ON THE SMALL RIGID BODY LIMIT IN 3D INCOMPRESSIBLE FLOWS

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Abstract. We consider the evolution of a small rigid body in an incompressible viscous fluid filling the whole space $\mathbb{R}^3$. The motion of the fluid is modelled by the Navier-Stokes equations, whereas the motion of the rigid body is described by the conservation law of linear and angular momentum. Under the assumption that the diameter of the rigid body tends to zero and that the density of the rigid body goes to infinity, we prove that the solution of the fluid-rigid body system converges to a solution of the Navier-Stokes equations in the full space without rigid body.

MSC classification: 35Q30, 74F10, 76D05.

1. Introduction and Statement of results

The motion of one or several rigid bodies in a liquid is a classical topic in fluid mechanics. We consider here the motion of an incompressible viscous fluid and a small smooth moving rigid body in the three-dimensional space. In this fluid-rigid body system, we will not take into account the gravity, and we suppose that the rigid body moves under the influence of the fluid. We study the asymptotic behavior of the fluid-rigid body system as the diameter of the rigid body tends to zero.

Assume that the whole three-dimensional space is occupied by an incompressible viscous fluid of viscosity $\nu > 0$ and by a rigid body of size ε. At the initial time, the domain of the rigid body $\mathcal{S}_0^\varepsilon$ is a small non-empty smooth compact simply-connected subset of $\mathbb{R}^3$ included in the ball $B(0,\varepsilon)$ and $\mathcal{F}_0^\varepsilon = \mathbb{R}^3 \setminus \mathcal{S}_0^\varepsilon$ is the domain of the fluid. We also denote by $\mathcal{S}^\varepsilon(t)$ the region occupied by the rigid body and by $\mathcal{F}^\varepsilon(t) = \mathbb{R}^3 \setminus \mathcal{S}^\varepsilon(t)$ the region occupied by the viscous fluid at time t.

In order to describe the motion of the rigid body, we need to specify its center of mass, that we denote by $h_\varepsilon(t)$ and a rotation matrix $\mathcal{R}(t) \in SO(3)$ which describes how the body rotates compared to the initial position. In other words, we have that

$$\mathcal{S}^\varepsilon(t) = \{x \in \mathbb{R}^3 \mid x = h_\varepsilon(t) + \mathcal{R}(t)x_0,\ x_0 \in \mathcal{S}_0^\varepsilon\}.$$

The velocity of the solid particle $x(t) = h_\varepsilon(t) + \mathcal{R}(t)x_0$ is given by

$$\begin{aligned} x'(t) &= h_\varepsilon'(t) + \mathcal{R}'(t)x_0 \\ &= h_\varepsilon'(t) + \mathcal{R}'(t)\mathcal{R}(t)^{-1}(x - h_\varepsilon(t)) \\ &= h_\varepsilon'(t) + \mathcal{R}'(t)\mathcal{R}(t)^{T}(x - h_\varepsilon(t)) \end{aligned}$$

where the superscript ^{T} denotes the transpose. Since $\mathcal{R}(t) \in SO(3)$, the matrix $\mathcal{R}'(t)\mathcal{R}(t)^T$ is skew-symmetric and can therefore be identified to a three-dimensional rotation vector $\omega_\varepsilon(t)$:

$$\mathcal{R}'(t)\mathcal{R}(t)^T z = \omega_\varepsilon(t) \times z, \quad z \in \mathbb{R}^3$$

where $\times$ denotes the standard cross product of vectors in $\mathbb{R}^3$. Therefore the velocity of the solid particle at x is given by

$$h'_\varepsilon(t) + \omega_\varepsilon(t) \times (x - h_\varepsilon(t)), \quad x \in \mathcal{S}^\varepsilon(t). \tag{1}$$

We assume that the rigid body is homogeneous of density ρ_ε. We denote its total mass by m_ε so that $m^\varepsilon = \rho_\varepsilon |\mathcal{S}_0^\varepsilon|$ where $|\mathcal{S}_0^\varepsilon|$ is the volume of the rigid body. We also introduce J^ε the matrix of inertia of the rigid body defined by

$$(J^\varepsilon a) \cdot b = \rho_\varepsilon \int_{\mathcal{S}^\varepsilon} \big(a \times (x - h_\varepsilon(t)\big) \cdot \big(b \times (x - h_\varepsilon(t)\big) dx, \text{ for any } a, b \in \mathbb{R}^3,$$

(see [11]).

We assume that the fluid is governed by the classical Navier-Stokes equations with no-slip boundary conditions on the boundary of the rigid body, and the dynamics of the rigid body is described by the equations of the balance of linear and angular momentum. We suppose that the fluid is homogeneous of constant density 1 to simplify the notations and we denote by $u_\varepsilon(t,x)$ the velocity of the fluid and by $p_\varepsilon(t,x)$ the pressure of the fluid. Moreover, we also denote by $\Sigma(u_\varepsilon, p_\varepsilon)$ the stress tensor of the fluid

$$\Sigma(u_\varepsilon, p_\varepsilon) = 2\nu D(u_\varepsilon) - p_\varepsilon I_3,$$

where I_3 is identity matrix of order 3 and $D(u_\varepsilon)$ is the deformation tensor

$$D(u_\varepsilon) := \frac{1}{2}\Big(\frac{\partial u_{\varepsilon,i}}{\partial x_j} + \frac{\partial u_{\varepsilon,j}}{\partial x_i}\Big)_{i,j} \quad i,j = 1,2,3. \tag{2}$$

With the notation introduced above, we have the following mathematical formulation for the fluid-rigid body system (see [5], [11] and [20]):

- Fluid equations:

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} + (u_\varepsilon \cdot \nabla) u_\varepsilon - \nu \Delta u_\varepsilon + \nabla p_\varepsilon &= 0 \text{ for } t \in (0, +\infty),\ x \in \mathcal{F}^\varepsilon(t). \\ \operatorname{div} u_\varepsilon &= 0 \text{ for } t \in (0, +\infty),\ x \in \mathcal{F}^\varepsilon(t). \end{aligned} \tag{3}$$

- Rigid body equations:

$$m^\varepsilon h''_\varepsilon(t) = -\int_{\partial \mathcal{S}^\varepsilon(t)} \Sigma(u_\varepsilon, p_\varepsilon) n_\varepsilon ds \quad \text{for } t \in (0, +\infty). \tag{4}$$

$$(J^\varepsilon \omega_\varepsilon)'(t) = -\int_{\partial \mathcal{S}^\varepsilon(t)} (x - h_\varepsilon) \times (\Sigma(u_\varepsilon, p_\varepsilon) n_\varepsilon) ds \quad \text{for } t \in (0, +\infty). \tag{5}$$

- Boundary conditions:

$$\begin{aligned} u_\varepsilon(t,x) = h'_\varepsilon(t) + \omega_\varepsilon(t) \times \big(x - h_\varepsilon(t)\big), &\text{ for } t \in (0, +\infty), x \in \partial \mathcal{S}^\varepsilon(t). \\ \lim_{|x| \to \infty} u_\varepsilon(t,x) &= 0 \quad \text{for } t \in [0, +\infty). \end{aligned} \tag{6}$$

In the above system, we have denoted by $n_\varepsilon(t,x)$ the unit normal vector to $\partial \mathcal{S}^\varepsilon$ pointing outside the fluid domain $\mathcal{F}^\varepsilon$. The first line in (6) is the Dirichlet boundary condition: the fluid velocity and the solid velocity must agree on the boundary of the body.

