UNIFORM TIME OF EXISTENCE FOR THE ALPHA EULER EQUATIONS

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Abstract. We consider the α-Euler equations on a bounded domain with Navier slip boundary conditions. Working with conormal Sobolev spaces, we show that if α is sufficiently small then the solution exists on a time interval uniform in α. In view of our previous result [4], this implies the convergence of the solutions of the α-Euler equations towards the solution of the incompressible Euler equation as $\alpha \to 0$. As a byproduct we obtain the well-posedness of the incompressible Euler equation in conormal spaces where only 4 derivatives need to be controlled.

1. Introduction

The aim of this paper is to consider the limit $\alpha \to 0$ for the following α-Euler equations:

$$\partial_t(u-\alpha\Delta u)+u\cdot\nabla(u-\alpha\Delta u)+\sum_j(u-\alpha\Delta u)_j\nabla u_j=-\nabla p,\qquad \operatorname{div} u=0. \tag{1}$$

These equations are the vanishing viscosity case of the second grade fluids introduced in [6]. Later, they were rediscovered via a geometric principle, and also via an averaging procedure in the standard incompressible Euler equations, see [11]. As a result, their significance and interest for the mathematicians greatly increased.

If we set $\alpha = 0$ in (1) we obtain the incompressible Euler equations:

$$\partial_t u+u\cdot\nabla u=-\nabla p,\qquad \operatorname{div} u=0.$$

A natural question that arises is whether the the solutions of the α-Euler equations converge as $\alpha \to 0$ towards a solution of the Euler equations.

In absence of boundaries this is a trivial matter because adding α to the PDE brings more regularity to the equation. Because there are no boundary terms when making the integrations by parts, one can make the same energy estimates as for the Euler equations and simply ignore the terms with α (see [9], and also [4] for a simpler proof). Once H^3 uniform bounds are obtained, one can easily pass to the limit with classical compactness methods.

The situation is completely different when boundaries are present. The main reason is that the α-Euler equations have an additional boundary condition with respect to the Euler equation. Indeed, for the Euler equation one needs to assume that the velocity is tangent to the boundary:

$$u\cdot n=0\quad \text{on } \partial\Omega$$

where Ω is the fluid domain. But for the α-Euler equations, it is necessary to assume either Dirichlet boundary conditions (the most physically relevant conditions) or the following Navier slip boundary conditions:

$$u\cdot n=0,\quad [D(u)n]\big|_{tan}=0\quad \text{on } \partial\Omega \tag{2}$$

where $D(u)$ is the deformation tensor defined by $D(u)=\frac{1}{2}\big((\nabla u)+(\nabla u)^t\big)$ and the subscript $_{tan}$ denotes the tangential part.

In the case of the Dirichlet boundary conditions, it was proved in dimension two that the expected convergence of solutions holds true in L^2, see [10]. Their idea was to adapt the Kato criteria for the vanishing viscosity limit and observe that the condition imposed by the Kato criteria is satisfied in the case of the limit $\alpha \to 0$. But in dimension three, this method does not work and the question is still open.

In the case of the Navier boundary conditions, a direct estimate on the difference between the solutions was proved in [4] implying a quite general result stating convergence in L^2 for the $\alpha \to 0$ limit. Unfortunately, that result has an important hypothesis: weak H^1 solutions for the α-Euler equations must exist on a time interval independent of α. This hypothesis is clearly verified in dimension two and also for axisymmetric solutions in dimension three because in both these cases the solutions are global. But in the general case of the dimension three, it is not at all clear why solutions should exist on a time interval independent of α. Indeed, the only type of solutions known to exist in this case are the strong solutions in H^3. These are obtained by making some H^3 a priori estimates on the velocity. Unfortunately, it is impossible to obtain H^3 bounds on the velocity uniformly in α (which would be required to obtain a time of existence uniform in α). Indeed, if such bounds would exist then by the result from [4] we would get that the solutions of the α-Euler equations converge weakly in H^3 to the solution of the Euler equation. But weak convergence in H^3 preserve the Navier boundary conditions so we would find that the solution of the Euler equation verifies the Navier boundary condition. That would be a contradiction because the Euler equation does not preserve in general the Navier boundary conditions.

So, in order to obtain a uniform time of existence some new solutions must be invented. The ideal result would be to prove existence of weak H^1 solutions. Unfortunately, even though H^1 energy estimates are available, we were not able to prove the existence of weak solutions from these estimates. We propose instead some sort of "strong solutions" whose regularity involve only one normal derivative and not two or more. As explained above, due to the difference in the boundary conditions it is impossible to control two normal derivatives of the velocity. We will use the so-called conormal spaces where the regularity is measured only via tangential derivatives. The conormal spaces are a well-known tool in the study of symmetric hyperbolic systems, see for instance [8, 14], and they were also recently used in the vanishing viscosity limit, see [12].

In order to state our results, we first explain our function spaces (the precise definition will be given in Section 3 below). We denote by H^m_{co} the space of square integrable functions such that all tangential derivatives of order $\leqslant m$ are also square integrable. The space X^m is the same as H^m_{co} except that we allow one of the derivatives to be non-tangential. The $W^{m,\infty}_{co}$ is the space of bounded functions such that all tangential derivatives of order $\leqslant m$ are also bounded. Let us also define $\omega^\alpha = \operatorname{curl} u - \alpha\Delta \operatorname{curl} u$.

Our main result is the following theorem. We will assume in the sequel that Ω is a smooth and bounded open set of $\mathbb{R}^3$.

Theorem 1 (uniform time of existence)**.** *Let u_0 be divergence free and verifying the Navier boundary conditions (2). Assume moreover that $u_0 \in L^2$ and $\omega_0^\alpha \in H^{m-1}_{co} \cap W^{1,\infty}_{co}$ where $m \geqslant 5$. There exists $\alpha_0 > 0$ sufficiently small and a time $T > 0$ independent of α such that for all $0 \leqslant \alpha \leqslant \alpha_0$ there exists a solution u of (1) and (2) bounded in $L^\infty(0,T; X^m \cap W^{1,\infty})$ independently of α. Moreover, the time existence T depends only on $\|u_0\|_{L^2}$, $\|\omega_0^\alpha\|_{W^{1,\infty}_{co}}$ and $\|\omega_0^\alpha\|_{H^{m-1}_{co}}$.*

Combining this theorem with [4, Theorem 5] immediately yields the following convergence result:

Theorem 2 (convergence)**.** *Let u_0 be divergence free and verifying the Navier boundary conditions (2). Assume that $u_0 \in H^3 \cap W^{4,\infty}_{co}$ and $\operatorname{curl}\Delta u_0 \in H^4_{co}$. Let $\overline{u}$ be the solution of the incompressible Euler equations with initial data u_0. There exists some time T independent of α and a solution u^α of (1) and (2) on $[0,T]$ with initial data u_0 such that*

$$\lim_{\alpha\to 0} \|u^\alpha - \overline{u}\|_{L^\infty(0,T;L^2)} = 0.$$

Finally, as a particular case of Theorem 1 (case $\alpha = 0$) we obtain a new existence result for the incompressible Euler equations.

Theorem 3. *Let u_0 be divergence free, tangent to the boundary and such that $u_0 \in X^4$ and $\operatorname{curl} u_0 \in W^{1,\infty}_{co}$. Then there exists a unique local in time solution u of the incompressible Euler equations with initial data u_0 such that $u \in L^\infty(0,T; X^4 \cap W^{1,\infty})$.*

An existence result for the Euler equations in conormal Sobolev spaces was also obtained in [12]. But our theorem imposes less regularity on the initial data. Indeed, in [12] the authors assume that $u_0 \in X^7$ while we only need $u_0 \in X^4$. Let us also observe that, compared to the classical existence result of H^3 solutions of the Euler equation, only one additional derivative is required in Theorem 3.

The structure of the paper is the following. In the next section we introduce some notation and prove an identity related to the Navier boundary conditions. In Section 3 we introduce the conormal spaces and show some inequalities in conormal spaces. In Section 4 we show some elliptic estimates in conormal spaces. We prove next in Section 5 the *a priori* estimates on the solutions of (1). In Section 6 we construct a sequence of approximate solutions and we use the *a priori* estimates from Section 5 to obtain Theorems 1 and 3. We end this paper with a final remark in Section 7.

2. Some notations and preliminary results

Let

$$\omega = \operatorname{curl} u \qquad \text{and} \qquad \omega^\alpha = \omega - \alpha\Delta\omega.$$

Applying the curl to relation (1) implies the following equation for the vorticity ω^α:

$$\partial_t \omega^\alpha + u\cdot\nabla\omega^\alpha - \omega^\alpha\cdot\nabla u = 0 \tag{3}$$

We denote by n a smooth vector field defined on $\overline{\Omega}$ such that its restriction to the boundary is the unitary exterior normal to the boundary. We assume moreover that $\|n\| = 1$ in a small neighborhood of the boundary $\Omega_\delta = \{x \in \overline{\Omega}\ ;\ d(x,\partial\Omega) \leqslant \delta\}$. We introduce a smooth function $d : \overline{\Omega} \to \mathbb{R}_+$ such that d never vanishes in Ω and such that $d(x) = d(x,\partial\Omega)$ for all $x \in \Omega_\delta$. In other words, d is a smooth version of $d(x,\partial\Omega)$.

For a vector field w we define

$$w_{tan} = w\times n \quad \text{and} \quad w_{nor} = w\cdot n.$$

We observe that for any vector fields w and $\widetilde{w}$ we have the following relation:

$$w\cdot\widetilde{w} = w_{tan}\cdot\widetilde{w}_{tan} + w_{nor}\widetilde{w}_{nor} \quad \text{on } \Omega_\delta.$$

More generally, the above relation holds true everywhere if one multiplies the LHS by $\|n\|^2$.

We now show (or recall) some identities related to the Navier boundary conditions.

Lemma 4. *Suppose that u is divergence free and verifies the Navier boundary conditions* (2). *Then*

$$\omega\times n = -2n\times\sum_i u_i(n\times\nabla)n_i \equiv F(u) \quad \text{on } \partial\Omega \tag{4}$$

and

$$n\cdot\partial_n\omega = (n\times\nabla)\cdot F(u) - (n\times\nabla)u \operatorname{div} n \equiv G(u,(n\times\nabla)u) \quad \text{on } \partial\Omega$$

Proof. Relation (4) was proved in [5, Eqn. (14)]. Next, we use that ω is divergence free and write

$$\begin{aligned}
(\partial_n\omega)\cdot n &= \sum_{i,j} n_i n_j \partial_i\omega_j \\
&= \sum_{i,j} n_i(n_j\partial_i - n_i\partial_j)\omega_j \\
&= \sum_{i,j}(n_j\partial_i - n_i\partial_j)(n_i\omega_j) - \sum_{i,j}\omega_j(n_j\partial_i - n_i\partial_j)n_i \\
&= \frac{1}{2}\sum_{i,j}(n_j\partial_i - n_i\partial_j)(n_i\omega_j - n_j\omega_i) - \omega\cdot n \operatorname{div} n + \sum_{i,j}\omega_j n_i\partial_j n_i \\
&= (n\times\nabla)\cdot(\omega\times n) - \omega\cdot n\operatorname{div} n + \frac{1}{2}\omega\cdot\nabla(\|n\|^2)
\end{aligned}$$

Using (4) and the identity $\omega\cdot n = (n\times\nabla)u$ and recalling that $\|n\|^2 = 1$ in the neighborhood of the boundary completes the proof of the lemma. □

3. Conormal Sobolev spaces

The conormal Sobolev spaces are defined by using a family of generator tangent vector fields. Here, in order to simplify the presentation we will use a particular family of generator tangent vector fields. We define it in the following way. Let $U_0 = \{x \in \overline{\Omega}\ ;\ d(x,\partial\Omega) < \delta\}$ and $U_1 = \{x \in \overline{\Omega}\ ;\ d(x,\partial\Omega) > \delta/2\}$ and $\varphi_0, \varphi_1 \in C_0^\infty(\overline{\Omega})$ be a partition of unity subordinated to the open cover of $\overline{\Omega}$ given by $\overline{\Omega} = U_0 \cup U_1$. We have that φ_0 is compactly supported in U_0 and is equal to 1 in $\Omega_{\delta/2}$. The function φ_1 is compactly supported in U_1 and is equal to 1 in Ω_δ^c. Since $\|n\| = 1$ on U_0, the set

$$
\begin{aligned}
\mathcal{Z} &= \left\{\varphi_0 \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix}, \varphi_0 \begin{pmatrix} n_3 \\ 0 \\ -n_1 \end{pmatrix}, \varphi_0 \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix}, \varphi_0 n d(x,\partial\Omega), \varphi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \varphi_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \varphi_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \\
&\equiv \{Z_1, \dots, Z_7\}
\end{aligned}
$$

is clearly a family of generator tangent vector fields.

