)}$ with $q(t) = \alpha\beta/((\alpha - \beta)t + \beta)$. Because $f$ is a bounded Lipschitz then the function $F$ is smoodth and we obtain

$$F'(t) = F(t)^{1-q(t)} \frac{q'}{q^2} \left( \mathbf{Ent}_{dx}\left( e^{q(t)\mathbf{Q_t} f} \right) - \int \frac{|\nabla q(t) \mathbf{Q_t} f|^2}{2q'(t)} e^{q(t)\mathbf{Q_t} f} dx \right). \tag{17}$$

Taking $x(t) = nq'(t)$ and using (17) and (15) then yields

$$F'(t) \leqslant F(t) \frac{n}{2} \frac{q'(t)}{q^2(t)} \log \left( \frac{1}{2\pi e^2} q'(t) \right).$$

Theorem 2.1 then follows from an integration over $t \in [0,1]$ and from the fact that

$$\int_0^1 \frac{n}{2}\frac{q'(t)}{q^2(t)} \log\left(\frac{1}{2\pi e^2} q'(t)\right) dt = \frac{n}{2}\frac{\beta-\alpha}{\beta\alpha} \log\left(\frac{\beta-\alpha}{\alpha\beta}\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha+\beta}{\beta-\alpha}}\right).$$

►

In the previous theorem, $\alpha$ and $\beta$ can be chosen arbitrary. When $\beta = \infty$, we obtain the following consequence.

**Corollary 2.2** *Let $f$ be a bounded Lipschitz function. Then for any $t > 0$ we have the following inequality,*

$$\left\| e^{\mathbf{Q_t} f} \right\|_\infty \leqslant \left\| e^f \right\|_1 \left(\frac{1}{2\pi t}\right)^{\frac{n}{2}}. \tag{18}$$

*In other words,*

$$\mathbf{Q_t} f(x) \leqslant \log \left\| e^f \right\|_1 + \frac{n}{2} \log\left(\frac{1}{2\pi t}\right),$$

*for any $x \in \mathbb{R}^n$.*

**Remark 2.3** The inequality (14) is optimal (see [ABC$^+$00]), and we can see that the inequality (13) is also optimal. When

$$\begin{cases} f(x) &= -ax^2, \text{ with } 0 < a < 1/2 \\ t &= 1 \\ \beta &= \alpha/(1-2a) \end{cases}$$

we can see easily that the inequality (13) is an equality.

## 2.2 Ultracontractivity and vanishing viscosity

The upper bound in inequality (13) of Theorem 2.1 can be proved using the optimal heat kernel bound. Let us explain now this method.

First we define the heat semigroup on $\mathbb{R}^n$. Let $g \in \mathbf{L}^p$ and denote $\mathbf{P_t} g$ the heat semigroup on $\mathbb{R}^n$ starting from $g$ defined by

$$\mathbf{P_t}\, g(x) = \int g(y) \frac{e^{-\frac{\|x-y\|^2}{2t}}}{(2\pi t)^{n/2}} dy, \tag{19}$$

for $x \in \mathbb{R}^n$. We can easily show the following inequality

$$\|\mathbf{P}_t g\|_\infty \leqslant \|g\|_1 \left(\frac{1}{2\pi t}\right)^{\frac{n}{2}}, \tag{20}$$

which is exactly the same bound of the inequality (18). Then after some calculations, like in [Bak94] or [Led00], we can obtain an ultracontractive estimate for the heat semigroup $(\mathbf{P_t})_{t\geqslant 0}$, that for any $q \leqslant p < 0$, $t > 0$, and for every povitive smooth function $g$

$$\|g\|_p \leqslant \|\mathbf{P_t} g\|_q \left(\frac{p-q}{4\pi t}\right)^{\frac{n}{2}\frac{p-q}{pq}} \frac{(1-q)^{\frac{n}{2}\left(1-\frac{1}{q}\right)}}{(1-p)^{\frac{n}{2}\left(1-\frac{1}{p}\right)}} \left(\frac{p}{q}\right)^{\frac{n}{2}\left(1-\frac{1}{p}-\frac{1}{q}\right)}, \tag{21}$$

where $\|h\|_p = \left(\int h^p\right)^{1/p}$ for $p < 0$ and $h \geqslant 0$.

Let $0 < \alpha \leqslant \beta$ and $\varepsilon > 0$ and let apply the inequality (21) for $p = -\varepsilon\alpha$, $q = -\varepsilon\beta$, $g = \exp(-f/\varepsilon)$ and the time $\varepsilon t/2$. We obtain

$$\left\|e^f\right\|_\alpha^{-\varepsilon} \leqslant \left\|\left(\mathbf{P}_{\varepsilon\mathbf{t/2}}\left(e^{-f/\varepsilon}\right)^{-\varepsilon}\right)\right\|_\beta^{-\varepsilon} \left(\frac{\beta-\alpha}{4\pi t}\right)^{\frac{n\varepsilon}{2}\frac{\beta\varepsilon-\alpha\varepsilon}{\alpha\beta\varepsilon^2}} \frac{(1+\beta\varepsilon)^{\frac{n\varepsilon}{2}\left(1+\frac{1}{\beta\varepsilon}\right)}}{(1+\alpha\varepsilon)^{\frac{n\varepsilon}{2}\left(1+\frac{1}{\alpha\varepsilon}\right)}} \left(\frac{\alpha}{\beta}\right)^{\frac{n\varepsilon}{2}\left(1+\frac{1}{\alpha\varepsilon}+\frac{1}{\beta\varepsilon}\right)}.$$