The system (3)-(6) should be completed by some initial conditions. As mentioned at the beginning, we assume that the initial position of the center of mass of the rigid body is in the origin. We denote by u_ε^0 the initial fluid velocity:

$$u_\varepsilon(0,x) = u_\varepsilon^0, \quad h_\varepsilon(0) = 0, \quad h'_\varepsilon(0) = l_\varepsilon^0, \quad \omega_\varepsilon(0) = \omega_\varepsilon^0. \tag{7}$$

The coupled system satisfies some L^2 energy estimates at least at the formal level. Taking the inner product of (3) with u_ε, integrating the result by parts and using the equations (4) and (5), we get the following energy estimate (see [11]):

$$\|u_\varepsilon(t)\|^2_{L^2(\mathcal{F}^\varepsilon(t))} + m^\varepsilon|h'_\varepsilon(t)|^2 + (J^\varepsilon\omega_\varepsilon(t))\cdot\omega_\varepsilon(t) + 4\nu\int_0^t \|D(u_\varepsilon)\|^2_{L^2(\mathcal{F}^\varepsilon(t))} \\ \leq \|u^0_\varepsilon\|^2_{L^2(\mathcal{F}^\varepsilon_0)} + m_\varepsilon|l^0_\varepsilon|^2 + \left(J^\varepsilon\omega^0_\varepsilon\right)\cdot\omega^0_\varepsilon. \tag{8}$$

The sum of the first three terms on the left-hand side of (8) is called the kinetic energy of the system at time t, while the forth term is called viscous dissipation. Obviously the initial kinetic energy is the right-hand side of (8).

Over the last few years, there were a lot of works dealing with the well-posedness of the fluid-rigid body system by using energy estimates. Both weak finite energy solutions (Leray solutions) and strong H^1 solutions were constructed. When the fluid is enclosed in a bounded region, the existence of solutions is proved under some constraints on the collisions between the rigid body and the boundary of the domain. When the domain of motion is the whole of $\mathbb{R}^3$ there is of course no such constraint. We give some references below but we would like to say that this list is not exhaustive. The existence of weak Leray solutions have been proved in [2], [6], [11] and [20] (see also the references therein). We refer to [3], [5] and [7] for results about strong H^1 solutions. The vanishing viscosity limit was considered in [22]. Let us also mention that the case of the dimension two was also considered in the literature, see for example [5], [13], [19] and [23].

The initial conditions should satisfy the following compatibility conditions (see [5]):

$$\begin{aligned} &u^0_\varepsilon \in L^2(\mathcal{F}^\varepsilon_0), \quad \operatorname{div} u^0_\varepsilon = 0 \ \text{ in } \ \mathcal{F}^\varepsilon_0, \\ &u^0_\varepsilon\cdot n_\varepsilon = (l^0_\varepsilon + \omega^0_\varepsilon\times x)\cdot n_\varepsilon \ \text{ on } \ \partial\mathcal{S}^\varepsilon_0. \end{aligned} \tag{9}$$

The second condition above is a weak version of the Dirichlet boundary condition in which only the normal components of the fluid velocity and of the solid velocity must agree on the boundary of the obstacle. This is in agreement with the usual theory of Leray solutions of the Navier-Stokes equations where the initial velocity is assumed to be only tangent to the boundary.

Before stating a result of existence of weak solutions for the motion of a rigid body in a fluid, let us introduce the global density and the global velocity, defined on the whole of $\mathbb{R}^3$. The fluid is homogeneous of constant density 1 while the rigid body is of density ρ_ε, so we can define the global density $\widetilde{\rho}_\varepsilon(t,x)$ as follows:

$$\widetilde{\rho}_\varepsilon(t,x) = \chi_{\mathcal{F}^\varepsilon(t)}(x) + \rho_\varepsilon\chi_{\mathcal{S}^\varepsilon(t)}(x), \ \text{ for } \ x\in\mathbb{R}^3$$

where we denote χ_A denotes the characteristic function of the set A. Moreover, recalling the formula for the velocity of the rigid body, see (1), one may define a global velocity $\widetilde{u}_\varepsilon$ by

$$\widetilde{u}_\varepsilon(t,x) = \begin{cases} u_\varepsilon(t,x) & \text{if } x\in\mathcal{F}^\varepsilon(t) \\ h'_\varepsilon(t) + \omega_\varepsilon(t)\times(x-h_\varepsilon(t)) & \text{if } x\in\mathcal{S}^\varepsilon(t). \end{cases}$$

Clearly, by conditions (9), we know that

$$\widetilde{u}^0_\varepsilon \in L^2(\mathbb{R}^3), \quad \operatorname{div}\widetilde{u}^0_\varepsilon = 0 \ \text{ in } \ \mathbb{R}^3.$$

Motivated by the energy estimates (8) and by the construction of $\widetilde{\rho}_\varepsilon$ and $\widetilde{u}_\varepsilon$ we introduce the following notion of weak solution (see [2], [6], [11] and [20]).

Definition 1. *A triplet* $(\widetilde{u}_\varepsilon, h_\varepsilon, \omega_\varepsilon)$ *is a weak Leray solution of* (3)–(7), *if*

- $\widetilde{u}_\varepsilon, h_\varepsilon, \omega_\varepsilon$ *satisfying*

$$h_\varepsilon \in W^{1,\infty}(\mathbb{R}_+;\mathbb{R}^3), \quad \omega_\varepsilon \in L^\infty(\mathbb{R}_+;\mathbb{R}^3),$$

$$u_\varepsilon \in L^\infty\left(\mathbb{R}_+; L^2(\mathcal{F}^\varepsilon)\right) \cap L^2_{loc}\left(\mathbb{R}_+; H^1(\mathcal{F}^\varepsilon)\right), \quad \widetilde{u}_\varepsilon \in C^0_w\left(\mathbb{R}_+; L^2(\mathbb{R}^3)\right);$$

- $\widetilde{u}_\varepsilon$ *is divergence free in the whole of* $\mathbb{R}^3$ *with* $D\widetilde{u}_\varepsilon(t,x) = 0$ *in* $\mathcal{S}^\varepsilon(t)$*;*
- $\widetilde{u}_\varepsilon$ *verifies the equation in the following sense:*

$$-\int_0^\infty \int_{\mathbb{R}^3} \widetilde{\rho}_\varepsilon \widetilde{u}_\varepsilon \cdot \left(\partial_t \varphi_\varepsilon + (\widetilde{u}_\varepsilon \cdot \nabla)\varphi_\varepsilon\right) + 2\nu \int_0^\infty \int_{\mathbb{R}^3} D(\widetilde{u}_\varepsilon) : D(\varphi_\varepsilon) = \int_{\mathbb{R}^3} \widetilde{\rho}_\varepsilon(0)\widetilde{u}^0_\varepsilon \cdot \varphi_\varepsilon(0).$$

for any test function $\varphi_\varepsilon \in W^{1,\infty}\left(\mathbb{R}_+; H^1_\sigma(\mathbb{R}^3)\right)$*, compactly supported in* $\mathbb{R}_+ \times \mathbb{R}^3$ *such that* $D\varphi_\varepsilon(t,x) = 0$ *in* $\mathcal{S}^\varepsilon(t)$*.*

One has the following result of existence of weak solutions of the initial-boundary value problem (3)–(7) in the sense defined above (see [2], [6], [11] and [20]).