If $\beta \in \mathbb{N}^7$ is a multi-index, we introduce the notation $\partial_Z^\beta = \partial_{Z_1}^{\beta_1} \dots \partial_{Z_7}^{\beta_7}$. For $m \in \mathbb{N}$, we introduce the so-called conormal Sobolev space H^m_{co} which consists of all square-integrable functions f such that $\partial_Z^\beta f \in L^2(\Omega)$ for all $|\beta| \leqslant m$. The norm on H^m_{co} is given by

$$\|f\|^2_{H^m_{co}} = \sum_{|\beta| \leqslant m} \|\partial_Z^\beta f\|^2_{L^2}.$$

We define in a similar manner $W^{m,\infty}_{co}$ by using the L^∞ norm instead of the L^2 norm. Finally, let X^m be defined by

$$X^m = \{f\ ;\ f \in H^m_{co} \text{ and } \nabla f \in H^{m-1}_{co}\}.$$

with norm

$$\|f\|_{X^m} = \|f\|_{H^m_{co}} + \|\nabla f\|_{H^{m-1}_{co}}.$$

It can be checked that the following identity holds true

$$n \times (n \times u) = (n_3u_2 - n_2u_3) \begin{pmatrix} 0 \\ -n_3 \\ n_2 \end{pmatrix} + (n_1u_3 - n_3u_1) \begin{pmatrix} n_3 \\ 0 \\ -n_1 \end{pmatrix} + (n_2u_1 - n_1u_2) \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix}$$

for any vector field u. So, in view of our definition of $\mathcal{Z}$, we have that

$$\varphi_0 n \times (n \times u) = (n_3u_2 - n_2u_3)Z_1 + (n_1u_3 - n_3u_1)Z_2 + (n_2u_1 - n_1u_2)Z_3.$$

Next, because of the identity $\|n\|^2 u = -n \times (n \times u) + n(n \cdot u)$ and since on the support of φ_0 we have that $\|n\| = 1$, we can decompose

$$
\begin{aligned}
\varphi_0 u &= -\varphi_0 n \times (n \times u) + \varphi_0 n(n \cdot u) \\
&= (n_2u_3 - n_3u_2)Z_1 + (n_3u_1 - n_1u_3)Z_2 + (n_1u_2 - n_2u_1)Z_3 + \frac{u \cdot n}{d} Z_4.
\end{aligned}
$$

We also trivially have that $\varphi_1 u = u_1Z_5 + u_2Z_6 + u_3Z_7$ and since $\varphi_0 + \varphi_1 = 1$ we finally deduce that the following decomposition holds true for any vector field u:

$$
\begin{aligned}
u &= \varphi_0 u + \varphi_1 u \\
&= (n_2u_3 - n_3u_2)Z_1 + (n_3u_1 - n_1u_3)Z_2 + (n_1u_2 - n_2u_1)Z_3 + \frac{u \cdot n}{d} Z_4 + u_1Z_5 + u_2Z_6 + u_3Z_7 \\
&\equiv \sum_{i=1}^7 \widetilde{u}_i Z_i.
\end{aligned}
\tag{5}
$$

A very important property of this "canonical decomposition" associated to the set $\mathcal{Z}$ of generator vector fields is listed in the following lemma.

Lemma 5. *Let u be a divergence free vector field tangent to the boundary. For any $m \in \mathbb{N}$ there exists a constant $C = C(m,\Omega)$ such that $\|\widetilde{u}_i\|_{H^m_{co}} \leqslant C\|u\|_{H^{m+1}_{co}}$ and $\|\widetilde{u}_i\|_{W^{m,\infty}_{co}} \leqslant C\|u\|_{W^{m+1,\infty}_{co}}$ for every $i \in \{1, \dots, 7\}$.*

Proof. From the explicit formulas for the $\widetilde{u}_i$ the assertion is obvious except for $\widetilde{u}_4$. Because u is tangent to the boundary, we can apply Lemma 6 below to $u \cdot n$ to deduce that

$$\|\widetilde{u}_4\|_{H^m_{co}} = \Big\|\frac{u\cdot n}{d}\Big\|_{H^m_{co}} \leqslant C(\|u\cdot n\|_{H^m_{co}} + \|n\cdot\nabla(u\cdot n)\|_{H^m_{co}}).$$

We have that

$$\begin{aligned} n\cdot\nabla(u\cdot n) &= \sum_{i,j} n_i\partial_i(n_j u_j) \\ &= \sum_{i,j} n_i\partial_i n_j u_j + \sum_{i,j} n_i n_j \partial_i u_j \\ &= n\cdot\nabla n\cdot u + \sum_{i,j} n_i(n_j\partial_i - n_i\partial_j)u_j + \|n\|^2 \operatorname{div} u \\ &= n\cdot\nabla n\cdot u + \sum_{i,j} n_i(n_j\partial_i - n_i\partial_j)u_j. \end{aligned} \tag{6}$$

Because $n_j\partial_i - n_i\partial_j$ are tangential derivatives, we immediately deduce that

$$\|n\cdot\nabla(u\cdot n)\|_{H^m_{co}} \leqslant C\|u\|_{H^{m+1}_{co}}$$

so

$$\|\widetilde{u}_4\|_{H^m_{co}} \leqslant C\|u\|_{H^{m+1}_{co}}.$$

A similar argument works for the $W^{m,\infty}_{co}$ spaces so the proof is completed. □

We show now the following easy lemma who was used in the proof of the previous lemma.

Lemma 6. *Let f be a function vanishing on the boundary of Ω. For each $m\in\mathbb{N}$ there exists a constant $C = C(m,\Omega)$ such that*

$$\Big\|\frac{f}{d}\Big\|_{H^m_{co}} \leqslant C(\|f\|_{H^m_{co}} + C\|n\cdot\nabla f\|_{H^m_{co}})$$

and

$$\Big\|\frac{f}{d}\Big\|_{W^{m,\infty}_{co}} \leqslant C(\|f\|_{W^{m,\infty}_{co}} + C\|n\cdot\nabla f\|_{W^{m,\infty}_{co}}).$$

Proof. The inequalities are obvious in a compact subset of Ω because in such a region d has a strictly positive uniform lower bound. We only need to prove something in the neighborhood of the boundary. Using local changes of coordinates combined with a partition of unity of the neighborhood of the boundary and recalling that the conormal spaces are invariant by changes of variables, we see that it suffices to prove the stated inequalities in the following setting:

- Ω is the upper-half of the unit ball $B_+ = \{x\in\mathbb{R}^3\ ;\ \|x\|<1 \text{ and } x_3>0\}$.
- f vanishes on the flat part of B_+: $f(x_1,x_2,0)=0$.
- the conormal spaces are constructed using the vector fields ∂_1, ∂_2 and $x_3\partial_3$.

So we need to prove that

$$\|f/x_3\|_{H^m_{co}} \leqslant C(\|f\|_{H^m_{co}} + C\|\partial_3 f\|_{H^m_{co}}) \quad\text{and}\quad \|f/x_3\|_{W^{m,\infty}_{co}} \leqslant C(\|f\|_{W^{m,\infty}_{co}} + C\|\partial_3 f\|_{W^{m,\infty}_{co}}).$$

These bounds are easy to prove since we can write by the Taylor formula

$$\frac{f}{x_3} = \int_0^1 \partial_3 f(x_1,x_2,tx_3)\,dt$$

so

$$\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}(f/x_3) = \int_0^1 \big(\partial_1^{\beta_1}\partial_2^{\beta_2}(tx_3\partial_3)^{\beta_3}\partial_3 f\big)(x_1,x_2,tx_3)\,dt$$

Taking the L^∞ norm yields

$$\|\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}(f/x_3)\|_{L^\infty} \leqslant \|\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}\partial_3 f\|_{L^\infty}$$

while taking the L^2 norm gives

$$\|\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}(f/x_3)\|_{L^2} \leqslant \int_0^1 \|\big(\partial_1^{\beta_1}\partial_2^{\beta_2}(tx_3\partial_3)^{\beta_3}\partial_3 f\big)(x_1,x_2,tx_3)\|_{L^2(dx)}\,dt$$
$$= \int_0^1 \|\big(\partial_1^{\beta_1}\partial_2^{\beta_2}(y_3\partial_{y_3})^{\beta_3}\partial_{y_3} f\big)(x_1,x_2,y_3)\|_{L^2(dx_1dx_2dy_3)}\frac{1}{\sqrt{t}}\,dt.$$

The last L^2 norm is not on the full domain B_+ (like the other L^2 norms). Because of the change of variables $y_3 = tx_3$, the domain of integration of the last L^2 norm is the subset of B_+ formed by the triples (x_1, x_2, tx_3) where $x \in B_+$. Since the L^2 norm is taken on a subset of B_+, we can bound it by the norm on the full B_+ obtaining in the end

$$\|\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}(f/x_3)\|_{L^2} \leqslant \|\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}\partial_3 f\|_{L^2}\int_0^1 \frac{1}{\sqrt{t}}\,dt = 2\|\partial_1^{\beta_1}\partial_2^{\beta_2}(x_3\partial_3)^{\beta_3}\partial_3 f\|_{L^2}.$$

This completes the proof of the lemma. □

The next result shows that the gradient of a divergence free vector field is controlled by the vorticity and by tangential derivatives only.

Lemma 7. *Let $k \in \mathbb{N}$ and u be a divergence free vector field. There exists a constant $C = C(k,\Omega) > 0$ such that*

$$\|\nabla u\|_{W^{k,\infty}_{co}} \leqslant C(\|\omega\|_{W^{k,\infty}_{co}} + \|u\|_{W^{k+1,\infty}_{co}})$$

where $\omega = \operatorname{curl} u$.

Proof. In the interior of Ω the bound is obvious, so we only need to prove it in the neighborhood of the boundary. We will prove it in Ω_δ where $\|n\| = 1$.

Because of the identities

$$\nabla = -\frac{n}{\|n\|^2}\times(n\times\nabla) + \frac{n}{\|n\|^2}(n\cdot\nabla)$$

and

$$u = -\frac{n}{\|n\|^2}\times(n\times u) + \frac{n}{\|n\|^2}(n\cdot u)$$

we observe that it suffices to bound $\|n\cdot\nabla(n\cdot u)\|_{W^{k,\infty}_{co}}$ and $\|n\cdot\nabla(n\times u)\|_{W^{k,\infty}_{co}}$. Thanks to (6) we have that

$$\|n\cdot\nabla(n\cdot u)\|_{W^{k,\infty}_{co}} \leqslant C\|u\|_{W^{k+1,\infty}_{co}}.$$

To bound $\|n\cdot\nabla(n\times u)\|_{W^{k,\infty}_{co}}$, let us consider for example the first component:

$$\big[n\cdot\nabla(n\times u)\big]_1 = \sum_i n_i\partial_i(n_2u_3 - n_3u_2)$$
$$= \sum_i n_i(\partial_i n_2 u_3 - \partial_i n_3 u_2) + \sum_i n_i(n_2\partial_i u_3 - n_3\partial_i u_2)$$
$$= (n\cdot\nabla n\times u)_1 + \sum_i n_i[(n_2\partial_i - n_i\partial_2)u_3 - (n_3\partial_i - n_i\partial_3)u_2] + \|n\|^2\omega_1.$$

We infer that

$$\|n\cdot\nabla(n\times u)\|_{W^{k,\infty}_{co}} \leqslant C(\|\omega\|_{W^{k,\infty}_{co}} + \|u\|_{W^{k+1,\infty}_{co}})$$

and this completes the proof. □

We end this section with the following technical results about the conormal Sobolev spaces:

Lemma 8. a) *For all $k \in \mathbb{N}$ and $|\beta_1| + |\beta_2| \leqslant k$ we have that*

(7) $$\|\partial_Z^{\beta_1} f\partial_Z^{\beta_2} g\|_{L^2} \leqslant C(\|f\|_{L^\infty}\|g\|_{H^k_{co}} + \|f\|_{H^k_{co}}\|g\|_{L^\infty})$$

and

(8) $$\|fg\|_{H^k_{co}} \leqslant C(\|f\|_{L^\infty}\|g\|_{H^k_{co}} + \|f\|_{H^k_{co}}\|g\|_{L^\infty}).$$

b) *The embedding $X^2 \subset L^\infty$ holds true.*

Proof. Relation (7) was proved in [12, Lemma 8]. Relation (8) follows from (7) and the Leibniz formula.

We prove now the embedding stated in item b). In the interior of Ω the H^m_{co} regularity is the same as the H^m regularity. Since in dimension three we have the embedding $H^2 \subset L^\infty$ the desired embedding holds true in a compact region of Ω. Therefore, we can assume that we are in the neighborhood of the boundary. Using a change of coordinates and a partition of unity, we can assume that the domain Ω is the half-plane $\Omega = \{x\, ; x_3 > 0\}$. Let us denote $x_h = (x_1, x_2)$ and take some $f \in X^2$. We have that f and $\nabla_h f \in H^1(\Omega)$. By the trace theorem, for all $x_3 \geqslant 0$ we have that $f(\cdot, x_3)$ and $\nabla_h f(\cdot, x_3) \in H^{\frac12}(\mathbb{R}^2)$ so $f(\cdot, x_3) \in H^{\frac32}(\mathbb{R}^2)$. The Sobolev embedding $H^{\frac32}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ completes the proof of item b). $\square$

4. Some ellipticity results in conormal spaces

We start with the following easy lemma relating velocity to vorticity in conormal spaces.