Taking the power $-1/\varepsilon$ and letting $\varepsilon$ tend to zero, we obtain, using the vanishing viscosity (equality (6)), the inequality (13) for any $\beta \geqslant \alpha > 0$, $t > 0$ and smooth function $f$.

Let us now present a third proof of the theorem 2.1, based on the Brunn-Minskowski inequality. This proof are interesting because we use only the definition (11) of $(\mathbf{Q_t})_{t\geqslant 0}$ and the geometry of $\mathbb{R}^n$.

### 2.3 Brunn-Minskowski inequality

We explain now the link between the geometry on $\mathbb{R}^n$ and the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$. Let us recall the theorem of Brunn-Minskowski, and let refer to [DG80] for a review, or [BL00] to see the link with the logarithmic Sobolev inequality.

Let $a, b > 0$, $a+b = 0$, and $u$, $v$, $w$ three non negative functions on $\mathbb{R}^n$. Assume that, for any $x, y \in \mathbb{R}^n$, we have

$$w(ax+by) \geqslant u(x)^a v(y)^b. \tag{22}$$

Then

$$\int w(x)dx \geqslant \left(\int u(x)dx\right)^a \left(\int v(x)dx\right)^b. \tag{23}$$

Let us now prove the Theorem 2.1 using the BRUNN-MINSKOWSKI inequality. Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. Set

$$\begin{cases} u(x) = \exp(\beta \mathbf{Q}_1 f(x)) \\ v(x) = \exp\left(-\dfrac{(\beta-\alpha)\beta}{2\alpha}|x|^2\right) \\ w(x) = \exp\left(\alpha f\left(\dfrac{\beta}{\alpha}x\right)\right), \end{cases}$$

and $a = \alpha/\beta$, $b = (\beta-\alpha)/\beta$. The HOPF-LAX formula enables to obtain easily that for any $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned} u(x)^a v(y)^b &\leqslant \exp\left(\alpha f(x-z) + \frac{\alpha}{2}|z|^2 - \frac{(\beta-\alpha)^2}{2\alpha}|y|^2\right) \qquad (24) \\ &\leqslant \exp\left(\alpha f\left\{\frac{\beta}{\alpha}\left(\frac{\alpha}{\beta}x - \frac{\beta-\alpha}{\beta}y\right)\right\}\right) \\ &= w(ax+by), \end{aligned}$$

where $z = -(\beta-\alpha)y/\alpha$. Inequality (24) implies that (22) is satisfied for the functions $u, v$ and $w$ and the constants $a$ and $b$. Note finally that BRUNN-MINSKOWSKI inequality (23) coincides with (13).

**Remark 2.4** This link between BRUNN-MINSKOWSKI inequality and the inequality (13) is not surprising. We know that Euclidean logarithmic SOBOLEV inequality for the LEBESGUE measure (inequality (14)) is equivalent to the logarithmic SOBOLEV inequality for the Gaussian measure (each can be obtained from the other, see for example [ABC$^+$00]). Besides, from [BL00], the Theorem of BRUNN-MINSKOWSKI implies the logarithmic SOBOLEV inequality for the Gaussian measure, so that the link between the inequality (13) and BRUNN-MINSKOWSKI's Theorem follows.

Let us notice that this proof uses the HOPF-LAX formula, equality (11), whereas the others proofs use the HAMILTON-JACOBI differential equation.

The following section presents some results on more general spaces satisfying an Euclidean-type SOBOLEV inequality.

## 3 Ultracontractive bounds under Euclidean-type Sobolev inequality

In this section, we present our main result connecting Euclidean-type SOBOLEV inequality with majoration of the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$. Let us first defined the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$ on a Riemannian manifold.

Let $M$ be a smooth complete Riemannian manifold of dimension $n$ with Riemannian metric $d$. If $f$ is a smooth function on $M$ (for example Lipschitz), the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$ is defined by the following equation

$$\left\{ \begin{array}{ll} \mathbf{Q_t} f(x) & = \inf_{y\in M}\left\{ f(y) + \frac{1}{2t} d(x,y)^2 \right\}, \quad t>0,\, x\in M, \\ \mathbf{Q}_0 f(x) & = f(x), \quad x\in M. \end{array}\right. \tag{25}$$

Following the argument in the classical Euclidean case, one shows similarly that $v = v(x,t) = \mathbf{Q_t} f(x)$ is a solution of the initial-value HAMILTON-JACOBI problem on the manifold $M$,

$$\left\{ \begin{array}{rl} \frac{\partial v}{\partial t}(x,t) + \frac{1}{2}|\nabla v(x,t)|^2 & = 0 \\ v(x,0) & = f(x), \end{array}\right. \tag{26}$$

where $|\nabla v|$ stands for the Riemannian lengh of the gradient of $v$ for the variable $x$. This semigroup is called the HOPF-LAX solution of HAMILTON-JACOBI equations. More details about HAMILTON-JACOBI equations may be found in [Bar94, Eva98].