Theorem 1. *Let* $\widetilde{u}^0_\varepsilon \in L^2(\mathbb{R}^3)$ *be divergence free and such that* $D\widetilde{u}^0_\varepsilon = 0$ *in* $\mathcal{S}^\varepsilon_0$*. Then, there exists at least one global weak solution* $(\widetilde{u}_\varepsilon, h_\varepsilon, \omega_\varepsilon)$ *of the initial-boundary value problem* (3)–(7) *in the sense of Definition 1. Moreover,* $\widetilde{u}_\varepsilon$ *satisfies the following energy estimate:*

$$\int_{\mathbb{R}^3} \widetilde{\rho}_\varepsilon |\widetilde{u}_\varepsilon(t)|^2 + 4\nu \int_0^t \int_{\mathbb{R}^3} |D(\widetilde{u}_\varepsilon)|^2 \leq \int_{\mathbb{R}^3} \widetilde{\rho}_\varepsilon(0)|\widetilde{u}^0_\varepsilon|^2 \quad \forall t > 0. \tag{10}$$

We now let $\varepsilon \to 0$ and wish to find the limit of the solution given in Theorem 1. Let us first review the literature available on related results.

In dimension two the literature is richer. Iftimie, Lopes Filho and Nussenzveig Lopes [15] proved convergence towards the Navier-Stokes equations in $\mathbb{R}^2$ in the case when the rigid body does not move. Lacave [16] considered the case of a thin obstacle tending to a curve. Dashti and Robinson [4] were the first to consider the case of an obstacle moving with the fluid, but it was observed in [1, Subsection 2.1] that the elliptic estimates of [4] are sometimes faulty. Recently, Lacave and Takahashi [18] considered a small disk moving under the influence of a two-dimensional viscous incompressible fluid. Under the condition that the density of the solid is independent of ε and assuming that the initial data is sufficiently small, they used the $L^p - L^q$ decay estimates of the semigroup associated to the fluid-rigid body system to deduce the convergence towards the solution of the Navier-Stokes equations in $\mathbb{R}^2$. In [12], the authors extended the result of [18] to the case of arbitrary shape of the body and with no restriction on the size of the initial data but assuming that the density of the obstacle is large.

In dimension three, Iftimie and Kelliher [14] considered the case of a fixed obstacle and proved convergence towards the Navier-Stokes equations in $\mathbb{R}^3$. Lacave [17] considered more general shrinking obstacles (for instance shrinking to a curve) but still fixed. When the obstacle is moving with the fluid, the limit $\varepsilon \to 0$ was considered in [21] in the case when the rigid body is a ball. Unfortunately, the elliptic estimates in that paper, see [21, Theorem 3.1], are not correct as was observed in [1, Subsection 2.1].

As far as we know, Theorem 2 below is the first result on the limit $\varepsilon \to 0$ in dimension three for a moving obstacle. We will essentially show that if the density of the rigid body goes to infinity, then the energy estimates are sufficient to pass to the limit in the weak formulation by using a truncation procedure. We obtain then the convergence of the solutions constructed in Theorem 1 to a solution of the Navier-Stokes equations in $\mathbb{R}^3$

under the assumption that the initial data $\widetilde{u}_\varepsilon^0$ is bounded in L^2. We do not need to impose any small data condition or any restriction on the shape of the body.

Let us now state the main result of this paper.

Theorem 2. *Let $\widetilde{u}_\varepsilon^0 \in L^2(\mathbb{R}^3)$ be divergence free and such that $D\widetilde{u}_\varepsilon^0 = 0$ in $\mathcal{S}_0^\varepsilon$. We assume that*

- *$\mathcal{S}_0^\varepsilon \subset B(0,\varepsilon)$;*
- *the mass m^ε of the rigid body satisfies that*

$$\frac{m^\varepsilon}{\varepsilon^3} \to \infty \quad \text{as } \varepsilon \to 0; \tag{11}$$

- *$\widetilde{u}_\varepsilon^0$ converges weakly in $L^2(\mathbb{R}^3)$ towards some u_0.*
- *$\sqrt{m^\varepsilon} h_\varepsilon'(0)$ and $\left(J^\varepsilon \omega_\varepsilon^0\right) \cdot \omega_\varepsilon^0$ are bounded uniformly in ε.*

Let $(\widetilde{u}_\varepsilon, h_\varepsilon, \omega_\varepsilon)$ be the global solution of the system (3)–(7) given by Theorem 1. Then there exists a subsequence $\widetilde{u}_{\varepsilon_k}$ of $\widetilde{u}_\varepsilon$ which converges

$$\widetilde{u}_{\varepsilon_k} \rightharpoonup u \quad \text{weak*} \text{ in } L^\infty\left(\mathbb{R}_+; L^2(\mathbb{R}^3)\right) \text{ and weakly in } L^2_{loc}\left(\mathbb{R}_+; H^1(\mathbb{R}^3)\right)$$

towards a solution u of the Navier-Stokes equations in $\mathbb{R}^3$ in the sense of distributions with initial data $u_0(x)$.

Moreover, suppose in addition that $\widetilde{u}_\varepsilon^0$ converges strongly in $L^2(\mathbb{R}^3)$ to $u_0(x)$ and that both $\sqrt{m^\varepsilon} h_\varepsilon'(0)$ and $\left(J^\varepsilon \omega_\varepsilon^0\right) \cdot \omega_\varepsilon^0$ converge to 0 as $\varepsilon \to 0$. Then the limit solution u satisfies the energy estimate

$$\forall t \geq 0 \quad \|u(t)\|^2_{L^2(\mathbb{R}^3)} + 4\nu \int_0^t \|D(u)\|^2_{L^2(\mathbb{R}^3)} \leq \|u_0\|^2_{L^2(\mathbb{R}^3)}.$$

Let us give a few remarks on the hypotheses of the Theorem above. First, if the rigid body shrinks isotropically to a point, then hypothesis (11) means that the density ρ_ε of the rigid body tends to infinity as $\varepsilon \to 0$. If the rigid body does not shrink isotropically to a point, then condition (11) is stronger than simply saying that the density of the rigid body goes to infinity. Indeed, since $\mathcal{S}_0^\varepsilon \subset B(0,\varepsilon)$ we have that $|\mathcal{S}_0^\varepsilon| \leq \frac{4\pi}{3}\varepsilon^3$ so $\rho_\varepsilon = \frac{m^\varepsilon}{|\mathcal{S}_0^\varepsilon|} \geq \frac{3m^\varepsilon}{4\pi\varepsilon^3} \to \infty$ as $\varepsilon \to 0$.