Lemma 9. *Let u be a divergence free vector field tangent to the boundary. There exists a constant $B_m = B(m, \Omega)$ such that the following inequality holds true:*

$$\|u\|_{X^{m+1}} \leqslant B_m(\|u\|_{L^2} + \|\omega\|_{H^m_{co}})$$

where $\omega = \operatorname{curl} u$.

Proof. Let ∂_Z^m be a tangential derivative of order m. We use [7, Proposition 1.4] to write

$$\begin{aligned}\|\nabla\partial_Z^m u\|_{L^2} &\leqslant C(\|\partial_Z^m u\|_{L^2} + \|\operatorname{curl}\partial_Z^m u\|_{L^2} + \|\operatorname{div}\partial_Z^m u\|_{L^2} + \|n\cdot\partial_Z^m u\|_{H^{1/2}(\partial\Omega)})\\ &\leqslant C(\|u\|_{H^m_{co}} + \|\omega\|_{H^m_{co}} + \|[\operatorname{curl}, \partial_Z^m]u\|_{L^2} + \|[\operatorname{div}, \partial_Z^m]u\|_{L^2} + \|[n\cdot, \partial_Z^m]u\|_{H^{1/2}(\partial\Omega)})\end{aligned}$$

where we used that u is divergence free and tangent to the boundary. Clearly

$$\|[\operatorname{curl}, \partial_Z^m]u\|_{L^2} \leqslant C\|u\|_{X^m}$$

and

$$\|[\operatorname{div}, \partial_Z^m]u\|_{L^2} \leqslant C\|u\|_{X^m}.$$

We observe now that $[n\cdot, \partial_Z^m]u$ is a combination of tangential derivatives of u of order $\leqslant m-1$. But if ∂_Z^{m-1} is a tangential derivative of order $\leqslant m-1$ then we have that

$$\|\partial_Z^{m-1}u\|_{H^{1/2}(\partial\Omega)} \leqslant C\|\partial_Z^{m-1}u\|_{H^1(\Omega)} \leqslant C\|u\|_{X^m}.$$

We infer from the above relations that the following estimate holds true:

$$\|u\|_{X^{m+1}} \leqslant C(\|u\|_{X^m} + \|\omega\|_{H^m_{co}}).$$

Clearly one can now iterate the argument and bound the term $\|u\|_{X^m}$ on the right-hand side. After m iterations we obtain the desired conclusion. $\square$

The main result of this section is the following elliptic estimate:

Proposition 10. *Let $m \in \mathbb{N}$. Suppose that u is divergence free and verifies the Navier boundary conditions* (2)*. There exists $\alpha_0 = \alpha_0(\Omega, m)$ and a constant $C > 0$ such that for all $\alpha \leqslant \alpha_0$ we have that*

$$\|u\|_{X^{m+1}} \leqslant C(\|u\|_{L^2} + \|\omega^\alpha\|_{H^m_{co}}).$$

Proof. We will in fact show that for $\alpha < \alpha_0$ (with $\alpha_0 < 1$ small enough to be chosen later) there exists a constant C such that

$$\|u\|^2_{X^{m+1}} + \alpha\|\omega\|^2_{X^{m+1}} + \alpha^2\|\Delta\omega\|^2_{H^m_{co}} \leqslant C(\|u\|^2_{L^2} + \|\omega^\alpha\|^2_{H^m_{co}}). \tag{9}$$

We proceed by induction. We consider first the case $m = 0$.

Case $m = 0$**.** Since $X^1 = H^1$ and $H^0_{co} = L^2$, we need to prove that if $\alpha \leqslant \alpha_0$ then

$$\|u\|^2_{H^1} + \alpha\|\omega\|^2_{H^1} + \alpha^2\|\Delta\omega\|^2_{L^2} \leqslant C(\|u\|^2_{L^2} + \|\omega^\alpha\|^2_{L^2})$$

for some constant $C = C(\alpha_0, \Omega)$.

Clearly

$$\begin{aligned}\|\omega^\alpha\|^2_{L^2} &= \|\omega\|^2_{L^2} + \alpha^2\|\Delta\omega\|^2_{L^2} - 2\alpha\int_\Omega \omega\cdot\Delta\omega \\ &= \|\omega\|^2_{L^2} + \alpha^2\|\Delta\omega\|^2_{L^2} + 2\alpha\|\nabla\omega\|^2_{L^2} - 2\alpha\int_{\partial\Omega}\omega\cdot\partial_n\omega\end{aligned}$$

We use Lemma 4 to write the boundary terms under the form:

$$\begin{aligned}\int_{\partial\Omega}\omega\cdot\partial_n\omega &= \int_{\partial\Omega}\omega_{tan}\cdot(\partial_n\omega)_{tan} + \int_{\partial\Omega}\omega_{nor}\,(\partial_n\omega)_{nor} \\ &= \int_{\partial\Omega}F(u)\cdot(\partial_n\omega)_{tan} + \int_{\partial\Omega}\omega_{nor}\,G(u,(n\times\nabla)u) \\ &\equiv I_1 + I_2.\end{aligned}$$

We go back to an integral on Ω by means of the Stokes formula:

$$I_2 = \int_{\partial\Omega}\omega_{nor}\,G(u,(n\times\nabla)u) = \int_{\partial\Omega}\|n\|^2\omega_{nor}\,G(u,(n\times\nabla)u) = \int_\Omega\sum_i\partial_i[n_i\omega_{nor}\,G(u,(n\times\nabla)u)]$$

so

$$|I_2| \leqslant C(\|\omega\|_{L^2}\|u\|_{H^2} + \|\omega\|_{H^1}\|u\|_{H^1}).$$

We use again the Stokes formula to write

$$\begin{aligned}\int_{\partial\Omega}F(u)\cdot(\partial_n\omega)_{tan} = \int_{\partial\Omega}F(u)\cdot(\sum_i n_i\partial_i\omega)_{tan} &= \int_{\partial\Omega}\sum_i n_iF(u)\cdot(\partial_i\omega)_{tan} \\ &= \int_\Omega\sum_i\partial_i[F(u)\cdot(\partial_i\omega)_{tan}].\end{aligned}$$

Expanding the last term above and separating the terms containing second order derivatives of ω, we observe that we can bound pointwise

$$|\sum_i\partial_i[F(u)\cdot(\partial_i\omega)_{tan}] - F(u)\cdot(\Delta\omega)_{tan}| \leqslant C(|u| + |\nabla u|)|\nabla\omega|.$$

We infer that we can bound

$$|I_1| \leqslant C\int_\Omega(|u| + |\nabla u|)|\nabla\omega| + C\int_\Omega|F(u)\cdot(\Delta\omega)_{tan}| \leqslant C\|u\|_{H^1}\|\nabla\omega\|_{L^2} + C\|u\|_{L^2}\|\Delta\omega\|_{L^2}.$$

The previous relations imply that

$$\Big|\int_{\partial\Omega}\omega\cdot\partial_n\omega\Big| \leqslant C\|\omega\|_{L^2}\|u\|_{H^2} + C\|\nabla\omega\|_{L^2}\|u\|_{H^1} + C\|u\|_{L^2}\|\Delta\omega\|_{L^2}$$

But we have that $\|u\|_{L^2} + \|\omega\|_{L^2} \simeq \|u\|_{H^1}$ and $\|u\|_{L^2} + \|\omega\|_{H^1} \simeq \|u\|_{H^2}$ (see [7, Proposition 1.4]), so we can further write that

$$\Big|\int_{\partial\Omega}\omega\cdot\partial_n\omega\Big| \leqslant C(\|u\|_{L^2} + \|\omega\|_{L^2})(\|\omega\|_{L^2} + \|\nabla\omega\|_{L^2}) + C\|u\|_{L^2}\|\Delta\omega\|_{L^2}$$

We conclude that

$$\begin{aligned}\|\omega^\alpha\|^2_{L^2} &\geqslant \|\omega\|^2_{L^2} + \alpha^2\|\Delta\omega\|^2_{L^2} + 2\alpha\|\nabla\omega\|^2_{L^2} - C\alpha\|u\|_{L^2}\|\Delta\omega\|_{L^2} \\ &\qquad - C\alpha(\|u\|_{L^2} + \|\omega\|_{L^2})(\|\omega\|_{L^2} + \|\nabla\omega\|_{L^2}) \\ &\geqslant (1 - C\alpha)\|\omega\|^2_{L^2} + \frac{\alpha^2}{2}\|\Delta\omega\|^2_{L^2} + \alpha\|\nabla\omega\|^2_{L^2} - C\|u\|^2_{L^2}.\end{aligned}$$

We finally obtain that

$$\|u\|_{L^2}^2+\varepsilon_0\|\omega^\alpha\|_{L^2}^2 \geqslant (1-C\varepsilon_0)\|u\|_{L^2}^2+\varepsilon_0(1-C\alpha)\|\omega\|_{L^2}^2+\frac{\varepsilon_0\alpha^2}{2}\|\Delta\omega\|_{L^2}^2+\varepsilon_0\alpha\|\nabla\omega\|_{L^2}^2$$
$$\geqslant C(\varepsilon_0,\alpha)(\|u\|_{H^1}^2+\alpha^2\|\Delta\omega\|_{L^2}^2+\alpha\|\nabla\omega\|_{L^2}^2)$$

provided that α and ε_0 are sufficiently small. This completes the proof in the case $m=0$.

We show now that step $m-1$ implies step m.

Step $m-1$ implies step m. We assume that we have proved

$$\|u\|_{X^m}^2+\alpha\|\omega\|_{X^m}^2+\alpha^2\|\Delta\omega\|_{H^{m-1}_{co}}^2 \leqslant K_{m-1}(\|u\|_{L^2}^2+\|\omega^\alpha\|_{H^{m-1}_{co}}^2) \tag{10}$$

for some constant K_{m-1} and we want to prove that

$$\|u\|_{X^{m+1}}^2+\alpha\|\omega\|_{X^{m+1}}^2+\alpha^2\|\Delta\omega\|_{H^{m}_{co}}^2 \leqslant K_{m}(\|u\|_{L^2}^2+\|\omega^\alpha\|_{H^{m}_{co}}^2) \tag{11}$$

for some other constant K_m.

Let $\partial_Z^m=\partial_Z^\beta$ be a tangential derivative of order less than m: $\beta\in\mathbb{N}^7$ verifies $|\beta|\leqslant m$.

If ∂_W is a tangential derivative, we will denote by ∂_W^t the transpose of ∂_W, *i.e.* if $\partial_W=\sum_i W_i\partial_i$ then $\partial_W^t f=-\sum_i\partial_i(W_i f)=-\operatorname{div} W f-\partial_W f$. Because ∂_W is a tangential derivative, we have that $\int_\Omega \partial_W f g=\int_\Omega f\partial_W^t g$ for all f and g without need to assume any boundary conditions on f and g.