**Theorem 3.1** *Let $(M,d)$ be a smooth Riemannian manifold and let $\mu$ be a measure on $M$ absolutely continuous with respect to the standard volume element on $M$.*

*Let $n\geqslant 3$. Suppose that $\mu$ satisfies the following Euclidean-type* SOBOLEV *inequality for a constant $a>0$,*

$$\|f\|^2_{\frac{2n}{n-2}} \leqslant a\|\nabla f\|_2^2, \tag{27}$$

*for any function $f$ with compact support. Then there exists a constant $k>0$ such that the measure $\mu$ satisfies the following inequality*

$$\left\| e^{\mathbf{Q_t} f}\right\|_\beta \leqslant \left\| e^f \right\|_\alpha \left( \frac{k\alpha(\beta-\alpha)}{t\beta}\right)^{\frac{n}{2}\frac{\beta-\alpha}{\beta\alpha}}, \tag{28}$$

*for any smooth function $f$, $t\geqslant 0$, $\alpha>0$ and $\beta\in[\alpha,+\infty[\cup\{+\infty\}$.*

*Conversely, let $k > 0$ and $\alpha > 0$. If the measure $\mu$ satisfies the inequality* (28) *for any smooth function $f$, $\beta \geqslant \alpha$ and $t \geqslant 0$ then there exists $a > 0$ such that $\mu$ satisfies an Euclidean-type* Sobolev *inequality* (27).

Let refer to [Heb99] for results about Euclidean-type Sobolev inequality. Taking $\beta = \infty$ and $\alpha = 1$ in the inequality (28), the following corollary holds.

**Corollary 3.2** *Under conditions of Theorem 3.1, for every function $f$ and $t > 0$, we find*

$$\left\| e^{\mathbf{Q_t} f} \right\|_\infty \leqslant \left\| e^f \right\|_1 \frac{k}{t^{n/2}}. \tag{29}$$

Let us notice that the ultracontractive bound on the inequality (29) is the same as the ultracontractive bound for the heat semigroup, inequality (20). As in the Varopoulos Theorem, we do not know at this time if the ultracontractive bound for the Hamilton-Jacobi solutions, inequality (29), is equivalent to the Euclidean-type Sobolev inequality.

To prove this result, like Theorem 2.1, we use the method developed by Davies and Bakry in [Dav90] and [Bak94], involving two main results. The first one links the Sobolev inequality with entropy-energy inequality. The second ones ensures the equivalence between the control of the Hamilton-Jacobi equations and the entropy-energy inequality. Let us define this latter inequality.

**Definition 3.3** *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a strictly increasing concave function. The measure $\mu$ on the manifold $M$ satisfies an entropy-energy inequality of function $\Phi$ if the following inequality holds for any smooth enough function $f$:*

$$\mathbf{Ent}_\mu(f^2) \leqslant \int f^2 d\mu \; \Phi\left( \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu} \right), \tag{30}$$

*where $\mathbf{Ent}_\mu(f^2) = \int f^2 \log f^2 d\mu - \int f^2 d\mu \log \int f^2 d\mu$.*

This inequality is a generalisation of the logarithmic Sobolev inequality, as (1) arises choosing $\Phi(x) = ax$ and $\mu$ as a probability measure. Further details on this inequality can be found in [Bak94, ABC$^+$00, BCL97].

The next results states the link between Sobolev inequality and entropy-energy inequality.

**Proposition 3.4** *Let $(M, d)$ be a smooth Riemannian manifold and let $\mu$ be a measure on $M$. Let $n \geqslant 3$. We suppose that the measure $\mu$ satisfies the following* Sobolev *inequality, with constants $a$ and $b$,*

$$\|f\|_{2n/(n-2)}^2 \leqslant a\|f\|_2^2 + b\|\nabla f\|_2^2,$$

*for any smooth function $f$ with compact support. Then the measure $\mu$ satisfies the entropy-energy inequality of function*

$$\Phi(x) = \frac{n}{2}\log(ax + b).$$

*Conversely, if the measure $\mu$ satisfies the entropy-energy inequality with $\Phi(x) = (n/2)\log(ax + b)$ then there exists $\lambda \geqslant 1$ such that the measure $\mu$ satisfies the* Sobolev *inequality with constants $\lambda a$ and $\lambda b$.*

Let refer for example to [BCLSC95] or [Bak94] for a proof of this result.

The next theorem gives the equivalence between the entropy-energy inequality and the control of the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$.