Next, the weak convergence of $\widetilde{u}_\varepsilon^0$ in $L^2(\mathbb{R}^3)$ implies its boundedness in $L^2(\mathbb{R}^3)$. Together with the hypothesis that $\sqrt{m^\varepsilon} h_\varepsilon'(0)$ and $\left(J^\varepsilon \omega_\varepsilon^0\right) \cdot \omega_\varepsilon^0$ are bounded uniformly in ε this implies that the right-hand side of (8) is bounded. Then (10) implies that $\sqrt{\widetilde{\rho}_\varepsilon}\widetilde{u}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3))$ and $D(\widetilde{u}_\varepsilon)$ is bounded in $L^2(\mathbb{R}_+; L^2(\mathbb{R}^3))$. We observed above that $\rho_\varepsilon \to \infty$ so we can assume that $\rho_\varepsilon \geq 1$. Then we have that $\widetilde{\rho}_\varepsilon \geq 1$ (recall that $\widetilde{\rho}_\varepsilon = 1$ in the fluid region and $\widetilde{\rho}_\varepsilon = \rho_\varepsilon$ in the solid region) so $\widetilde{u}_\varepsilon$ is bounded in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3))$. We infer that

$$\widetilde{u}_\varepsilon(t,x) \text{ is bounded in } L^\infty\left(\mathbb{R}_+; L^2(\mathbb{R}^3)\right) \cap L^2_{\text{loc}}\left(\mathbb{R}_+; H^1(\mathbb{R}^3)\right) \tag{12}$$

and we will see that this is all we need to pass to the limit in our PDE. We require neither the Dirichlet boundary conditions nor the special form of $\widetilde{u}_\varepsilon$ inside the rigid body. All we need is the above boundedness and the fact that the Navier-Stokes equations are satisfied in the exterior of the ball $B(h_\varepsilon(t), \varepsilon)$. More precisely, we can prove the following more general statement.

Theorem 3. *Let $v_\varepsilon(t,x)$ be a divergence free vector field bounded independently of ε in*

$$L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^3)) \cap C^0_w(\mathbb{R}_+; L^2(\mathbb{R}^3)).$$

We make the following assumptions:

- *The vector field v_ε verifies the Navier-Stokes equations*

$$\partial_t v_\varepsilon - \nu\Delta v_\varepsilon + v_\varepsilon\cdot\nabla v_\varepsilon = -\nabla\pi_\varepsilon \tag{13}$$

in the exterior of the ball $B(h_\varepsilon(t),\varepsilon)$ with initial data $v_\varepsilon(0,x)$ in the following sense:

$$-\int_0^\infty\int_{\mathbb{R}^3} v_\varepsilon\cdot\partial_t\varphi + \nu\int_0^\infty\int_{\mathbb{R}^3}\nabla v_\varepsilon : \nabla\varphi + \int_0^T\int_{\mathbb{R}^3} v_\varepsilon\cdot\nabla v_\varepsilon\cdot\varphi = \int_{\mathbb{R}^3} v_\varepsilon(0)\cdot\varphi(0)$$

for every test function $\varphi\in W^{1,\infty}\big(\mathbb{R}_+\times\mathbb{R}^3\big)$ which is divergence free, compactly supported in $\mathbb{R}_+\times\mathbb{R}^3$ and such that for all t the function $x\mapsto\varphi(t,x)$ is smooth and compactly supported in the set $\{|x-h_\varepsilon(t)|>\varepsilon\}$.

- *The initial data $v_\varepsilon(0,x)$ is divergence free, square integrable and converges weakly to some $v_0(x)$ in $L^2(\mathbb{R}^3)$.*
- *The center of the ball verifies $h_\varepsilon\in W^{1,\infty}(\mathbb{R}_+;\mathbb{R}^3)$ and $\varepsilon^{\frac32}h_\varepsilon'(t)\to 0$ strongly in $L^\infty_{loc}(\mathbb{R}_+)$ when $\varepsilon\to 0$.*

Then there exists a subsequence of v_ε which converges weak$*$ in $L^\infty_{loc}(\mathbb{R}_+;L^2(\mathbb{R}^3))$ and weakly in $L^2_{loc}(\mathbb{R}_+;H^1(\mathbb{R}^3))$ to a solution v of the Navier-Stokes equations in $\mathbb{R}^3$ in the sense of distributions with initial data $v_0(x)$.

Moreover, if we assume in addition that $v_\varepsilon(0,x)$ converges strongly in L^2 to $v_0(x)$ and that the following energy estimate holds true for v_ε

$$\forall t\geq 0\quad \|v_\varepsilon(t)\|^2_{L^2(\mathbb{R}^3\setminus B(h_\varepsilon(t),\varepsilon))}+4\nu\int_0^t\|D(v_\varepsilon)\|^2_{L^2(\mathbb{R}^3\setminus B(h_\varepsilon(t),\varepsilon))}\leq\|v_\varepsilon(0)\|^2_{L^2(\mathbb{R}^3)}+o(1) \tag{14}$$

as $\varepsilon\to 0$, then the limit solution v satisfies the following energy estimate

$$\forall t\geq 0\quad \|v(t)\|^2_{L^2(\mathbb{R}^3)}+4\nu\int_0^t\|D(v)\|^2_{L^2(\mathbb{R}^3)}\leq\|v_0\|^2_{L^2(\mathbb{R}^3)}\,. \tag{15}$$

Let us observe that Theorem 2 follows from Theorem 3 applied for $v_\varepsilon(t,x)=\widetilde{u}_\varepsilon(t,x)$. Indeed, from (12) we have that v_ε is bounded in $L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^3))\cap L^2_{\rm loc}(\mathbb{R}_+;H^1(\mathbb{R}^3))$. Next we obviously have that $\widetilde{u}_\varepsilon$ verifies the Navier-Stokes equations in the exterior of the ball $B(h_\varepsilon(t),\varepsilon)$ and so does v_ε. We observed that the right-hand side of (8) is bounded so $\sqrt{m^\varepsilon}h_\varepsilon'$ is bounded. From (11) we infer that $\varepsilon^{\frac32}h_\varepsilon'(t)\to 0$ strongly in $L^\infty(\mathbb{R}_+)$ when $\varepsilon\to 0$. Then all the hypothesis of the first part of Theorem 3 is verified and the first part of Theorem 2 follows.