We have that

$$\|\partial_Z^m\omega^\alpha\|_{L^2}^2=\|\partial_Z^m\omega\|_{L^2}^2+\alpha^2\|\partial_Z^m\Delta\omega\|_{L^2}^2-2\alpha\int_\Omega \partial_Z^m\omega\cdot\partial_Z^m\Delta\omega$$

We perform now several integrations by parts:

$$\begin{aligned}-\int_\Omega \partial_Z^m\omega\cdot\partial_Z^m\Delta\omega &= -\int_\Omega(\partial_Z^m)^t\partial_Z^m\omega\cdot\Delta\omega\\ &=\int_\Omega\nabla(\partial_Z^m)^t\partial_Z^m\omega\cdot\nabla\omega-\int_{\partial\Omega}(\partial_Z^m)^t\partial_Z^m\omega\cdot\partial_n\omega\end{aligned} \tag{12}$$

We wish now to commute the gradient with ∂_Z^m. Repeatedly using the formula

$$\int_\Omega \partial_i\partial_W^t f\ g=\int_\Omega\partial_i f\ \partial_W g-\int_\Omega fg\ \partial_i\operatorname{div} w-\sum_j\int_\Omega\partial_j f\ g\ \partial_i w_j \tag{13}$$

we observe that we can write

$$\int_\Omega\nabla(\partial_Z^m)^t\partial_Z^m\omega\cdot\nabla\omega=\int_\Omega\nabla\partial_Z^m\omega\cdot\partial_Z^m\nabla\omega+I_1$$

where

$$|I_1|\leqslant C\|\omega\|_{X^{m+1}}\|\omega\|_{X^m}.$$

Moreover,

$$\int_\Omega\nabla\partial_Z^m\omega\cdot\partial_Z^m\nabla\omega=\frac12\|\nabla\partial_Z^m\omega\|_{L^2}^2+\frac12\|\partial_Z^m\nabla\omega\|_{L^2}^2-\frac12\|[\nabla,\partial_Z^m]\omega\|_{L^2}^2$$

where the last term can be bounded by

$$\|[\nabla,\partial_Z^m]\omega\|_{L^2}\leqslant C\|\omega\|_{X^m}^2.$$

It remains to estimate the boundary term in (12). To do that, we proceed as in the case $m=0$ by decomposing $\omega=\omega_{tan}+\omega_{nor}$ and writing

$$\begin{aligned}\int_{\partial\Omega}(\partial_Z^m)^t\partial_Z^m\omega\cdot\partial_n\omega&=\int_{\partial\Omega}[(\partial_Z^m)^t\partial_Z^m\omega]_{tan}\cdot(\partial_n\omega)_{tan}+\int_{\partial\Omega}[(\partial_Z^m)^t\partial_Z^m\omega]_{nor}\,(\partial_n\omega)_{nor}\\ &\equiv J_1+J_2\end{aligned}$$

Using Lemma 4 and the Stokes formula we can write

$$
\begin{aligned}
J_2 &= \int_{\partial\Omega} n\cdot(\partial_Z^m)^t\partial_Z^m\omega\; G(u,(n\times\nabla)u)\\
&= \sum_i\int_\Omega \partial_i\big[(\partial_Z^m)^t\partial_Z^m\omega_i\; G(u,(n\times\nabla)u)\big]\\
&= \sum_i\int_\Omega \partial_i\big[(\partial_Z^m)^t\partial_Z^m\omega_i\big]\; G(u,(n\times\nabla)u)+\sum_i\int_\Omega (\partial_Z^m)^t\partial_Z^m\omega_i\;\partial_i\big[G(u,(n\times\nabla)u)\big]\\
&= \sum_i\int_\Omega \partial_i\big[(\partial_Z^m)^t\partial_Z^m\omega_i\big]\; G(u,(n\times\nabla)u)+\sum_i\int_\Omega \partial_Z^{m+1}\omega_i\;\partial_Z^{m-1}\partial_i\big[G(u,(n\times\nabla)u)\big]\\
&\equiv J_{21}+J_{22}
\end{aligned}
$$

where ∂_Z^{m+1} denotes a tangential derivative of order $\leqslant m+1$ and ∂_Z^{m-1} denotes a tangential derivative of order $\leqslant m-1$. Clearly

$$|J_{22}|\leqslant C\|\partial_Z^{m+1}\omega\|_{L^2}\|\partial_Z^{m-1}\nabla[G(u,(n\times\nabla)u)]\|_{L^2}\leqslant C\|\omega\|_{H^{m+1}_{co}}\|u\|_{X^{m+1}}$$

Repeatedly using relation (13) we can also bound

$$|J_{21}|\leqslant C\|\omega\|_{X^{m+1}}\|u\|_{H^{m+1}_{co}}$$

We go now to the estimate of the term J_1. Recalling that in the neighborhood of the boundary we have the decomposition $\omega=n\times\omega_{tan}+\omega_{nor}\,n$, we can write

$$
\begin{aligned}
J_1 = \int_{\partial\Omega}&[(\partial_Z^m)^t\partial_Z^m\omega]_{tan}\cdot(\partial_n\omega)_{tan}=\int_{\partial\Omega}[(\partial_Z^m)^t\partial_Z^m(n\times\omega_{tan}+\omega_{nor}\,n)]_{tan}\cdot(\partial_n\omega)_{tan}\\
&=\int_{\partial\Omega}[(\partial_Z^m)^t\partial_Z^m(n\times\omega_{tan})]_{tan}\cdot(\partial_n\omega)_{tan}+\int_{\partial\Omega}[(\partial_Z^m)^t\partial_Z^m(\omega_{nor}\,n)]_{tan}\cdot(\partial_n\omega)_{tan}\equiv J_{11}+J_{12}.
\end{aligned}
$$

Using Lemma 4, the fact that ∂_Z is a tangential derivative and that ∂_Z^t is $-\partial_Z$ plus a zero order term, we deduce that $(\partial_Z^m)^t\partial_Z^m(n\times\omega_{tan})=(\partial_Z^m)^t\partial_Z^m(n\times F(u))=(\partial_Z^m)^t\partial_Z^m F(u)_{tan}$ on the boundary. We infer that

$$
\begin{aligned}
J_{11} &= \sum_i\int_{\partial\Omega} n_i[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\cdot(\partial_i\omega)_{tan}\\
&= \sum_i\int_\Omega \partial_i\left\{[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\cdot(\partial_i\omega)_{tan}\right\}\\
&= \sum_i\int_\Omega \partial_i\left\{[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\right\}\cdot(\partial_i\omega)_{tan}+\sum_i\int_\Omega[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\cdot\partial_i[(\partial_i\omega)_{tan}]\\
&= \sum_i\int_\Omega \partial_i\left\{[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\right\}\cdot(\partial_i\omega)_{tan}+\sum_i\int_\Omega[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\cdot(\Delta\omega)_{tan}\\
&\qquad\qquad+\sum_i\int_\Omega[(\partial_Z^m)^t\partial_Z^m F(u)_{tan}]_{tan}\cdot(\partial_i\omega\times\partial_i n)\\
&\equiv J_{111}+J_{112}+J_{113}.
\end{aligned}
$$

Using relation (13) m times we can bound

$$|J_{111}|\leqslant C\|u\|_{X^{m+1}}\|\omega\|_{X^m}$$

Integrating by parts m times allows to estimate

$$|J_{112}|\leqslant C\|u\|_{H^m_{co}}\|\Delta\omega\|_{H^m_{co}}$$

and

$$|J_{113}|\leqslant C\|u\|_{H^m_{co}}\|\omega\|_{X^{m+1}}.$$

This completes the estimate of the term J_{11}. We claim that exactly the same estimates hold true for the term J_{12}. Indeed, the key point that allowed us to estimate J_{11} is the fact that thanks to Lemma 4, on the boundary the expression $[(\partial_Z^m)^t\partial_Z^m(\omega_{nor}\, n)]_{tan}$ can be written as a combination of tangential derivatives of u of order $2m$ at most. But exactly the same holds true for the expression $[(\partial_Z^m)^t\partial_Z^m(\omega_{nor}\, n)]_{tan}$. Indeed, because of the identity $\omega_{nor} = \omega\cdot n = (n\times\nabla)u$ and thanks to the Leibniz formula, we can write

$$[(\partial_Z^m)^t\partial_Z^m(\omega_{nor}\, n)]_{tan} = (\partial_Z^m)^t\partial_Z^m((n\times\nabla)u\, n)\times n = (\partial_Z^m)^t\partial_Z^m((n\times\nabla)u)\, n\times n + \Gamma$$

where the expression Γ is a linear combination of tangential derivatives of u of order $2m$ at most. The first term on the right-hand side vanishes, so we can conclude that the estimates we proved for J_{11} hold true for J_{12} as well.

From the previous estimates we infer that

$$\begin{aligned}\|\partial_Z^m\omega^\alpha\|_{L^2}^2 \geqslant \|\partial_Z^m\omega\|_{L^2}^2 &+ \alpha^2\|\partial_Z^m\Delta\omega\|_{L^2}^2 + \alpha\|\nabla\partial_Z^m\omega\|_{L^2}^2 + \alpha\|\partial_Z^m\nabla\omega\|_{L^2}^2\\ &- C\alpha(\|\omega\|_{X^m}\|\omega\|_{X^{m+1}} + \|\omega\|_{H^{m+1}_{co}}\|u\|_{X^{m+1}} + \|\omega\|_{X^{m+1}}\|u\|_{H^{m+1}_{co}}\\ &\qquad + \|u\|_{X^{m+1}}\|\omega\|_{X^m} + \|u\|_{H^m_{co}}\|\Delta\omega\|_{H^m_{co}})\end{aligned}$$

Summing over all possible choices of ∂_Z^m we get

$$\begin{aligned}\|\omega^\alpha\|_{H^m_{co}}^2 &\geqslant \|\omega\|_{H^m_{co}}^2 + \alpha^2\|\Delta\omega\|_{H^m_{co}}^2 + \alpha\|\nabla\omega\|_{H^m_{co}}^2 - C\alpha(\|\omega\|_{X^m}\|\omega\|_{X^{m+1}} + \|\omega\|_{H^{m+1}_{co}}\|u\|_{X^{m+1}}\\ &\qquad + \|\omega\|_{X^{m+1}}\|u\|_{H^{m+1}_{co}} + \|u\|_{X^{m+1}}\|\omega\|_{X^m} + \|u\|_{H^m_{co}}\|\Delta\omega\|_{H^m_{co}})\\ &= \|\omega\|_{H^m_{co}}^2 + \alpha^2\|\Delta\omega\|_{H^m_{co}}^2 + \alpha\|\nabla\omega\|_{H^m_{co}}^2 - C\alpha R\end{aligned}$$

where

$$R = \|\omega\|_{X^m}\|\omega\|_{X^{m+1}} + \|\omega\|_{H^{m+1}_{co}}\|u\|_{X^{m+1}} + \|\omega\|_{X^{m+1}}\|u\|_{H^{m+1}_{co}} + \|u\|_{X^{m+1}}\|\omega\|_{X^m} + \|u\|_{H^m_{co}}\|\Delta\omega\|_{H^m_{co}}.$$

To prove (11) it clearly suffices to show that there exists $\varepsilon > 0$ and K'_m such that

$$\|u\|_{X^{m+1}}^2 + \alpha\|\omega\|_{X^{m+1}}^2 + \alpha^2\|\Delta\omega\|_{H^m_{co}}^2 \leqslant K'_m(\|u\|_{L^2}^2 + \|\omega^\alpha\|_{H^{m-1}_{co}}^2 + \varepsilon\|\omega^\alpha\|_{H^m_{co}}^2) \tag{14}$$

Using (10) we have that

$$\begin{aligned}\|u\|_{L^2}^2 + \|\omega^\alpha\|_{H^{m-1}_{co}}^2 + \varepsilon\|\omega^\alpha\|_{H^m_{co}}^2 &\geqslant \frac{1}{K_{m-1}}(\|u\|_{X^m}^2 + \alpha\|\omega\|_{X^m}^2 + \alpha^2\|\Delta\omega\|_{H^{m-1}_{co}}^2)\\ &\qquad + \varepsilon\|\omega\|_{H^m_{co}}^2 + \varepsilon\alpha^2\|\Delta\omega\|_{H^m_{co}}^2 + \varepsilon\alpha\|\nabla\omega\|_{H^m_{co}}^2 - C\alpha\varepsilon R\end{aligned}$$

Thanks to Lemma 9 we can estimate

$$\frac{1}{K_{m-1}}\|u\|_{X^m}^2 + \varepsilon\|\omega\|_{H^m_{co}}^2 = \frac{1}{2K_{m-1}}\|u\|_{X^m}^2 + \frac{1}{2K_{m-1}}\|u\|_{X^m}^2 + \varepsilon\|\omega\|_{H^m_{co}}^2 \geqslant \frac{1}{2K_{m-1}}\|u\|_{X^m}^2 + \frac{\varepsilon}{B_m^2}\|u\|_{X^{m+1}}^2$$

provided that $\varepsilon \leqslant \frac{1}{2K_{m-1}}$ which we will assume to hold true in what follows. Writing also

$$\frac{1}{K_{m-1}}\|\omega\|_{X^m}^2 + \varepsilon\|\nabla\omega\|_{H^m_{co}}^2 \geqslant \frac{1}{2K_{m-1}}\|\omega\|_{X^m}^2 + C_1\varepsilon\|\omega\|_{X^{m+1}}^2$$

we infer from the above relations that

$$\begin{aligned}\|u\|_{L^2}^2 + \|\omega^\alpha\|_{H^{m-1}_{co}}^2 + \varepsilon\|\omega^\alpha\|_{H^m_{co}}^2 &\geqslant \frac{1}{2K_{m-1}}(\|u\|_{X^m}^2 + \alpha\|\omega\|_{X^m}^2 + \alpha^2\|\Delta\omega\|_{H^{m-1}_{co}}^2)\\ &\qquad + C_2\varepsilon(\|u\|_{X^{m+1}}^2 + \alpha\|\omega\|_{X^{m+1}}^2 + \alpha^2\|\Delta\omega\|_{H^m_{co}}^2) - C\alpha\varepsilon R.\end{aligned} \tag{15}$$

It remains to estimate the term $C\alpha\varepsilon R$. We bound first

$$\begin{aligned}R &= \|\omega\|_{X^m}\|\omega\|_{X^{m+1}} + \|\omega\|_{H^{m+1}_{co}}\|u\|_{X^{m+1}} + \|\omega\|_{X^{m+1}}\|u\|_{H^{m+1}_{co}} + \|u\|_{X^{m+1}}\|\omega\|_{X^m} + \|u\|_{H^m_{co}}\|\Delta\omega\|_{H^m_{co}}\\ &\leqslant C(\|\omega\|_{X^m}\|\omega\|_{X^{m+1}} + \|\omega\|_{X^{m+1}}\|u\|_{X^{m+1}} + \|u\|_{X^m}\|\Delta\omega\|_{H^m_{co}})\end{aligned}$$