**Theorem 3.5** *Let $\mu$ be a non negative measure on the manifold $M$. Suppose that $\mu$ satisfies an entropy-energy inequality of function $\Phi$. Let $c > 0$ and let $q_c$ denote the stricly increasing non-negative function satisfying the following differential equation on $[0, t_0]$ $(t_0 > 0)$,*

$$\frac{2}{q_c'} = \Phi'(cq_c^2). \tag{31}$$

*Then, for any $c > 0$, the following inequality is satisfied for any smooth function $f$,*

$$\left\|e^{\mathbf{Q_t}f}\right\|_{q_c(t)} \leqslant \left\|e^f\right\|_{q_c(0)} e^{A(t)} \text{ with } A(t) = \int_{q_c(0)}^{q_c(t)} \frac{\psi(cy^2)}{y^2}dy, \tag{32}$$

*where $\psi(x) = \Phi(x) - x\Phi'(x)$ and $t \in [0, t_0]$.*

*Conversely, if inequality (32) is satisfied for any $c > 0$, then the measure $\mu$ satisfies the entropy-energy inequality of function $\Phi$.*

***Proof***

◀ Suppose that the measure $\mu$ satisfies the entropy-energy inequality with function $\Phi$. Let $g$ be a bounded function on $M$; then for any $x > 0$, the concavity of the function $\Phi$ implies that

$$\mathbf{Ent}_\mu(g^2) \leqslant \Phi'(x)\int |\nabla g|^2 d\mu + \psi(x)\int g^2 d\mu, \tag{33}$$

where $\psi(x) = \Phi(x) - x\Phi'(x)$.

Let $f$ be a smooth function on $M$ and consider $\mathbf{Q_t} f$ defined by the equation (25). Set $F(t) = \left\| e^{\mathbf{Q_t} f} \right\|_{q(t)}$, where $q$ is an increasing function satisfying $2/q' \in \mathrm{Im}\Phi'$. Taking the derivative in time $t$ of $F$, one gets

$$F'(t) = F(t)^{1-q(t)} \frac{q'(t)}{q^2(t)} \left( \mathbf{Ent}_\mu\left(e^{q(t)\mathbf{Q_t} f}\right) - \frac{1}{2q'(t)} \int |\nabla q(t) \mathbf{Q_t} f|^2 e^{q(t)\mathbf{Q_t} f} d\mu \right).$$

Inequality (33) applied to $g = \exp(q\mathbf{Q_t} f/2)$ and to the function $x(t)$ satisfying $1/q'(t) = \Phi'(x(t))/2$ gives

$$\frac{F'(t)}{F(t)} \leqslant \frac{q'(t)}{q^2(t)} \psi(x(t)).$$

Integrating over $t$ implies

$$\left\| e^{\mathbf{Q_t} f} \right\|_{q(t)} \leqslant \left\| e^{f} \right\|_{q(0)} e^{A(t)}, \tag{34}$$

where

$$A(t) = \int_0^t \frac{q'(s)}{q^2(s)} \psi\left( \Phi'^{-1}\left( \frac{2}{q'(s)} \right) \right) ds. \tag{35}$$

Let now consider $c > 0$. There exists $t_0 > 0$ such that $q_c$ satisfies the differential equation (31) in the space $[0, t_0]$. Then changing the variables in the equation (34) yields equation (32).

Let us prove the converse. Let $x$ be positive and take $c = x$. There exists $t_0 > 0$ such that the function $q_c$ is the solution of the differential equation (31) in the space $[0, t_0]$, which satisfies the condition $q_c(0) = 1$. Then we obtain the equality $q_c'(0) = 2/\Phi'(x)$. Considering $F(t) = \left\| e^{\mathbf{Q_t} f} \right\|_{q_c(t)}$, inequality (32) leads to $F(t) \leqslant F(0) e^{A(t)}$ (forall $t \in [0, t_0]$). After derivation in zero, we find

$$\frac{F'(0)}{F(0)} \leqslant A'(0).$$

And we obtain, after calculation,

$$\mathbf{Ent}_\mu\left(e^f\right) \leqslant \frac{\Phi'(x)}{4} \int |\nabla f|^2 e^f d\mu + \psi(x) \int e^f d\mu. \tag{36}$$

Taking at this step $g = \exp(f/2)$ in (36) and optimising over $x > 0$ yields the entropy-energy inequality of function $\Phi$. ▶

Theorem 3.5 is a generalisation of Theorem 2.1 of [BGL00]. When the measure $\mu$ is a probability measure satisfying the logarithmic SOBOLEV inequality (1), we have $\Phi(x) = (2/\rho)x$ and $\psi = 0$.

Let us notice that we can obtain Theorem 2.1 by Theorem 3.5. Inserting $q(t) = \alpha\beta/((\alpha-\beta)t+\beta)$ and $c = n(\beta-\alpha)/(4\alpha\beta)$ in (32) then implies (13).

Besides, when the entropy-energy inequality holds, Theorem 3.5 gives a control of the norm of the operator $(\exp \mathbf{Q_t})$. This control depends on $\Phi$ and some illustrations will be provided in the following section, which shows the influence of the sign of $\psi$.