Let us now assume in addition the L^2 strong convergence of $\widetilde{u}^0_\varepsilon$ towards u_0 and let us prove (14). We have the L^2 strong convergence of $v_\varepsilon(0,x)$ to v_0. Recalling that the matrix of inertia is non-negative we can ignore the second and the third terms in (8) to estimate

$$\begin{aligned}
&\|v_\varepsilon(t)\|^2_{L^2(\mathbb{R}^3\setminus B(h_\varepsilon(t),\varepsilon))}+4\nu\int_0^t\|D(v_\varepsilon)\|^2_{L^2(\mathbb{R}^3\setminus B(h_\varepsilon(t),\varepsilon))}\\
&\qquad=\|u_\varepsilon(t)\|^2_{L^2(\mathbb{R}^3\setminus B(h_\varepsilon(t),\varepsilon))}+4\nu\int_0^t\|D(u_\varepsilon)\|^2_{L^2(\mathbb{R}^3\setminus B(h_\varepsilon(t),\varepsilon))}\\
&\qquad\leq\|u_\varepsilon(t)\|^2_{L^2(\mathcal{F}^\varepsilon(t))}+4\nu\int_0^t\|D(u_\varepsilon)\|^2_{L^2(\mathcal{F}^\varepsilon(t))}\\
&\qquad\leq\|u^0_\varepsilon\|^2_{L^2(\mathcal{F}^\varepsilon(t))}+m_\varepsilon|l^0_\varepsilon|^2+\big(J^\varepsilon\omega^0_\varepsilon\big)\cdot\omega^0_\varepsilon\\
&\qquad\leq\|v_\varepsilon(0)\|^2_{L^2(\mathbb{R}^3)}+m_\varepsilon|l^0_\varepsilon|^2+\big(J^\varepsilon\omega^0_\varepsilon\big)\cdot\omega^0_\varepsilon.
\end{aligned}$$

By hypothesis $m_\varepsilon|l^0_\varepsilon|^2+\big(J^\varepsilon\omega^0_\varepsilon\big)\cdot\omega^0_\varepsilon\to 0$ so (14) follows.

The passing to the limit stated in Theorem 3 uses the boundedness of v_ε in the energy space $L^\infty\left(\mathbb{R}_+; L^2(\mathbb{R}^3)\right) \cap L^2_{\text{loc}}\left(\mathbb{R}_+; H^1(\mathbb{R}^3)\right)$ and the construction of a cut-off φ_ε supported in the exterior of the ball $\overline{B}(h_\varepsilon(t), \varepsilon)$. We multiply (13) with the cut-off φ_ε, and then pass to the limit by means of classical compactness methods. The main obstruction is that, when the rigid body moves under the influence of the fluid, not only the velocity depends on time, but also the cut-off function. Time derivative estimates for v_ε are not easy to obtain and, once obtained, it is not easy to pass to the limit in the term with the time derivative.

Let us comment now on the hypothesis of large density assumed in Theorem 2. This hypothesis is not very satisfactory, it would be more natural to assume that the density of the obstacle is constant like in the paper of Lacave and Takahashi [18] in dimension two. That would allow for instance to include the case of "tracer particles" to follow a fluid flow. Unfortunately we are forced to assume here that the density of the obstacle is large. But if we look at the proof we realize that the large density hypothesis is not really necessary. What we require is a bound for the velocity of the obstacle, namely $h'_\varepsilon = o(\varepsilon^{-\frac{3}{2}})$. The large density hypothesis is only a mean to obtain this bound from the energy estimates (8) (see the paragraph following Theorem 3). In dimension two, Lacave and Takahashi [18] were able to use some estimates for the Stokes-rigid body semigroup in order to derive estimates for h_ε without resorting to the energy estimates. But in dimension three we were not able to find another way to bound the velocity of the obstacle.

Another issue which is definitely worth investigating is to find the limit of h_ε, the position of the obstacle. The first step in doing this would require some estimates for h_ε independent of ε. Here we only rely on energy estimates and the energy estimates (8) give a bound on $m^\varepsilon |h'_\varepsilon(t)|^2$. So, if we want to obtain estimates for h_ε uniform in ε we need the mass of the particle to be bounded from below by a constant independent of ε. Therefore the density needs to be larger than $C\varepsilon^{-3}$, which is considerably more restrictive than just saying that the density should go to infinity. So our method cannot be used to find the limit of the position of the obstacle. Let us observe that the semigroup method of [18] allows to find uniform estimates for h_ε in dimension two in the case of constant density. However, even [18] does not find the limit of h_ε. Finally, we remark that in the inviscid case in dimension two one can find an equation for the limit of h_ε in various situations, see for instance [8], [9] and [10]. This is due to the particular form (transport equation) of the equation of the vorticity which implies many conservation laws for the vorticity. This is not available neither for the Navier-Stokes equations, nor for the case of the dimension three.

The rest of the paper is organized as follows. In Section 2, we introduce some notation and present some preliminary results. The construction of the cut-off near the rigid body is given in Section 3. We show the strong convergence by means of temporal estimates in Section 4 and pass to the limit to conclude our proof in Section 5.

2. Notation and Preliminary results

In this section, we will introduce some notations and preliminary results.

For a sufficiently regular vector field $u : \mathbb{R}^3 \to \mathbb{R}^3$, we denote by ∇u the second order tensor field whose components $(\nabla u)_{ij}$ are given by $\partial u_j / \partial x_i$, and by $D(u)$ the symmetric part of ∇u (see (2)). The double dot product $M : N$ of two matrices $M = (m_{ij})$ and $N = (n_{ij})$ denotes the quantity $\sum_{i,j} m_{ij} n_{ij}$.

For function spaces, we shall use standard notations L^p and H^m to denote the usual Lebesgue and Sobolev spaces. C^m_b denotes the set of bounded functions whose first m

derivatives are bounded functions. We add subscripts 0 and σ to these spaces to specify that their elements are compactly supported and divergence free, respectively. For instance, the notation $C^\infty_{0,\sigma}$ defines the space of smooth, compactly supported and divergence free vector fields on $\mathbb{R}^3$.

In addition, unless we specify the domain, all function spaces and norms are considered to be taken on $\mathbb{R}^3$ in the x variable. For the t variable, we use the notation $\mathbb{R}_+ = [0,\infty)$ and emphasize that the endpoint 0 belongs to $\mathbb{R}_+$. Throughout this article, we denote by C a generic constant whose value can change from one line to another.

Let $\varphi \in C^1_b(\mathbb{R}_+; C^\infty_{0,\sigma})$. The stream function ψ of φ is defined by the following formula:

$$\psi(x) = -\int_{\mathbb{R}^3} \frac{x-y}{4\pi|x-y|^3} \times \varphi(y)dy$$

where $\times$ denotes the standard cross product of vectors in $\mathbb{R}^3$.