We use next Lemma 9 to write $\|u\|_{X^{m+1}}\leqslant C(\|u\|_{X^m}+\|\omega\|_{X^m})$ and deduce that

$$\begin{aligned}C\alpha\varepsilon R&\leqslant C\alpha\varepsilon\|\omega\|_{X^{m+1}}(\|\omega\|_{X^m}+\|u\|_{X^m})+C\alpha\varepsilon\|u\|_{X^m}\|\Delta\omega\|_{H^m_{co}}\\&\leqslant\frac{C_2\varepsilon}{2}(\alpha\|\omega\|^2_{X^{m+1}}+\alpha^2\|\Delta\omega\|^2_{H^m_{co}})+C\varepsilon(1+\alpha)\|u\|^2_{X^m}+C\alpha\varepsilon\|\omega\|^2_{X^m}.\end{aligned}$$

Using this bound in (15) implies that

$$\begin{aligned}\|u\|^2_{L^2}+\|\omega^\alpha\|^2_{H^{m-1}_{co}}+\varepsilon\|\omega^\alpha\|^2_{H^m_{co}}&\geqslant\frac{C_2\varepsilon}{2}(\|u\|^2_{X^{m+1}}+\alpha\|\omega\|^2_{X^{m+1}}+\alpha^2\|\Delta\omega\|^2_{H^m_{co}})\\&\quad+\Big(\frac{1}{2K_{m-1}}-C\varepsilon(1+\alpha)\Big)\|u\|^2_{X^m}+\alpha\Big(\frac{1}{2K_{m-1}}-C\varepsilon\Big)\|\omega\|^2_{X^m}\\&\geqslant\frac{C_2\varepsilon}{2}(\|u\|^2_{X^{m+1}}+\alpha\|\omega\|^2_{X^{m+1}}+\alpha^2\|\Delta\omega\|^2_{H^m_{co}})\end{aligned}$$

provided that ε is sufficiently small. The above relation implies that (14) holds true. This completes the proof. $\square$

We will also need some $W^{1,\infty}_{co}$ elliptic estimates for the operator $1-\alpha\Delta$ in the setting of the conormal Sobolev spaces. We start with an L^∞ bound.

Lemma 11. *There exists a constant C independent of α such that the following relation holds true:*

$$\|h\|_{L^\infty}+\sqrt{\alpha}\|\nabla h\|_{L^\infty}+\alpha\|\Delta h\|_{L^\infty}\leqslant C(\|h-\alpha\Delta h\|_{L^\infty}+\|h\|_{L^\infty(\partial\Omega)}+\sqrt{\alpha}\|h\|_{W^{1,\infty}(\partial\Omega)}+\alpha\|h\|_{W^{2,\infty}(\partial\Omega)}).$$

Proof. We assume first that h vanishes on the boundary of Ω. In this case, it was proved in [2, Lemma A.2] the following inequality:

$$\|\nabla h\|^2_{L^\infty}\leqslant C_1\|h\|_{L^\infty}\|\Delta h\|_{L^\infty}.$$

From the maximum principle we have that

$$\|h\|_{L^\infty}\leqslant\|h-\alpha\Delta h\|_{L^\infty}$$

so

$$\|\Delta h\|_{L^\infty}=\frac{1}{\alpha}\|h-\alpha\Delta h-h\|_{L^\infty}\leqslant\frac{1}{\alpha}(\|h-\alpha\Delta h\|_{L^\infty}+\|h\|_{L^\infty})\leqslant\frac{2}{\alpha}\|h-\alpha\Delta h\|_{L^\infty}.$$

We conclude that

$$\alpha\|\nabla h\|^2_{L^\infty}\leqslant 2C_1\|h\|_{L^\infty}\|h-\alpha\Delta h\|_{L^\infty}\leqslant 2C_1\|h-\alpha\Delta h\|^2_{L^\infty}\tag{16}$$

which completes the proof in the case when h vanishes on the boundary.

We consider now the general case. Let H be a $W^{2,\infty}$ extension of $h\big|_{\partial\Omega}$ to Ω such that $\|H\|_{W^{k,\infty}(\Omega)}\leqslant C\|h\|_{W^{k,\infty}(\partial\Omega)}$ for all $k\in\{0,1,2\}$, where C depends only on Ω. Because $h-H$ vanishes on the boundary, we can apply relation (16) to $h-H$ to obtain:

$$\begin{aligned}\sqrt{\alpha}\|\nabla(h-H)\|_{L^\infty}\leqslant C\|h-H-\alpha\Delta(h-H)\|_{L^\infty}&\leqslant C\|h-\alpha\Delta h\|_{L^\infty}+C\|H\|_{L^\infty}+C\alpha\|H\|_{W^{2,\infty}}\\&\leqslant C\|h-\alpha\Delta h\|_{L^\infty}+C\|h\|_{L^\infty(\partial\Omega)}+C\alpha\|h\|_{W^{2,\infty}(\partial\Omega)}.\end{aligned}$$

We infer that

$$\begin{aligned}\sqrt{\alpha}\|\nabla h\|_{L^\infty}&\leqslant\sqrt{\alpha}\|\nabla H\|_{L^\infty}+C\|h-\alpha\Delta h\|_{L^\infty}+C\|h\|_{L^\infty(\partial\Omega)}+C\alpha\|h\|_{W^{2,\infty}(\partial\Omega)}\\&\leqslant C\|h-\alpha\Delta h\|_{L^\infty}+C\|h\|_{L^\infty(\partial\Omega)}+C\sqrt{\alpha}\|h\|_{W^{1,\infty}(\partial\Omega)}+C\alpha\|h\|_{W^{2,\infty}(\partial\Omega)}.\end{aligned}$$

The L^∞ bound for h follows from the maximum principle and the L^∞ bound for Δh is obvious from the triangle inequality: $\alpha\|\Delta h\|_{L^\infty}\leqslant\|h\|_{L^\infty}+\|h-\alpha\Delta h\|_{L^\infty}$. This completes the proof. $\square$

We can now prove the $W^{1,\infty}_{co}$ estimates for $1-\alpha\Delta$.

Lemma 12. *Suppose that u is divergence free and verifies the Navier boundary conditions* (2). *There exists $\alpha_0=\alpha_0(\Omega)$ and a constant $C=C(\Omega)>0$ such that for all $\alpha\leqslant\alpha_0$ we have that*

$$\|\omega\|_{W^{1,\infty}_{co}}\leqslant C(\|\omega^\alpha\|_{W^{1,\infty}_{co}}+\|u\|_{W^{2,\infty}_{co}}+\sqrt{\alpha}\|u\|_{W^{3,\infty}_{co}}+\alpha\|u\|_{W^{4,\infty}_{co}}).$$

Proof. Recall that

$$\omega - \alpha\Delta\omega = \omega^\alpha. \tag{17}$$

We observe first that

$$\omega\cdot n = (n\times\nabla)\cdot u.$$

Because of the identity $\omega\|n\|^2 = n(\omega\cdot n) - (\omega\times n)\times n$ and using relation (4) we observe that

$$\omega = n(n\times\nabla)\cdot u - F(u)\times n \quad \text{on } \partial\Omega.$$

Therefore, for $k\in\mathbb{N}$, we have the bound

$$\|\omega\|_{W^{k,\infty}(\partial\Omega)} \leqslant C\|u\|_{W^{k+1,\infty}_{co}}. \tag{18}$$

We use Lemma 11 to deduce that

(19)

$$\begin{aligned}\|\omega\|_{L^\infty} + \sqrt{\alpha}\|\omega\|_{W^{1,\infty}} + \alpha\|\Delta\omega\|_{L^\infty} &\leqslant C(\|\omega^\alpha\|_{L^\infty} + \|\omega\|_{L^\infty(\partial\Omega)} + \sqrt{\alpha}\|\omega\|_{W^{1,\infty}(\partial\Omega)} + \alpha\|\omega\|_{W^{2,\infty}(\partial\Omega)})\\ &\leqslant C(\|\omega^\alpha\|_{L^\infty} + \|u\|_{W^{1,\infty}_{co}} + \sqrt{\alpha}\|u\|_{W^{2,\infty}_{co}} + \alpha\|u\|_{W^{3,\infty}_{co}}).\end{aligned}$$

Next we apply a tangential derivative ∂_Z to (17) and obtain

$$\partial_Z\omega - \alpha\Delta\partial_Z\omega = \partial_z\omega^\alpha + \alpha[\partial_Z,\Delta]\omega.$$

As above, we deduce from Lemma 11 the following inequality:

(20)

$$\|\partial_Z\omega\|_{L^\infty} + \sqrt{\alpha}\|\nabla\partial_Z\omega\|_{L^\infty} \leqslant C(\|\partial_Z\omega^\alpha\|_{L^\infty} + \alpha\|[\partial_Z,\Delta]\omega\|_{L^\infty} + \|u\|_{W^{2,\infty}_{co}} + \sqrt{\alpha}\|u\|_{W^{3,\infty}_{co}} + \alpha\|u\|_{W^{4,\infty}_{co}}).$$

We prove now that the following estimate holds true:

$$\|\omega\|_{W^{2,\infty}} \leqslant C(\|\omega\|_{W^{1,\infty}} + \|\nabla\omega\|_{W^{1,\infty}_{co}} + \|\Delta\omega\|_{L^\infty}). \tag{21}$$

The inequality is obvious in $\Omega\setminus\Omega_\delta$, so we only need to prove it on Ω_δ. But in this region we have that $\|n\|=1$ so

$$\nabla = -n\times(n\times\nabla) + n\partial_n$$

where $\partial_n = n\cdot\nabla$. Because $n\times\nabla$ is a tangential derivative, to prove (21) it suffices to show that

$$\|\partial_n^2\omega\|_{W^{2,\infty}(\Omega_\delta)} \leqslant C(\|\omega\|_{W^{1,\infty}} + \|\nabla\omega\|_{W^{1,\infty}_{co}} + \|\Delta\omega\|_{L^\infty}). \tag{22}$$

But

$$\Delta = \nabla\cdot\nabla = \big(n\times(n\times\nabla) - n\partial_n\big)\cdot\big(n\times(n\times\nabla) - n\partial_n\big)$$

and

$$(n\partial_n)\cdot(n\partial_n) = n\cdot\partial_n n\partial_n + \|n\|^2\partial_n^2 = \frac{1}{2}\partial_n(\|n\|^2) + \|n\|^2\partial_n^2 = \partial_n^2 \quad \text{on } \Omega_\delta$$

because $\|n\|=1$ on Ω_δ. This observation immediately implies relation (22), so (21) is proved.

Next, since $[\partial_Z,\Delta]\omega$ is a linear combination of derivatives of second order or less of ω, we can bound

$$\begin{aligned}\alpha\|[\partial_Z,\Delta]\omega\|_{L^\infty} &\leqslant C\alpha\|\omega\|_{W^{2,\infty}} \leqslant C\alpha(\|\omega\|_{W^{1,\infty}} + \|\nabla\omega\|_{W^{1,\infty}_{co}} + \|\Delta\omega\|_{L^\infty})\\ &\leqslant C\alpha(\|\omega\|_{W^{1,\infty}} + \|\nabla\omega\|_{W^{1,\infty}_{co}}) + C(\|\omega^\alpha\|_{L^\infty} + \|u\|_{W^{1,\infty}_{co}} + \sqrt{\alpha}\|u\|_{W^{2,\infty}_{co}} + \alpha\|u\|_{W^{3,\infty}_{co}}).\end{aligned}$$

Using this relation in the bound for $\partial_Z\omega$ given in (20), adding to the bound for ω given in (19) and summing over all tangential derivatives ∂_Z implies

$$\begin{aligned}\|\omega\|_{W^{1,\infty}_{co}} + \sqrt{\alpha}\|\omega\|_{W^{1,\infty}} + \sqrt{\alpha}\|\nabla\omega\|_{W^{1,\infty}_{co}} &\leqslant C\alpha(\|\omega\|_{W^{1,\infty}} + \|\nabla\omega\|_{W^{1,\infty}_{co}})\\ &\quad + C(\|\omega^\alpha\|_{W^{1,\infty}_{co}} + \|u\|_{W^{2,\infty}_{co}} + \sqrt{\alpha}\|u\|_{W^{3,\infty}_{co}} + \alpha\|u\|_{W^{4,\infty}_{co}}).\end{aligned}$$

If α is sufficiently small, the first term on the right-hand side can be absorbed in the left-hand side and the conclusion follows. $\square$

5. A priori estimates

In this section, we prove some *a priori* estimates for Theorems 1 and 3. These *a priori* estimates will be used in conjunction with an approximation procedure to yield the rigorous existence of the solutions in the next section.