Let us now present a proof of Theorem 3.1 using Proposition 3.4 and Theorem 3.5.

***Proof of Theorem 3.1***

◀ Suppose that the measure $\mu$ satisfies the inequality (27) with $a > 0$. Then by Proposition 3.4, $\mu$ satisfies the entropy-energy inequality with the function

$$\Phi(x) = \frac{n}{2}\log(ax).$$

Let $c = a(\beta-1)/2$. The function $q_c(t) = \beta/((1-\beta)t+\beta)$ satisfies the differential equation (31) in the space $[0,\infty[$. Applying Theorem 3.5 with the function $\Phi$ and the constant $c$ leads after some easy calculus to the following inequality

$$\left\| e^{\mathbf{Q_t} f} \right\|_\beta \leqslant \left\| e^f \right\|_1 \left( \frac{k(\beta-1)}{t} \right)^{\frac{n}{2}\frac{\beta-1}{\beta}} \left( \frac{1}{\beta} \right)^{\frac{n}{2}\frac{\beta+1}{\beta}},$$

where $k = nae/4$ and for any smooth function $f$. As $\beta \geqslant 1$, we know that

$$\left( \frac{1}{\beta} \right)^{\frac{\beta+1}{\beta-1}} \leqslant \frac{1}{\beta}.$$

At the light of the previous inequality we find the inequality (28) for the constant $k = nae/4$, for every $\beta \geqslant 1$ and $\alpha = 1$. Using the property for the semigroup $(\mathbf{Q_t})_{t\geqslant 0}$ that

$$\mathbf{Q_t}(\lambda f) = \lambda \mathbf{Q}_{\lambda t} f, \tag{37}$$

for every $\lambda > 0$ and $f$, inequality (28) is obtained for every $\beta$ and $\alpha$ such that $\beta \geqslant \alpha > 0$, as well as inequality (29), as a particular case of (28).

Let us now prove the converse. Let $k > 0$ and $\alpha > 0$. Suppose that for every $\beta \in ]\alpha, +\infty[ \cup \{+\infty\}$ (28) holds for every $t \geqslant 0$. By (37) one can assume that $\alpha = 1$.

Let $f$ be a smooth function and take $F(t) = \left\|e^{\mathbf{Q_t} f}\right\|_\beta$ where $\beta$ is a function of $t$ such that $\beta(0) = 1$, $\beta'(0) > 0$ and $\beta \geqslant 1$. Inequality (28) implies that

$$\frac{F'(0)}{F(0)} \leqslant g'(0), \tag{38}$$

where

$$g(t) = -\frac{n}{2}\frac{\beta - 1}{\beta} \log\left(\frac{t\beta}{k(\beta - 1)}\right).$$

As $g'0) = (n/2)\beta'(0) \log(\beta'(0)k)$, taking $x > 0$ and choosing the function $\beta$ such that $\beta'(0) = 4x/n$, ($\beta(t) = 1 + 4xt/n$ for example) transform inequality (38) as

$$\mathbf{Ent}_\mu(f^2) \leqslant \frac{n}{8x} \int |\nabla f|^2 e^f d\mu + \frac{n}{2} \log\left(\frac{k4x}{n}\right) \int e^f d\mu.$$

Optimising over $x$ implies the entropy-energy inequality for $\Phi(x) = (n/2) \log ax$ where $a = k4e/n$. The proof is then achieved using the converse of Proposition 3.4. ▶

The first and the most important example is the Lebesgue measure on $\mathbb{R}^n$. This example is presented in Theorem 2.1 of Section 2.

And like in the $\mathbb{R}^n$ case, section 2.2, the vanishing viscosity for an ultracontractive semigroup can be used to prove Theorem 3.1.

We go to see in the next section that properties remain almost identical in the case of a general Sobolev inequality.

# 4 Ultraconstractive bounds under other Sobolev inequality

## 4.1 Main results in this case

The aim of the following theorem is to present the case where the measure $\mu$, on the manifold $M$, satisfies a Sobolev inequality with a local term, where $b > 0$ in inequality (8).

**Theorem 4.1** *Let $M$ be a complete Riemannian manifold of Riemannian metric $d$. Let $\mu$ be a measure on $M$ absolutely continuous with respect to the standard volume element on $M$. Let $n \geqslant 3$. We suppose that the measure $\mu$ satisfies the following* Sobolev *inequality*

$$\|f\|^2_{\frac{2n}{n-2}} \leqslant a\|\nabla f\|_2^2 + b\|f\|_2^2, \tag{39}$$

*where a and b are two constants and* $\|\cdot\|_\alpha$ *is the* $\mathbf{L}^\alpha$*-norm for the measure* $\mu$*.*

*Then there exist a constant* $k > 0$ *such that the measure* $\mu$ *satisfies the following inequality ,*

$$\left\| e^{\mathbf{Q_t} f} \right\|_\infty \leqslant \left\| e^f \right\|_1 \frac{k}{t^{\frac{n}{2}}}. \tag{40}$$

*for any* $t \in ]0,1]$ *and every smooth function* $f$*.*

This theorem is a consequence of the following proposition.