Because φ is divergence free, we have that $\operatorname{curl}\psi = \varphi$ and $\psi = \operatorname{curl}\Delta^{-1}\varphi$. Furthermore, ψ is smooth, $\psi \in C^1_b(\mathbb{R}_+; C^\infty)$, and vanishes at infinity. Moreover, we have the following well-known estimate:

$$\|\nabla\psi(t,\cdot)\|_{H^2} \le C\,\|\varphi(t,\cdot)\|_{H^2}\,. \tag{16}$$

In our case, in order to deal with the singularity in h_ε, we need to have a stream function vanishing in h_ε. The stream function ψ defined above has no reason to vanish in h_ε, so we are led to introduce a modified stream function ψ_ε. We define

$$\psi_\varepsilon(t,x) = \psi(t,x) - \psi(t,h_\varepsilon(t)). \tag{17}$$

Clearly $\psi_\varepsilon(t,h_\varepsilon(t)) = 0$. We collect in the following lemma some useful properties of the modified stream function.

Lemma 1. *Let $\varphi \in C^1_b(\mathbb{R}_+; C^\infty_{0,\sigma})$ and define the modified stream function ψ_ε as in* (17). *We have that:*

(i) $\psi_\varepsilon \in W^{1,\infty}(\mathbb{R}_+; C^\infty)$ *and* $\operatorname{curl}\psi_\varepsilon = \varphi$.

(ii) There exists a universal constant $C > 0$ such that for all $R > 0$ we have that

$$\|\psi_\varepsilon(t,\cdot)\|_{L^\infty(B(h_\varepsilon(t),R))} \le CR\,\|\varphi(t,\cdot)\|_{H^2} \tag{18}$$

for all $t \ge 0$ and

$$\|\partial_t\psi_\varepsilon(t,\cdot)\|_{L^\infty(B(h_\varepsilon(t),R))} \le C\left(R\,\|\partial_t\varphi(t,\cdot)\|_{H^2} + |h'_\varepsilon(t)|\,\|\varphi(t,\cdot)\|_{H^2}\right)$$

for almost all $t \ge 0$.

Proof. We observe first that $\operatorname{curl}\psi_\varepsilon = \operatorname{curl}\psi = \varphi$. Moreover, $\psi \in C^1_b(\mathbb{R}_+; C^\infty)$ and $h_\varepsilon \in W^{1,\infty}(\mathbb{R}_+)$ imply that $\psi_\varepsilon \in W^{1,\infty}(\mathbb{R}_+; C^\infty)$. This proves part *(i)*.

Next, to prove *(ii)* we use the mean value theorem to estimate

$$\begin{aligned}
\|\psi_\varepsilon(t,x)\|_{L^\infty(B(h_\varepsilon(t),R))} &= \|\psi(t,x) - \psi(t,h_\varepsilon(t))\|_{L^\infty(B(h_\varepsilon(t),R))} \\
&\le |x - h_\varepsilon(t)|\,\|\nabla\psi(t,x)\|_{L^\infty(B(h_\varepsilon(t),R))} \\
&\le R\,\|\nabla\psi\|_{H^2} \\
&\le CR\,\|\varphi\|_{H^2}
\end{aligned}$$

where we used relation (16). This proves (18).

We recall now that h_ε is Lipschitz in time so it is almost everywhere differentiable in time. Let t be a time where h_ε is differentiable. We write

$$\begin{aligned}\partial_t\psi_\varepsilon(t,x) &= \partial_t(\psi(t,x)-\psi(t,h_\varepsilon(t)))\\ &= \partial_t\psi(t,x)-\partial_t\psi(t,h_\varepsilon(t))-h_\varepsilon'(t)\cdot\nabla\psi(t,h_\varepsilon(t)).\end{aligned}$$

We can bound

$$\begin{aligned}\|\partial_t\psi_\varepsilon(t,\cdot)\|_{L^\infty(B(h_\varepsilon(t),R))} &\le \|\partial_t\psi(t,x)-\partial_t\psi(t,h_\varepsilon(t))\|_{L^\infty(B(h_\varepsilon(t),R))}+|h_\varepsilon'(t)|\,\|\nabla\psi(t,\cdot)\|_{L^\infty}\\ &\le |x-h_\varepsilon(t)|\,\|\partial_t\nabla\psi(t,\cdot)\|_{L^\infty}+|h_\varepsilon'(t)|\,\|\nabla\psi(t,\cdot)\|_{L^\infty}\\ &\le C\left(R\,\|\partial_t\nabla\psi(t,\cdot)\|_{H^2}+|h_\varepsilon'(t)|\,\|\nabla\psi(t,\cdot)\|_{H^2}\right)\\ &\le C\left(R\,\|\partial_t\varphi(t,\cdot)\|_{H^2}+|h_\varepsilon'(t)|\,\|\varphi(t,\cdot)\|_{H^2}\right).\end{aligned}$$

This completes the proof of the lemma. □

3. Cut-off near the rigid body

In this section, we will construct a cut-off φ_ε near the rigid body, which will be used as a test function in the procedure of passing to the limit in Section 5.

Firstly, we construct a cut-off function $\eta_\varepsilon(t,x)$ near the ball $B(h_\varepsilon(t),\varepsilon)$. Let $\eta(x)\in C^\infty(\mathbb{R}^3;[0,1])$ be a function such that

$$\eta(x):\mathbb{R}^3\to[0,1],\quad \eta(x)=\begin{cases}0 & \text{if } |x|\le\frac{3}{2}\\ 1 & \text{if } |x|\ge 2\end{cases}$$

The function $\eta(x)$ is a cut-off function in the neighborhood of the unit ball $B(0,1)$. A cut-off $\eta_\varepsilon(t,x)$ in the neighborhood of the domain $B(h_\varepsilon(t),\varepsilon)$ is the following function

$$\eta_\varepsilon(t,x)=\eta\left(\frac{x-h_\varepsilon(t)}{\varepsilon}\right)=\begin{cases}0 & \text{if } |x-h_\varepsilon(t)|\le\frac{3}{2}\varepsilon\\ 1 & \text{if } |x-h_\varepsilon(t)|\ge 2\varepsilon.\end{cases}. \tag{19}$$

Notice that $\eta_\varepsilon(t,x)$ is a space-time function while the function $\eta(x)$ only has a space variable. We state some properties of this new cut-off in the following lemma.

Lemma 2. *The cut-off function η_ε satisfies*

(i) $\eta_\varepsilon\in W^{1,\infty}(\mathbb{R}_+;C^\infty)$;
(ii) η_ε *vanishes in the neighborhood of the ball* $B(h_\varepsilon(t),\varepsilon)$;
(iii) For any real number $q\ge 1$ *there exists a constant* $C=C(q)$ *such that*

$$\|\eta_\varepsilon(t,\cdot)\|_{L^\infty}=1,\quad \|\eta_\varepsilon(t,\cdot)-1\|_{L^q}\le C\varepsilon^{\frac{3}{q}},\quad \|\nabla\eta_\varepsilon(t,\cdot)\|_{L^q}\le C\varepsilon^{\frac{3-q}{q}},\quad \left\|\nabla^2\eta_\varepsilon(t,\cdot)\right\|_{L^q}\le C\varepsilon^{\frac{3-2q}{q}}.$$

Proof. Since h_ε is Lipschitz part *(i)* follows immediately. Part *(ii)* is also obvious. We prove now part *(iii)*.