We do first the *a priori* estimates required for Theorem 1. We start by making H^{m-1}_{co} estimates on the equation verified by the vorticity given in (3). We apply ∂_Z^β to (3), multiply by $\partial_Z^\beta \omega^\alpha$, we sum over $|\beta| \leqslant m-1$ and we integrate in space to obtain that

$$\frac{1}{2}\partial_t \|\omega^\alpha\|^2_{H^{m-1}_{co}} = -\sum_{|\beta|\leqslant m-1}\int_\Omega \partial_Z^\beta(u\cdot\nabla\omega^\alpha)\partial_Z^\beta\omega^\alpha + \sum_{|\beta|\leqslant m-1}\int_\Omega \partial_Z^\beta(\omega^\alpha\cdot\nabla u)\partial_Z^\beta\omega^\alpha \equiv I_1 + I_2.$$

We first bound I_2 by using Lemma 8, item a):

$$|I_2| \leqslant C\|\omega^\alpha\cdot\nabla u\|_{H^{m-1}_{co}}\|\omega^\alpha\|_{H^{m-1}_{co}} \leqslant C\|\omega^\alpha\|_{H^{m-1}_{co}}(\|\omega^\alpha\|_{H^{m-1}_{co}}\|\nabla u\|_{L^\infty} + \|\nabla u\|_{H^{m-1}_{co}}\|\omega^\alpha\|_{L^\infty})$$

To bound I_1, we use the decomposition from relation (5), $u = \sum\limits_{i=1}^{7}\widetilde{u}_i Z_i$, and write

$$\begin{aligned}
-I_1 &= \sum_{|\beta|\leqslant m-1}\int_\Omega \partial_Z^\beta(u\cdot\nabla\omega^\alpha)\partial_Z^\beta\omega^\alpha \\
&= \sum_{|\beta|\leqslant m-1}\sum_i\int_\Omega \partial_Z^\beta(\widetilde{u}_i\partial_{Z_i}\omega^\alpha)\partial_Z^\beta\omega^\alpha \\
&= \sum_{|\beta|\leqslant m-1}\sum_i\int_\Omega \widetilde{u}_i\partial_Z^\beta\partial_{Z_i}\omega^\alpha\partial_Z^\beta\omega^\alpha + I_{11} \\
&= \sum_{|\beta|\leqslant m-1}\sum_i\int_\Omega \widetilde{u}_i\partial_{Z_i}\partial_Z^\beta\omega^\alpha\partial_Z^\beta\omega^\alpha + \sum_{|\beta|\leqslant m-1}\sum_i\int_\Omega \widetilde{u}_i[\partial_Z^\beta,\partial_{Z_i}]\omega^\alpha\partial_Z^\beta\omega^\alpha + I_{11} \\
&= \sum_{|\beta|\leqslant m-1}\int_\Omega u\cdot\nabla\partial_Z^\beta\omega^\alpha\partial_Z^\beta\omega^\alpha + \sum_{|\beta|\leqslant m-1}\sum_i\int_\Omega \widetilde{u}_i[\partial_Z^\beta,\partial_{Z_i}]\omega^\alpha\partial_Z^\beta\omega^\alpha + I_{11} \\
&\equiv I_{12} + I_{13} + I_{11},
\end{aligned}$$

where

$$I_{11} = \sum_{|\beta|\leqslant m-1}\sum_i\int_\Omega\big[\partial_Z^\beta(\widetilde{u}_i\partial_{Z_i}\omega^\alpha) - \widetilde{u}_i\partial_Z^\beta\partial_{Z_i}\omega^\alpha\big]\partial_Z^\beta\omega^\alpha.$$

Now, an integration by parts using that u is divergence free and tangent to the boundary immediately yields that $I_{12} = 0$. Next, we observe that $[\partial_Z^\beta, \partial_{Z_i}]$ is a tangential derivative of order $\leqslant m-1$ so we can bound

$$|I_{13}| \leqslant \sum_{|\beta|\leqslant m-1}\sum_i\|\widetilde{u}_i\|_{L^\infty}\|[\partial_Z^\beta,\partial_{Z_i}]\omega^\alpha\|_{L^2}\|\partial_Z^\beta\omega^\alpha\|_{L^2} \leqslant C\|u\|_{W^{1,\infty}_{co}}\|\omega^\alpha\|^2_{H^{m-1}_{co}}$$

where we used Lemma 5 to bound $\|\widetilde{u}_i\|_{L^\infty} \leqslant C\|u\|_{W^{1,\infty}_{co}}$.

We estimate now I_{11}. We remark that it can be written as a sum of terms of the form

$$\int_\Omega \partial_Z^{\gamma_1}\widetilde{u}\;\partial_Z^{\gamma_2}\omega^\alpha\;\partial_Z^\beta\omega^\alpha \quad \text{with } 1\leqslant|\beta|\leqslant m-1,\; |\gamma_1|+|\gamma_2|\leqslant m,\; |\gamma_1|,|\gamma_2|\geqslant 1.$$

We now estimate a term of the form given above. Since $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, we can write $\partial_Z^{\gamma_1}\widetilde{u} = \partial_Z^{\gamma_3}\partial_{Z_j}\widetilde{u}$ and $\partial_Z^{\gamma_2}\omega^\alpha = \partial_Z^{\gamma_4}\partial_{Z_k}\omega^\alpha$ for some j and k. Clearly $|\gamma_3|+|\gamma_4| \leqslant m-2$. Using Lemma 8, item

a) with $k=m-2$ and Lemma 5 we observe that we can bound

$$\Big|\int_\Omega \partial_Z^{\gamma_1}\widetilde{u}\ \partial_Z^{\gamma_2}\omega^\alpha\ \partial_Z^\beta\omega^\alpha\Big| \leqslant \|\partial_Z^{\gamma_3}\partial_{Z_j}\widetilde{u}\ \partial_Z^{\gamma_4}\partial_{Z_k}\omega^\alpha\|_{L^2}\|\partial_Z^\beta\omega^\alpha\|_{L^2}$$
$$\leqslant C(\|\partial_{Z_j}\widetilde{u}\|_{L^\infty}\|\partial_{Z_k}\omega^\alpha\|_{H^{m-2}_{co}}+\|\partial_{Z_j}\widetilde{u}\|_{H^{m-2}_{co}}\|\partial_{Z_k}\omega^\alpha\|_{L^\infty})\|\omega^\alpha\|_{H^{m-1}_{co}}$$
$$\leqslant C(\|u\|_{W^{2,\infty}_{co}}\|\omega^\alpha\|_{H^{m-1}_{co}}+\|u\|_{H^m_{co}}\|\omega^\alpha\|_{W^{1,\infty}_{co}})\|\omega^\alpha\|_{H^{m-1}_{co}}$$

We obtain from the previous relations the following differential inequality for the H^{m-1}_{co} norm of ω^α:

$$\partial_t\|\omega^\alpha\|^2_{H^{m-1}_{co}} \leqslant C\|\omega^\alpha\|^2_{H^{m-1}_{co}}(\|\nabla u\|_{L^\infty}+\|u\|_{W^{2,\infty}_{co}})+C\|\omega^\alpha\|_{H^{m-1}_{co}}\|u\|_{X^m}\|\omega^\alpha\|_{W^{1,\infty}_{co}}. \tag{23}$$

We recall now that the quantity $\|u\|^2_{L^2}+2\alpha\|D(u)\|^2_{L^2}$ is conserved. Let us introduce the following norm:

$$\|u\|^2_{Y^m}\equiv\|u\|^2_{L^2}+2\alpha\|D(u)\|^2_{L^2}+\|\omega^\alpha\|^2_{H^{m-1}_{co}}.$$

Then from Proposition 10 we have that $\|u\|_{X^m}\leqslant C\|u\|_{Y^m}$. From (23) we infer that

$$\partial_t\|u\|^2_{Y^m}\leqslant C\|u\|^2_{Y^m}(\|\nabla u\|_{L^\infty}+\|u\|_{W^{2,\infty}_{co}}+\|\omega^\alpha\|_{W^{1,\infty}_{co}}).$$

From Lemma 7 we deduce that

$$\|\nabla u\|_{L^\infty}+\|u\|_{W^{2,\infty}_{co}}\leqslant C(\|\omega\|_{L^\infty}+\|u\|_{W^{2,\infty}_{co}})$$

From the maximum principle applied to the operator $1-\alpha\Delta$ and using relation (18) we deduce that

$$\|\omega\|_{L^\infty}\leqslant\|\omega^\alpha\|_{L^\infty}+\|\omega\|_{L^\infty(\partial\Omega)}\leqslant\|\omega^\alpha\|_{L^\infty}+C\|u\|_{W^{1,\infty}_{co}}$$

so that

$$\|\nabla u\|_{L^\infty}+\|u\|_{W^{2,\infty}_{co}}\leqslant C(\|\omega^\alpha\|_{L^\infty}+\|u\|_{W^{2,\infty}_{co}})\leqslant C(\|\omega^\alpha\|_{L^\infty}+\|u\|_{X^4})\leqslant C(\|\omega^\alpha\|_{L^\infty}+\|u\|_{Y^m})$$

where we used the embedding $X^2\subset L^\infty$ proved in Lemma 8, item b). We conclude that

$$\partial_t\|u\|_{Y^m}\leqslant C\|u\|^2_{Y^m}+C\|u\|_{Y^m}\|\omega^\alpha\|_{W^{1,\infty}_{co}}. \tag{24}$$

It remains to estimate the $W^{1,\infty}_{co}$ norm of ω^α. To do that, we use the equation for ω^α given in (3). We view it as a transport equation with source term $\omega^\alpha\cdot\nabla u$. We have that

$$\begin{aligned}\|\omega^\alpha(t)\|_{L^\infty}&\leqslant\|\omega^\alpha_0\|_{L^\infty}+\int_0^t\|\omega^\alpha(s)\|_{L^\infty}\|\nabla u(s)\|_{L^\infty}ds\\&\leqslant\|\omega^\alpha_0\|_{L^\infty}+C\int_0^t\|\omega^\alpha(s)\|_{L^\infty}(\|\omega^\alpha(s)\|_{L^\infty}+\|u(s)\|_{Y^m})ds\end{aligned} \tag{25}$$

Next, we apply a tangential derivative ∂_Z to (3) and recall the decomposition $u=\sum\limits_{i=1}^7\widetilde{u}_iZ_i$ to obtain

$$\partial_t\partial_Z\omega^\alpha+\partial_Z(\sum_i\widetilde{u}_i\partial_{Z_i}\omega^\alpha)-\partial_Z(\omega^\alpha\cdot\nabla u)=0$$

so

$$\partial_t\partial_Z\omega^\alpha+u\cdot\nabla\partial_Z\omega^\alpha=-\sum_i\partial_Z\widetilde{u}_i\partial_{Z_i}\omega^\alpha-\sum_i\widetilde{u}_i[\partial_Z,\partial_{Z_i}]\omega^\alpha+\partial_Z(\omega^\alpha\cdot\nabla u).$$

We infer that

$$\|\partial_Z\omega^\alpha(t)\|_{L^\infty}\leqslant\|\partial_Z\omega^\alpha_0\|_{L^\infty}+\int_0^t\|\sum_i\partial_Z\widetilde{u}_i\partial_{Z_i}\omega^\alpha+\sum_i\widetilde{u}_i[\partial_Z,\partial_{Z_i}]\omega^\alpha-\partial_Z(\omega^\alpha\cdot\nabla u)\|_{L^\infty}.$$

Summing over all Z and using also (25) we get the following bound for the $W^{1,\infty}_{co}$ norm of ω^α:

$$\begin{aligned}\|\omega^\alpha(t)\|_{W^{1,\infty}_{co}}\leqslant\|\omega^\alpha_0\|_{W^{1,\infty}_{co}}&+C\int_0^t\|\omega^\alpha(s)\|_{L^\infty}(\|\omega^\alpha(s)\|_{L^\infty}+\|u(s)\|_{Y^m})ds\\&+C\int_0^t(\|\widetilde{u}(s)\|_{W^{1,\infty}_{co}}+\|\nabla u(s)\|_{W^{1,\infty}_{co}})\|\omega^\alpha(s)\|_{W^{1,\infty}_{co}}\,ds.\end{aligned}$$

Next, we estimate $\|\widetilde{u}\|_{W^{1,\infty}_{co}} \leqslant C\|u\|_{W^{2,\infty}_{co}} \leqslant C\|u\|_{X^4} \leqslant C\|u\|_{Y^m}$. It remains to bound $\|\nabla u\|_{W^{1,\infty}_{co}}$. To do so, we use Lemma 7 and Lemma 12 to write

$$\begin{aligned}\|\nabla u\|_{W^{1,\infty}_{co}} &\leqslant C(\|\omega\|_{W^{1,\infty}_{co}} + \|u\|_{W^{2,\infty}_{co}})\\ &\leqslant C(\|\omega^\alpha\|_{W^{1,\infty}_{co}} + \|u\|_{W^{2,\infty}_{co}} + \sqrt{\alpha}\|u\|_{W^{3,\infty}_{co}} + \alpha\|u\|_{W^{4,\infty}_{co}})\\ &\leqslant C(\|\omega^\alpha\|_{W^{1,\infty}_{co}} + \|u\|_{Y^m} + \sqrt{\alpha}\|u\|_{W^{4,\infty}_{co}}).\end{aligned}$$