**Proposition 4.2** *Let* $\mu$ *be a measure on* $M$*. Assume that the measure* $\mu$ *satisfies the entropy-energy inequality with function* $\Phi(x) = (n/2)\log(ax+b)$*,* $(a, b > 0)$*. Let* $m \in \mathbb{N}$ *and* $K \in [m\pi, m\pi + \pi/2]$*.*

*Then, for any function* $f$*,* $(\mathbf{Q_t})_{t\geqslant 0}$ *satisfies the following inequality for every* $t, u > 0$ *such that* $tu + K \in [m\pi, m\pi + \pi/2]$*,*

$$\left\| e^{\mathbf{Q_t} f} \right\|_{\frac{4b}{una}\tan(tu+K)} \leqslant \left\| e^f \right\|_{\frac{4b}{una}\tan K} \exp\left(A(t)\right), \tag{41}$$

*where*

$$A(t) = \frac{n^2 au}{8b}\left( \frac{\log\left(\frac{\cos^2(tu+K)}{b}\right)}{\tan(tu+K)} - \frac{\log\left(\frac{\cos^2 K}{b}\right)}{\tan K} \right) + \frac{n^2 u^2 a}{8b}. \tag{42}$$

*Conversely, if there exists* $m \in \mathbb{N}$ *such that inequalities* (41) *and* (42) *hold for any* $K \in [m\pi, m\pi + \pi/2]$ *and* $t, u$ *such that* $tu + K \in [m\pi, m\pi + \pi/2]$*, then the measure* $\mu$ *satisfies an entropy-energy inequality with function* $\varphi$*.*

This proposition is a simple consequence of Theorem 3.5, when we have $\Phi(x) = n/2\log(ax+b)$, which enable us to prove Theorem 4.1.

***Proof of Theorem 4.1***

◀ Let $\mu$ be a measure satisfying the Sobolev inequality (39). Proposition 3.4 ensures that the measure $\mu$ satisfies the entropy-energy inequality with function $\Phi(x) = n/2\log(ax+b)$. Let us now apply the previous proposition for $t = 1$ and $K = \pi/2 - u$. The following inequality then arises

$$\left\| e^{\mathbf{Q_1} f} \right\|_\infty \leqslant \left\| e^f \right\|_{4b/(una\tan u)} e^{A(1)},$$

where

$$A(1) = -\frac{n^2 au}{8b}\tan u\left(\log\frac{\sin^2 u}{b}\right) + \frac{n^2 u^2 a}{8b}.$$

Due to the property (16), the following inequality holds, for any $t > 0$ and every smooth function $f$,

$$\left\| e^{\mathbf{Q_t} f} \right\|_\infty \leqslant \left\| e^f \right\|_1 e^{\varphi(t)}, \tag{43}$$

where

$$\varphi(t) = \frac{1}{t}\left\{ -\frac{n}{2}\log\left(\frac{\sin^2(\psi(t))}{b}\right) + \frac{n^2\psi(t)^2 a}{8b}\right\},$$

and the function $\psi$ is defined by the following formula, for every $t \geqslant 0$

$$\psi(t)\tan\psi(t) = \frac{4b}{na}t.$$

Using the definition of $\psi$ we prove that there exist $C > 0$, such that for every $t \in ]0, 1]$ we have

$$\frac{1}{C}\sqrt{t} \geqslant \psi(t) \leqslant C\sqrt{t}.$$

This inequality implies that there exists $C' > 0$ such that for every $t \in ]0, 1]$,

$$\varphi(t) \leqslant -\frac{n}{2}\log t + C'. \tag{44}$$

Inequalities (43) and (44) lead to the theorem 4.1. ▶

**Remark 4.3** Like in the previous section, we do not know if the ultracontractive bound given by the inequality (40) is equivalent to the Sobolev inequality (39).

## 4.2 Particular case

Let us now describe the special case when $b = 1$ in the Sobolev inequality. Suppose that the Riemannian manifold $M$ is compact and let $\mu$ be a probability measure absolutely continuous with respect to te standard volume element on $M$. If the measure $\mu$ satisfies a Sobolev inequality, then we know that we can choose the constant $b = 1$, (see for example [Bak94, ABC$^+$00]).

An example is the unit sphere $\mathbb{S}^n$ of dimension $n$ in $\mathbb{R}^{n+1}$. Let $n > 2$. The probability measure $\mu$ of the volume, satisfies the following optimal Sobolev inequality

$$\|f\|_{2n/(n-2)}^2 \leqslant \frac{4}{n(n-2)}\|\nabla f\|_2^2 + \|f\|_2^2.$$

More generally, let consider a Riemannian manifold $M$ of dimension $n > 2$. If the Ricci curvature is bounded below by a constant $\rho > 0$, then the following

SOBOLEV inequality holds for the probability measure of the volume, see [Ili83, Bak94, ABC+00]

$$\|f\|_{2n/(n-2)}^2 \leqslant \frac{4(n-1)}{n(n-2)\rho}\|\nabla f\|_2^2 + \|f\|_2^2.$$

In this particular case, the following proposition can be stated.