Clearly $\|\eta_\varepsilon(t,\cdot)\|_{L^\infty}=1$. Next

$$\|\eta_\varepsilon(t,x)-1\|_{L^q}=\left\|\eta\left(\frac{x-h_\varepsilon(t)}{\varepsilon}\right)-1\right\|_{L^q}=\varepsilon^{\frac{3}{q}}\,\|\eta(x)-1\|_{L^q}\le C\varepsilon^{\frac{3}{q}}.$$

Notice that $\nabla\eta(x)$ and $\nabla^2\eta(x)$ are bounded functions supported in the annulus $\{\frac{3}{2} < |x| < 2\}$. So

$$\|\nabla\eta_\varepsilon(t,x)\|_{L^q} = \left\|\frac{1}{\varepsilon}\nabla\eta\left(\frac{x-h_\varepsilon(t)}{\varepsilon}\right)\right\|_{L^q} = \varepsilon^{\frac{3-q}{q}}\|\nabla\eta(x)\|_{L^q} \le C\varepsilon^{\frac{3-q}{q}},$$

$$\left\|\nabla^2\eta_\varepsilon(t,x)\right\|_{L^q} = \left\|\frac{1}{\varepsilon^2}\nabla^2\eta\left(\frac{x-h_\varepsilon(t)}{\varepsilon}\right)\right\|_{L^q} = \varepsilon^{\frac{3-2q}{q}}\left\|\nabla^2\eta(x)\right\|_{L^q} \le C\varepsilon^{\frac{3-2q}{q}}.$$

This completes the proof of the lemma. □

Given a test function $\varphi \in C^1_b(\mathbb{R}_+; C^\infty_{0,\sigma})$, we use the cutoff η_ε and the modified stream function ψ_ε defined in Section 2 (see relation (17)) to construct a new test function φ_ε which vanishes in the neighborhood of the ball $B(h_\varepsilon(t),\varepsilon)$. We define

$$\varphi_\varepsilon = \operatorname{curl}(\eta_\varepsilon\psi_\varepsilon). \tag{20}$$

We notice that this new test function φ_ε depends on time even if φ is assumed to be constant in time. We state some properties of φ_ε in the following lemma:

Lemma 3. *The test function φ_ε has the following properties:*

(i) $\varphi_\varepsilon \in W^{1,\infty}(\mathbb{R}_+; C^\infty_{0,\sigma})$ and φ_ε vanishes in the neighborhood of $B(h_\varepsilon(t),\varepsilon)$;

(ii) for all $T > 0$ we have that $\varphi_\varepsilon \to \varphi$ strongly in $L^\infty(0,T;H^1)$ as $\varepsilon \to 0$;

(iii) there exists a universal constant C such that for all $T > 0$

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;H^1)} \le C\|\varphi\|_{L^\infty(0,T;H^2)}.$$

Proof. The various norms used below are in the x variable unless otherwise stated.

Clearly, η_ε and ψ_ε are $W^{1,\infty}$ in time and smooth in space, so φ_ε has the same properties. The function φ_ε is a curl so it is divergence free. Because η_ε vanishes in the neighborhood of $B(h_\varepsilon(t),\varepsilon)$, so is φ_ε. The compact support in space follows immediately once we recall that $\operatorname{curl}\psi_\varepsilon = \varphi$ and observe that

$$\varphi_\varepsilon = \operatorname{curl}(\eta_\varepsilon\psi_\varepsilon) = \eta_\varepsilon\varphi + \nabla\eta_\varepsilon \times \psi_\varepsilon$$

Claim *(i)* is proved.

To prove *(ii)*, we observe that $\operatorname{supp}\nabla\eta_\varepsilon \subset \{|x-h_\varepsilon(t)| \le 2\varepsilon\}$ and we estimate

$$\begin{aligned}
\|\varphi_\varepsilon - \varphi\|_{L^2} &\le \|(\eta_\varepsilon - 1)\varphi\|_{L^2} + \|\nabla\eta_\varepsilon \times \psi_\varepsilon\|_{L^2} \\
&\le \|\eta_\varepsilon - 1\|_{L^2}\|\varphi\|_{L^\infty} + \|\nabla\eta_\varepsilon\|_{L^2}\|\psi_\varepsilon\|_{L^\infty(B(h_\varepsilon(t),2\varepsilon))} \\
&\le C\left(\varepsilon^{\frac{3}{2}}\|\varphi\|_{L^\infty} + \varepsilon^{\frac{3}{2}}\|\varphi\|_{H^2}\right) \\
&\le C\varepsilon^{\frac{3}{2}}\|\varphi\|_{H^2}
\end{aligned}$$

where we used Lemmas 1 and 2.

Taking the supremum on $[0,T]$ we infer that

$$\|\varphi_\varepsilon - \varphi\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{\frac{3}{2}}\|\varphi\|_{L^\infty(0,T;H^2)}. \tag{21}$$

Next, by the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow W^{1,6}(\mathbb{R}^3)$ and by Lemma 2 we estimate

$$\begin{aligned}
\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^2} &= \left\|\nabla\big((\eta_\varepsilon - 1)\varphi + \nabla\eta_\varepsilon \times \psi_\varepsilon\big)\right\|_{L^2} \\
&\le \|\eta_\varepsilon - 1\|_{L^3}\|\nabla\varphi\|_{L^6} + \|\nabla\eta_\varepsilon\|_{L^2}\|\varphi\|_{L^\infty} + \|\nabla\eta_\varepsilon\|_{L^2}\|\nabla\psi_\varepsilon\|_{L^\infty} \\
&\qquad + \left\|\nabla^2\eta_\varepsilon\right\|_{L^2}\|\psi_\varepsilon\|_{L^\infty(B(h_\varepsilon(t),2\varepsilon))} \\
&\le C\left(\varepsilon\|\varphi\|_{W^{1,6}} + \varepsilon^{\frac{1}{2}}\|\varphi\|_{L^\infty} + \varepsilon^{\frac{1}{2}}\|\nabla\psi_\varepsilon\|_{L^\infty} + C\varepsilon^{-\frac{1}{2}}\|\psi_\varepsilon\|_{L^\infty(B(h_\varepsilon(t),2\varepsilon))}\right).
\end{aligned}$$

From relations (17) and (16) we get that

$$\|\nabla\psi_\varepsilon\|_{L^\infty} = \|\nabla\psi\|_{L^\infty} \le C\,\|\nabla\psi\|_{H^2} \le C\|\varphi\|_{H^2}.$$

From Lemma 1 we have that

$$\|\psi_\varepsilon\|_{L^\infty(B(h_\varepsilon(t),2\varepsilon))} \le C\varepsilon\|\varphi\|_{H^2}.$$

We conclude from the above relations that

$$\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^2} \le C\varepsilon^{\frac{1}{2}}\|\varphi\|_{H^2}.$$