The last term on the right-hand side can be estimated using Lemma 9, the relation (9) and the embedding $X^2 \subset L^\infty$:

$$\sqrt{\alpha}\|u\|_{W^{4,\infty}_{co}} \leqslant C\sqrt{\alpha}\|u\|_{X^6} \leqslant C\sqrt{\alpha}(\|u\|_{L^2} + \|\omega\|_{H^5_{co}}) \leqslant C(\|u\|_{L^2} + \|\omega^\alpha\|_{H^4_{co}}) \leqslant C\|u\|_{Y^m}. \tag{26}$$

where we used that $m \geqslant 5$. We conclude that

$$\|\omega^\alpha(t)\|_{W^{1,\infty}_{co}} \leqslant \|\omega^\alpha_0\|_{W^{1,\infty}_{co}} + C\int_0^t (\|\omega^\alpha\|_{W^{1,\infty}_{co}} + \|u\|_{Y^m})\|\omega^\alpha(s)\|_{W^{1,\infty}_{co}}\, ds.$$

Combining the above relation with (24) integrated in time implies that the quantity

$$F(t) = \|u(t)\|_{Y^m} + \|\omega^\alpha(t)\|_{W^{1,\infty}_{co}}$$

verifies the following relation

$$\begin{aligned}F(t) &\leqslant C\|u_0\|_{L^2} + C\sqrt{\alpha}\|\nabla u_0\|_{L^2} + C\|\omega^\alpha_0\|_{H^{m-1}_{co}} + C\|\omega^\alpha_0\|_{W^{1,\infty}_{co}} + C\int_0^t F^2(s)\, ds\\ &\leqslant C\|u_0\|_{L^2} + C\|\omega^\alpha_0\|_{H^{m-1}_{co}} + C\|\omega^\alpha_0\|_{W^{1,\infty}_{co}} + C\int_0^t F^2(s)\, ds\end{aligned}$$

where we also used Proposition 10. Clearly this implies a bound uniform in α for F on a time interval uniform in α provided that the quantity

$$\|u_0\|_{L^2} + \|\omega^\alpha_0\|_{H^{m-1}_{co}} + \|\omega^\alpha_0\|_{W^{1,\infty}_{co}}$$

is bounded uniformly in α. We finally observe that $\|u\|_{W^{1,\infty}} \leqslant CF$ and this completes the proof of the *a priori* estimates for Theorem 1.

To prove the *a priori* estimates for Theorem 3, let us simply set $\alpha = 0$ in Theorem 1 (which is allowed). Then the minimal required hypothesis becomes $u_0 \in L^2$ and $\omega_0 \in H^4 \cap W^{1,\infty}_{co}$ which in view of Lemma 9 is equivalent to $u_0 \in X^5$ and $\omega_0 \in W^{1,\infty}_{co}$. This is not good enough because in Theorem 3 we assumed only $u_0 \in X^4$ and $\omega_0 \in W^{1,\infty}_{co}$. But if we go back to the proof of Theorem 1, it is easy to see that the hypothesis $m \geqslant 5$ was used only in relation (26). In the rest of the proof the hypothesis $m \geqslant 4$ is sufficient. But when $\alpha = 0$ the relation (26) is not required in the proof (and moreover it is trivially verified because the left-hand side vanishes). So in the case $\alpha = 0$ we can choose $m = 4$. Theorem 3 follows.

6. Approximation procedure

In this section we construct an approximation procedure that will allow to turn the *a priori* estimates from the previous section in a rigorous result of existence of solutions. We need to approximate the initial data by a sequence of smooth vector fields which belong to and are bounded in the same function spaces as u_0. That is in conormal spaces. Density results for conormal spaces are known, see for example [13, 14]. But these density results are not well adapted to divergence free vector fields. In fact, they are even false for divergence free vector fields. Indeed, it is proved in [13, 14] that C^∞_0 is dense in H^m_{co}. A similar density result can't be true for divergence free vector fields because a divergence free vector field has a normal trace at the boundary. If that normal trace is not vanishing, then no sequence of C^∞_0 divergence free vector fields can converge to this vector field. In our case, a new approximation procedure must be invented and it is not at all obvious how to proceed.

Let $\mathbb{P}$ be the Leray projector, *i.e.* the L^2 orthogonal projection on the space of divergence free vector fields tangent to the boundary.

Lemma 13. *Let $m \geqslant 2$ and $\omega \in H^{m-1}_{co}(\Omega)$ be a divergence free vector field. Then $\omega - \mathbb{P}\omega \in H^{m-1}(\Omega)$. Suppose in addition that $\omega \in W^{1,\infty}_{co}$, that $m \geqslant 4$ and that there exists some ψ such that $\omega = \operatorname{curl}\psi$. Then there exist two vector fields ψ_1 and ψ_2 such that:*

$$\omega = \operatorname{curl}(\psi_1 + \psi_2) \quad \textit{and} \quad \psi_1 + \psi_2 = \psi - \nabla p \quad \textit{for some } p, \tag{27}$$

$$\psi_1 \in X^m \cap W^{2,\infty}_{co}, \quad \nabla\psi_1 \in W^{1,\infty}_{co}, \quad \operatorname{div}\psi_1 = 0, \quad \psi_1 \times n = 0 \textit{ on } \partial\Omega, \tag{28}$$
$$\psi_2 \in H^m(\Omega).$$

Proof. We show first that $\omega \cdot n \in X^{m-1}$. Because $H^{m-1} = H^{m-1}_{co}$ in the interior of Ω, it suffices to show it in Ω_δ. But in that region we have that $\|n\| = 1$, so

$$\nabla = -n \times (n \times \nabla) + n\partial_n.$$

We infer that

$$-[n \times (n \times \nabla)] \cdot \omega + n \cdot \partial_n\omega = \operatorname{div}\omega = 0$$

Clearly $n \cdot \partial_n\omega = \partial_n(\omega \cdot n) - \partial_n n \cdot \omega$ so

$$\partial_n(\omega \cdot n) = \partial_n n \cdot \omega + [n \times (n \times \nabla)] \cdot \omega.$$

The right-hand side belongs to H^{m-2}_{co}. We infer that $\nabla(\omega \cdot n) \in H^{m-2}_{co}$ so $\omega \cdot n \in X^{m-1}$.

Now, let ∂_Z^{m-2} be a tangential derivative of order $\leqslant m-2$. Because $\omega \cdot n \in X^{m-1}$ we have that $\partial_Z^{m-2}(\omega \cdot n) \in H^1(\Omega)$ so $\partial_Z^{m-2}(\omega \cdot n)\big|_{\partial\Omega} \in H^{\frac12}(\partial\Omega)$. We conclude that $\omega \cdot n\big|_{\partial\Omega} \in H^{m-\frac32}(\partial\Omega)$.

Next, from the properties of the Leray projector we know that there exists some $q \in H^1(\Omega)$ such that

$$\omega - \mathbb{P}\omega = \nabla q.$$

Recall that $\mathbb{P}\omega$ is divergence free and tangent to the boundary. Applying the divergence and taking the trace to the boundary of the above relation, we observe that q verifies the following Neumann problem for the laplacian:

$$\Delta q = 0 \quad \text{in } \Omega$$
$$\partial_n q = \omega \cdot n \quad \text{on } \partial\Omega.$$

Because $\omega \cdot n\big|_{\partial\Omega} \in H^{m-\frac32}(\partial\Omega)$, the classical regularity results for the Neumann problem of the laplacian imply that $q \in H^m(\Omega)$. This completes the proof of the first part of the lemma.

To prove the second part, let us define $w = \omega - \mathbb{P}\omega$. From the first part of the lemma we know that $w \in H^{m-1}$. Since $m \geqslant 4$, by Sobolev embedding we have that $H^{m-1} \subset W^{1,\infty}$ so we have in particular that $w \in H^{m-1}_{co} \cap W^{1,\infty}_{co}$. Since ω also belongs to this space, we infer that $\mathbb{P}\omega \in H^{m-1}_{co} \cap W^{1,\infty}_{co}$.

Next, since $\mathbb{P}\omega$ is divergence free and tangent to the boundary one can apply [3, Theorem 2.1] to find two vector fields $\overline{\psi}$ and Y such that

$$\mathbb{P}\omega = \operatorname{curl}\overline{\psi} + Y, \quad \overline{\psi}\big|_{\partial\Omega} = 0,$$
$$\operatorname{div} Y = 0, \quad \operatorname{curl} Y = 0, \quad Y \cdot n\big|_{\partial\Omega} = 0.$$

The vector field Y is obviously smooth (as a consequence of [7, Proposition 1.4] for example). Let h be the solution of

$$\Delta h = \operatorname{div}\overline{\psi} \quad \text{in } \Omega$$
$$h = 0 \quad \text{on } \partial\Omega$$

and let us define

$$\psi_1 = \overline{\psi} - \nabla h.$$

Because h vanishes on the boundary and $n\times\nabla$ are tangential derivatives, one has that $n\times\nabla h=0$ on the boundary. From the relations above one can readily check that ψ_1 has the following properties:

$$\operatorname{curl}\psi_1=\mathbb{P}\omega-Y,\quad \operatorname{div}\psi_1=0 \quad\text{and}\quad \psi_1\times n=0 \text{ on } \partial\Omega.$$

Because Y is smooth and $\mathbb{P}\omega\in H^{m-1}_{co}$ we infer that $\operatorname{curl}\psi_1\in H^{m-1}_{co}$. As in Lemma 9, one can deduce that $\psi_1\in X^m$. Indeed, the only difference between the setting of that lemma and the present setting is that in Lemma 9 the vector field is tangent to the boundary while here it is normal to the boundary. Nevertheless, the proof goes through by replacing the elliptic estimate given in [7, Proposition 1.4] with the elliptic estimate corresponding to normal vector fields given for instance in [1, Corollary 2.15]. So we can conclude that $\psi_1\in X^m$. From the embedding $X^2\subset L^\infty$ we further obtain that $\psi_1\in W^{2,\infty}_{co}$. Since $\operatorname{curl}\psi_1\in W^{1,\infty}_{co}$ and ψ_1 is divergence free, we infer from Lemma 7 that $\nabla\psi_1\in W^{1,\infty}_{co}$. Relation (28) is completely proved.

We define next

$$\psi_2=\mathbb{P}(\psi-\psi_1).$$

From the properties of the Leray projector we know that there is some p such that

$$\psi-\psi_1-\psi_2=\psi-\psi_1-\mathbb{P}(\psi-\psi_1)=\nabla p.$$

Taking the curl of the above equality shows that relation (27) holds true. Finally, we observe that

$$\operatorname{curl}\psi_2=\operatorname{curl}\psi-\operatorname{curl}\psi_1=\omega-\mathbb{P}\omega+Y=w+Y\in H^{m-1}(\Omega).$$

Recalling that ψ_2 is also divergence free and tangent to the boundary, we can apply [7, Proposition 1.4] to deduce that $\psi_2\in H^m$. This completes the proof. ☐

Proposition 14. *Let u be a divergence free vector field verifying the Navier boundary conditions (2) and such that $u\in H^2$ and $\omega^\alpha\in H^{m-1}_{co}\cap W^{1,\infty}_{co}$ where $m\geqslant 4$. There exists a sequence of smooth divergence free vector fields u_n verifying the Navier boundary conditions such that $u_n\to u$ in H^2 and such that*

$$\|u_n\|_{L^2}+\|\omega^\alpha_n\|_{H^{m-1}_{co}}+\|\omega^\alpha_n\|_{W^{1,\infty}_{co}}\leqslant C(\|u\|_{L^2}+\|\omega^\alpha\|_{H^{m-1}_{co}}+\|\omega^\alpha\|_{W^{1,\infty}_{co}}) \tag{29}$$

for some constant $C=C(m,\Omega)$.