**Proposition 4.4** *Suppose that the measure $\mu$ satisfies the following* SOBOLEV *inequality*

$$\|f\|_{2n/(n-2)}^2 \leqslant a\|\nabla f\|_2^2 + \|f\|_2^2.$$

*Then we obtain the following estimate*

$$\mathbf{Q_t} f(x) \leqslant \int f d\mu + \frac{\pi^2 n^2 a}{16t}, \tag{45}$$

*for any $x \in M$ and $t > 0$.*

***Proof***

◀ Proposition 4.2 is applied with $b = 1$. Using the property (16), we just have to prove the inequality (45) for $t = 1$. Let take $t = 1$, $K = 0$ and $u = \pi/2$ in inequality (41). Then equation $\left\|e^f\right\|_0 = \exp\left(\int f d\mu\right)$, ledas to inequality (45). ▶

## 4.3 Application to transportation inequality

Let $(M, d)$ be a Riemannian manifold. Let us recall the definition of the distance $T_2$. Let $\mu$ and $\nu$ two probability measures on $M$. We denote

$$T_2(\mu,\nu) = \inf\left\{\int \frac{d(x,y)^2}{2} d\pi(x,y)\right\}, \tag{46}$$

where the infimum is taken over the set of measures $\pi$ on $M \times M$ such that $\pi$ has two margins $\mu$ and $\nu$. Let recall that by the OTTO-VILLANI's theorem, (see [OV00] and [BGL00]), a logarithmic SOBOLEV inequality implies a linear transportation inequality. In the same way, we obtain the following result about SOBOLEV inequality.

**Theorem 4.5** *Let $\mu$ be a probability measure on $M$, which is absolutely continuous with respect to the standard volume element on $M$. Let $n \geqslant 3$. Suppose that $\mu$ satisfies the following* SOBOLEV *inequality*

$$\|f\|_{\frac{2n}{n-2}}^2 \leqslant a\|\nabla f\|_2^2 + \|f\|_2^2.$$

*Let $V$ be the function defined for $x > 0$, by*

$$V(x) = \frac{n^2 a}{8}\left(\arctan\sqrt{e^{\frac{2x}{n}} - 1}\right)^2. \tag{47}$$

*Then the measure $\mu$ satisfies the following transportation inequality*

$$T_2(gd\mu, d\mu) \leqslant V(\mathbf{Ent}_\mu(g)), \tag{48}$$

*for any smooth function $g$, density of probability with respect to the measure $\mu$.*

Let us notice that $V$ looks like the function arctan, is increasing and bounded by $n^2 a\pi^2/32$.

***Proof***

◀ This result is based on Proposition 4.2. Let consider $m = 0$, $K = 0$ and $t = 1$. Then the following inequality holds, for any $0 < u < \pi$,

$$\left\| e^{\mathbf{Q}f} \right\|_{\frac{4}{una}\tan(u)} \leqslant \exp\left(\int f d\mu\right)\exp(A),$$

where $\mathbf{Q} = \mathbf{Q}_1$ and

$$A = \frac{n^2 au}{4}\frac{\log|\cos(u/2)|}{\tan(u/2)} + \frac{n^2 au^2}{8}.$$

Let $x > 0$ and $u = \arctan\sqrt{\exp(2x/n) - 1}$. The following equation follows straightforwad

$$\frac{4}{una}\tan(u) = \frac{1}{V'(x)}.$$

Defining $\Lambda(x) = V(x) - xV'(x)$, and using

$$\log\cos\arctan\sqrt{\exp\left(e^{\frac{2x}{n}} - 1\right)} = -\frac{x}{n},$$

we obtain after some calculus the following inequality

$$\int \exp\left(\frac{\mathbf{Q}f}{V'(x)}\right)d\mu \leqslant \exp\left(\frac{\int f d\mu}{V'(x)} + \frac{\Lambda(x)}{V'(x)}\right),$$

for any $x > 0$. Let $g$ be a density of probability with respect to the measure $\mu$. As we have

$$\int \exp\left(\frac{\mathbf{Q}f}{V'(x)} - \frac{\int f d\mu}{V'(x)} - \frac{\Lambda(x)}{V'(x)}\right)d\mu \leqslant 1,$$

we can write

$$\int g\mathbf{Q}f d\mu - \int f d\mu \leqslant V'(x)\mathbf{Ent}_\mu(g) + \Lambda(x),$$

for any Lipschitz function $f$. Optimising over all Lipschitz functions $f$ and over $x > 0$, we obtain the transportation inequality (48) as a consequence of the theorem of KANTOROVICH-RUBINSTEIN (see [ABC+00]). ▶

In the classical case, a transportation inequality gives a concentration inequality, see for example [Mar96a, Mar96b] or the chapter 8 of [ABC+00]. In this case we find the following estimate of the diameter.

**Corollary 4.6** *Suppose that the probability measure $\mu$ satisfies the following* SOBOLEV *inequality, for $n \geqslant 3$,*

$$\|f\|^2_{2n(n-2)} \leqslant a\|\nabla f\|_2^2 + \|f\|_2^2.$$

*Let define $D = \sup\{d(x,y)\,/\,x, y \in M\}$, the diameter of the manifold $M$. Then one has*

$$D \leqslant \frac{n\pi}{2}\sqrt{a}. \tag{49}$$

***Proof***

◀ The transport inequality (48) leads to

$$\sqrt{\frac{D^2}{2}} \leqslant \sup_{r>0}\left(\sqrt{V(\varphi_A)} + \sqrt{V(\varphi_{A_r^c})}\right),$$

where $A \subset M$, $A_r^c$ is the complementary of the $r$-neighbourhood of $A$, $\varphi_A = \mathbb{1}_A/\mu(A)$ and $V$ is the function defined by the equation (47). We obtain also

$$D \leqslant \sqrt{8\|V\|_\infty}, \tag{50}$$

so that (49) holds. ▶

The estimates specified by (49) are not optimal. In the case of the unit sphere, one has $D \leqslant \pi\sqrt{n/(n-2)}$ also $\pi$. In [Bak94], BAKRY finds, using also entropy-energy inequality, $D \leqslant \pi\sqrt{n/(n-1)}$ which is more accurate and BAKRY-LEDOUX prove in [BL96] $D \leqslant \pi$ under SOBOLEV.

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## References


[ABC$^+$00] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses **10**, Société Mathématique de France, Paris, 2000.

[Bak94] D. Bakry, ≪ L'hypercontractivité et son utilisation en théorie des semigroupes ≫, in *Lectures on probability theory. École d'été de probabilités de St-Flour 1992*, Lecture Notes in Math. **1581**, Springer, Berlin, 1994, pp. 1–114.

[Bar94] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer-Verlag, Paris, 1994.

[BCL97] D. Bakry, D. Concordet, M. Ledoux, ≪ Optimal heat kernel bounds under logarithmic Sobolev inequalities ≫, ESAIM Probab. Statist. **1** (1997), 391–407 (electronic).

[BCLSC95] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, ≪ Sobolev inequalities in disguise ≫, Indiana Univ. Math. J. **44** (1995), no. 4, 1033–1074.

[BGL00] S. Bobkov, I. Gentil, M. Ledoux, ≪ Hypercontractivity of Hamilton-Jacobi equations ≫, 2000, à paraître in J. Math. Pu. Appli.

[BL96] D. Bakry, M. Ledoux, ≪ Sobolev inequalities and Myers's diameter theorem for an abstract Markov generator ≫, Duke Math. J. **85** (1996), no. 1, 253–270.

[BL00] S. G. Bobkov, M. Ledoux, ≪ From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities ≫, Geom. Funct. Anal. **10** (2000), no. 5, 1028–1052.

[Car91] E. A. Carlen, ≪ Superadditivity of Fisher's information and logarithmic Sobolev inequalities ≫, J. Funct. Anal. **101** (1991), no. 1, 194–211.

[Dav90] E. B. Davies, *Heat kernels and spectral theory*, Cambridge University Press, Cambridge, 1990.

[DG80] S. Das Gupta, « Brunn-Minkowski inequality and its aftermath », J. Multivariate Anal. **10** (1980), no. 3, 296–318.

[Eva98] L. C. Evans, *Partial differential equations*, American Mathematical Society, Providence, RI, 1998.

[Gro75] L. Gross, « Logarithmic Sobolev inequalities », Amer. J. Math. **97** (1975), no. 4, 1061–1083.

[Heb99] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, New York University Courant Institute of Mathematical Sciences, New York, 1999.

[Ili83] S. Ilias, « Constantes explicites pour les inégalités de Sobolev sur les variétés riemanniennes compactes », Ann. Inst. Fourier (Grenoble) **33** (1983), no. 2, 151–165.

[Led00] M. Ledoux, « The geometry of Markov diffusion generators », Ann. Fac. Sci. Toulouse Math. (6) **9** (2000), no. 2, 305–366, probability theory.

[Mar96a] K. Marton, « Bounding $\overline{d}$-distance by informational divergence: a method to prove measure concentration », Ann. Probab. **24** (1996), no. 2, 857–866.

[Mar96b] K. Marton, « A measure concentration inequality for contracting Markov chains », Geom. Funct. Anal. **6** (1996), no. 3, 556–571.

[OV00] F. Otto, C. Villani, « Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality », J. Funct. Anal. **173** (2000), no. 2, 361–400.

[Var84] N. T. Varopoulos, « Une généralisation du théorème de Hardy-Littlewood-Sobolev pour les espaces de Dirichlet », C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 14, 651–654.

[Var85] N. T. Varopoulos, « Hardy-Littlewood theory for semigroups », J. Funct. Anal. **63** (1985), no. 2, 240–260.

[Var91] N. T. Varopoulos, « Analysis and geometry on groups », in *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, Math. Soc. Japan, Tokyo, 1991, pp. 951–957.