Proof. Let $v=u-\alpha\Delta u$ so that $\omega^\alpha=\operatorname{curl}v$. Because ω^α is divergence free, we can apply the previous lemma to ω^α to deduce the existence of some vector fields ψ_1 and ψ_2 such that

$$\omega^\alpha=\operatorname{curl}(\psi_1+\psi_2)\quad\text{and}\quad \psi_1+\psi_2=v-\nabla p\quad\text{for some } p,$$
$$\psi_1\in X^m\cap W^{2,\infty}_{co},\quad \nabla\psi_1\in W^{1,\infty}_{co},\quad \operatorname{div}\psi_1=0,\quad \psi_1\times n=0 \text{ on } \partial\Omega,$$
$$\psi_2\in H^m(\Omega).$$

Let $\varphi:\mathbb{R}_+\to[0,1]$ such that $\varphi(s)=1$ pour $s>1$ and $\varphi(s)=0$ for $s<1/2$. We define $\varphi_\varepsilon(x)=\varphi(d/\varepsilon)$ and $\psi_1^\varepsilon=\varphi_\varepsilon\psi_1$. Clearly $\psi_1^\varepsilon\to\psi_1$ in L^2 as $\varepsilon\to0$. Moreover, we claim that $\operatorname{curl}\psi_1^\varepsilon$ is bounded in $H^{m-1}_{co}\cap W^{1,\infty}_{co}$ uniformly in ε. To prove this, we start by writing

$$\operatorname{curl}\psi_1^\varepsilon=\varphi_\varepsilon\operatorname{curl}\psi_1-\psi_1\times\nabla\varphi_\varepsilon=\varphi_\varepsilon\operatorname{curl}\psi_1-\frac{1}{\varepsilon}\psi_1\times\nabla d\ \varphi'\Big(\frac{d}{\varepsilon}\Big).$$

We remark now that for every $k\in\mathbb{N}$ the functions φ_ε are bounded in $W^{k,\infty}_{co}$ uniformly in ε. Indeed, if ∂_Z is a tangential derivative, we have that

$$\partial_Z\varphi_\varepsilon=\frac{\partial_Z d}{\varepsilon}\varphi'\Big(\frac{d}{\varepsilon}\Big).$$

Since d vanishes on the boundary and ∂_Z is a tangential derivative we have that $\partial_Z d$ vanishes on the boundary. Because the support of $\varphi'(d/\varepsilon)$ is included in Ω_ε for ε sufficiently small, the mean value theorem implies that $|\partial_Z d|\leqslant C\varepsilon\|d\|_{W^{2,\infty}(\Omega_\delta)}$ on the support of $\varphi'(d/\varepsilon)$ (we assumed that ε is sufficiently small). So $\partial_Z\varphi_\varepsilon$ is uniformly bounded in ε and a similar argument works for the higher order tangential derivatives of φ_ε.

Since φ_ε is bounded in $W^{k,\infty}_{co}$ uniformly in ε and $\operatorname{curl}\psi_1\in H^{m-1}_{co}\cap W^{1,\infty}_{co}$, the Leibniz formula immediately implies that $\varphi_\varepsilon\operatorname{curl}\psi_1$ is bounded in $H^{m-1}_{co}\cap W^{1,\infty}_{co}$ uniformly in ε.

We remark next that since d is constant on the boundary, its gradient is normal to the boundary. But ψ_1 is also normal to the boundary, so $\psi_1 \times \nabla d$ vanishes on the boundary. We can therefore apply Lemma 6 to deduce that

$$\begin{aligned} C\|\frac{\psi_1 \times \nabla d}{d}\|_{H^{m-1}_{co}(\Omega_\varepsilon)\cap W^{1,\infty}_{co}(\Omega_\varepsilon)} &\leqslant C(\|\psi_1 \times \nabla d\|_{H^{m-1}_{co}\cap W^{1,\infty}_{co}} + \|\partial_n(\psi_1 \times \nabla d)\|_{H^{m-1}_{co}\cap W^{1,\infty}_{co}}) \\ &\leqslant C(\|\psi_1\|_{X^m} + \|\psi_1\|_{W^{1,\infty}_{co}} + \|\nabla \psi_1\|_{W^{1,\infty}_{co}}). \end{aligned}$$

As above, one can easily check that $\frac{d}{\varepsilon}\varphi'(\frac{d}{\varepsilon})$ is bounded independently of ε in any $W^{k,\infty}_{co}$. We conclude by the Leibniz formula that $\frac{1}{\varepsilon}\psi_1 \times \nabla d\ \varphi'(\frac{d}{\varepsilon}) = \frac{\psi_1 \times \nabla d}{d}\ \frac{d}{\varepsilon}\varphi'(\frac{d}{\varepsilon})$ is bounded independently of ε in $H^{m-1}_{co} \cap W^{1,\infty}_{co}$.

We infer from the previous relations that $\operatorname{curl}\psi_1^\varepsilon$ is bounded independently of ε in $H^{m-1}_{co} \cap W^{1,\infty}_{co}$. Next, since ψ_1^ε is compactly supported in Ω and belongs to X^m we infer that is also belongs to H^m. Regularizing ψ_1^ε by means of convolution with an approximation of the identity and letting $\varepsilon \to 0$ afterwards, one can construct a sequence ψ_1^n of vector fields such that $\psi_1^n \to \psi_1$ in L^2 and such that $\operatorname{curl}\psi_1^n$ is bounded in $H^{m-1}_{co} \cap W^{1,\infty}_{co}$.

Next, by density of smooth functions in H^m, there exists a sequence of smooth vector fields ψ_2^n such that $\psi_2^n \to \psi_2$ in H^m. Since $m \geqslant 4$ we have the Sobolev embedding $H^m \subset W^{2,\infty}$ so $\operatorname{curl}\psi_2^n$ is bounded in $H^{m-1} \cap W^{1,\infty}$. Let $v_n = \psi_1^n + \psi_2^n$. Then $v_n \to \psi_1 + \psi_2$ in L^2 and $\operatorname{curl} v_n$ is bounded in $H^{m-1}_{co} \cap W^{1,\infty}_{co}$. Let u_n be the solution of the following Stokes problem:

$$u_n - \alpha\Delta u_n = v_n + \nabla p_n, \quad \operatorname{div} u_n = 0, \quad u_n \text{ verifies the Navier boundary conditions (2).}$$

Since $\psi_1 + \psi_2 = v - \nabla p$, we observe that u verifies the following Stokes problem:

$$u - \alpha\Delta u = \psi_1 + \psi_2 + \nabla p, \quad \operatorname{div} u = 0, \quad u \text{ verifies the Navier boundary conditions (2).}$$

But regularity results for the above Stokes problem are known. We can deduce for instance from [5, Theorem 3] that $\|u_n - u\|_{H^2} \leqslant \|v_n - \psi_1 - \psi_2\|_{L^2} \to 0$. Since v_n is smooth, the same theorem also implies that u_n is smooth. Moreover, $\operatorname{curl}(u_n - \alpha\Delta u_n) = \operatorname{curl} v_n$ is bounded in $H^{m-1}_{co} \cap W^{1,\infty}_{co}$. One can also easily keep track of the estimates in the above arguments and deduce that relation (29) holds true for some constant C. The sequence u_n has all required properties and this completes the proof. $\square$

From the previous proposition, we deduce the existence of a sequence of smooth velocity fields u_0^n verifying the Navier boundary conditions such that $u_0^n \to u_0$ in H^2 and such that $\omega_0^{\alpha,n}$ is bounded in $H^{m-1}_{co} \cap W^{1,\infty}_{co}$. Using the result of [5], one can construct a local solution u^n with initial velocity u_0^n. This solution is smooth. Indeed, even though the result of [5] is stated only in H^3 it easily goes through to any H^m with $m \geqslant 3$. Moreover, the blow-up of the solution cannot occur while the Lipschitz norm of the solution is bounded. On these smooth solutions, the *a priori* estimates proved in the previous section are valid. In particular, we have a control of the Lipschitz norm of the solution on a time interval which depends only on $\|u_0^n\|_{L^2}$, $\|\omega_0^{\alpha,n}\|_{H^{m-1}_{co}}$ and $\|\omega_0^{\alpha,n}\|_{W^{1,\infty}_{co}}$. Since these quantities are bounded independently of n, the time existence of u^n has a lower bound independent of n. Finally, given that the solutions are bounded in the Lipschitz norm, passing to the limit as $n \to \infty$ on this time interval is quite simple and standard. This completes the proof of Theorem 1. As mentioned in the introduction, Theorem 2 is a direct consequence of Theorem 1 and of [4, Theorem 5]. Finally, to complete the proof of Theorem 3 a similar argument can be invoked provided that we can construct a sequence of smooth velocity fields with similar properties. This is performed in the next proposition.

Proposition 15. *Let $u \in X^m$, $m \geqslant 4$, be a divergence free vector field tangent to the boundary such that $\omega \in W^{1,\infty}_{co}$. There exists a sequence of smooth divergence free vector fields u_n tangent to the boundary such that $u_n \to u$ in L^2 and such that*

$$\|u_n\|_{X^m} + \|\omega_n\|_{W^{1,\infty}_{co}} \leqslant C(\|u\|_{X^m} + \|\omega\|_{W^{1,\infty}_{co}})$$

for some constant $C = C(m, \Omega)$.

Proof. Let u_α be the solution of the Stokes problem

$$u_\alpha - \alpha\Delta u_\alpha = u + \nabla p_\alpha, \quad \operatorname{div} u_\alpha = 0, \quad u_\alpha \text{ verifies the Navier boundary conditions (2).}$$

Multiplying the above PDE by $u - u_\alpha$ and integrating implies after an integration by parts that

$$\begin{aligned}\|u_\alpha - u\|_{L^2}^2 + 2\alpha\|D(u_\alpha - u)\|_{L^2}^2 = 2\alpha\int_\Omega D(u_\alpha - u)\cdot D(u) &\leqslant 2\alpha\|D(u_\alpha - u)\|_{L^2}\|D(u)\|_{L^2}\\ &\leqslant \alpha\|D(u_\alpha - u)\|_{L^2}^2 + \alpha\|D(u)\|_{L^2}^2.\end{aligned}$$

Therefore $u_\alpha \to u$ in L^2 as $\alpha \to 0$. Moreover, multiplying the equation of u_α by u_α immediately shows that

$$\|u_\alpha\|_{L^2} \leqslant \|u\|_{L^2}.$$

We observe now that $\operatorname{curl} u_\alpha - \alpha\Delta \operatorname{curl} u_\alpha = \omega \in H^{m-1}_{co} \cap W^{1,\infty}_{co}$, so one can apply Proposition 14 to construct an approximating sequence u^n_α such that

$$\begin{aligned}\|u^n_\alpha\|_{L^2} + \|\operatorname{curl}(u^n_\alpha - \alpha\Delta u^n_\alpha)\|_{H^{m-1}_{co}\cap W^{1,\infty}_{co}} &\leqslant C(\|u_\alpha\|_{L^2} + \|\operatorname{curl}(u_\alpha - \alpha\Delta u_\alpha)\|_{H^{m-1}_{co}\cap W^{1,\infty}_{co}})\\ &\leqslant C(\|u\|_{L^2} + \|\omega\|_{H^{m-1}_{co}\cap W^{1,\infty}_{co}}).\end{aligned}$$

Proposition 10 and Lemma 12 imply that

$$\|u^n_\alpha\|_{X^m} + \|\operatorname{curl} u^n_\alpha\|_{W^{1,\infty}_{co}} \leqslant C(\|u\|_{X^m} + \|\omega\|_{W^{1,\infty}_{co}}).$$

Because $u^n_\alpha \to u_\alpha$ in L^2 as $n \to \infty$ and $u_\alpha \to u$ in L^2 as $\alpha \to 0$, the conclusion follows immediately. □

7. A final remark

The aim of this paper was to prove that the hypothesis of [4, Theorem 5] (the existence of a uniform time of existence of solutions) is verified in dimension three. We would like to remark that [4, Theorem 5] is stated not only for the α-Euler equations, but more generally for the following second grade fluid equations:

$$\partial_t(u - \alpha\Delta u) - \nu\Delta u + u\cdot\nabla(u - \alpha\Delta u) + \sum_j (u - \alpha\Delta u)_j \nabla u_j = -\nabla p, \qquad \operatorname{div} u = 0. \tag{30}$$

More precisely, it was proved in [4, Theorem 5] that when both α, ν converge to 0, the solutions of the above equation converge towards the solution of the Euler equation provided that they exist on the same time interval. So the natural question to investigate would be the existence of the solutions of (30) on a time interval independent of α and ν. It is very easy to see that the answer is positive if $\nu \leqslant C\alpha$. Indeed, the vorticity equation can be written under the form

$$\partial_t\omega^\alpha + \frac{\nu}{\alpha}\omega^\alpha - \frac{\nu}{\alpha}\omega + u\cdot\nabla\omega^\alpha - \omega^\alpha\cdot\nabla u = 0.$$

If ν/α is bounded, the two additional terms are not worse than the others so similar estimates hold true giving the same results. We choose not to state this small improvement because we believe the result to be true without the condition $\nu \leqslant C\alpha$. Indeed, it is true when $\nu = 0$ by our result and also when $\alpha = 0$ by the result of [12] so it should also be true when $\alpha, \nu \to 0$ independently of the relative sizes. But proving it would be long and technical, although we presume doable. It would include both cases $\alpha = 0$ and $\nu = 0$ so the proof would need to encompass both our arguments and those of [12], which would make it very long and technically involved. In this case, it is not clear that the end justifies the means. Instead of stating a partial and obviously non-optimal result for the $\alpha, \nu \to 0$ limit, we preferred to prove a complete result for the limit $\alpha \to 0$.

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