Taking the supremum on $[0,T]$ we deduce that

$$\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^\infty(0,T;L^2)} \le C\varepsilon^{\frac{1}{2}}\,\|\varphi\|_{L^\infty(0,T;H^2)}\,. \tag{22}$$

We conclude form (21) and (22) that

$$\|\varphi_\varepsilon - \varphi\|_{L^\infty(0,T;H^1)} \le C\varepsilon^{\frac{1}{2}}\,\|\varphi\|_{L^\infty(0,T;H^2)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This proves *(ii)*. To prove *(iii)* we simply bound

$$\|\varphi_\varepsilon\|_{L^\infty(0,T;H^1)} \le \|\varphi\|_{L^\infty(0,T;H^1)} + \|\varphi_\varepsilon - \varphi\|_{L^\infty(0,T;H^1)} \le C\|\varphi\|_{L^\infty(0,T;H^2)}.$$

This completes the proof of the lemma. □

4. Temporal estimate and strong convergence

The aim of this section is to derive a temporal estimate and to prove the strong convergence of some sub-sequence of v_ε in $L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3)$. We will prove the following result.

Proposition 1. *There exists a sub-sequence v_{ε_k} of v_ε which converges strongly in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R}^3)$.*

It suffices to prove that for any $T > 0$ there exists a sub-sequence v_{ε_k} of v_ε which converges strongly in $L^2(0,T;L^2_{\text{loc}}(\mathbb{R}^3))$. A diagonal extraction then allows to choose the same subsequence for all times T. We choose some finite time T and for the rest of this section we assume that $t \in [0,T]$.

The main idea is to use the Arzelà–Ascoli theorem. Let $\varphi \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ be a test function which does not depend on the time. By the definition of the modified stream function, we observe that even if φ is constant in time, $\psi_\varepsilon(t,x)$ still depends on the time through $h_\varepsilon(t)$. We construct a family of φ_ε as in Section 3, so that φ_ε is time-dependent and satisfies Lemma 3.

We first bound

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi_\varepsilon(t,x)\,dx\Big| &= \Big|\int_{\mathbb{R}^3} v_\varepsilon\cdot\operatorname{curl}(\eta_\varepsilon\psi_\varepsilon)\,dx\Big| \\
&= \Big|\int_{\mathbb{R}^3} v_\varepsilon\cdot(\eta_\varepsilon\varphi + \nabla\eta_\varepsilon\times\psi_\varepsilon)\,dx\Big| \\
&\le \|v_\varepsilon\|_{L^2}\,\|\eta_\varepsilon\|_{L^\infty}\,\|\varphi\|_{L^2} + \|v_\varepsilon\|_{L^2}\,\|\nabla\eta_\varepsilon\|_{L^2}\,\|\psi_\varepsilon\|_{L^\infty(B(h_\varepsilon(t),2\varepsilon))} \\
&\le C\left(\|v_\varepsilon\|_{L^2}\,\|\varphi\|_{L^2} + \varepsilon^{\frac{3}{2}}\,\|v_\varepsilon\|_{L^2}\,\|\varphi\|_{H^2}\right) \\
&\le C\,\|v_\varepsilon\|_{L^2}\,\|\varphi\|_{H^2}\,.
\end{aligned}$$

where we used Lemma 1 and Lemma 2. The boundedness of v_ε in $L^2(\mathbb{R}^3)$ implies that there exists a constant C_1 independent of ε and t such that

$$\Big|\int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi_\varepsilon(t,x)\,dx\Big| \le C_1\,\|\varphi\|_{H^2}\,.$$

We infer that, for t fixed and $\varphi \in C^\infty_{0,\sigma}$, the map

$$\varphi \mapsto \int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi_\varepsilon(t,x)\,dx \in \mathbb{R}$$

is linear and continuous for the H^2 norm. Then, there exists some $\Xi_\varepsilon(t) \in H^{-2}_\sigma$ such that

$$\langle \Xi_\varepsilon(t), \varphi\rangle = \int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi_\varepsilon(t,x)\,dx \qquad \forall \varphi \in H^2_\sigma.$$

Moreover

$$\|\Xi_\varepsilon(t)\|_{H^{-2}} \le C_1 \quad \forall t \ge 0. \tag{23}$$

From Lemma 3 , we know that φ_ε vanishes in the neighborhood of $B(h_\varepsilon(t),\varepsilon)$, so it is compactly supported in the exterior of this ball. Therefore it can be used as test function in (13). Multiplying (13) by φ_ε and integrating in space and in time from s to t yields

$$\int_s^t\int_{\mathbb{R}^3} \partial_\tau v_\varepsilon\cdot\varphi_\varepsilon + \nu\int_s^t\int_{\mathbb{R}^3}\nabla v_\varepsilon : \nabla\varphi_\varepsilon + \int_s^t\int_{\mathbb{R}^3} v_\varepsilon\cdot\nabla v_\varepsilon\cdot\varphi_\varepsilon = 0.$$

After integrating by parts in time the first term above and using the definition of $\langle\Xi_\varepsilon(t),\varphi\rangle$, we obtain that

$$\langle \Xi_\varepsilon(t)-\Xi_\varepsilon(s),\varphi\rangle = \int_s^t\int_{\mathbb{R}^3} v_\varepsilon\cdot\partial_\tau\varphi_\varepsilon - \nu\int_s^t\int_{\mathbb{R}^3}\nabla v_\varepsilon : \nabla\varphi_\varepsilon - \int_s^t\int_{\mathbb{R}^3} v_\varepsilon\cdot\nabla v_\varepsilon\cdot\varphi_\varepsilon. \tag{24}$$

To bound the second term in the right-hand side above, we recall that v_ε is bounded independently of ε in $L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$. Thus, by the Hölder inequality and by Lemma 3, we deduce that

$$\begin{aligned}\Big|\nu\int_s^t\int_{\mathbb{R}^3}\nabla v_\varepsilon : \nabla\varphi_\varepsilon\Big| &\le \nu\int_s^t \|\nabla v_\varepsilon\|_{L^2}\,\|\nabla\varphi_\varepsilon\|_{L^2}\\ &\le C\nu(t-s)^{\frac12}\,\|\varphi\|_{H^2}\,\|v_\varepsilon\|_{L^2(0,T;H^1)}\\ &\le C\nu(t-s)^{\frac12}\,\|\varphi\|_{H^2}\,.\end{aligned}$$

Next, we estimate the non-linear term in (24) by the Hölder inequality and by the Gagliardo-Nirenberg inequality $\|v_\varepsilon\|_{L^3} \le C\,\|v_\varepsilon\|_{L^2}^{\frac12}\,\|\nabla v_\varepsilon\|_{L^2}^{\frac12}$,

$$\begin{aligned}\Big|\int_s^t\int_{\mathbb{R}^3} v_\varepsilon\cdot\nabla v_\varepsilon\cdot\varphi_\varepsilon\Big| &\le \int_s^t \|v_\varepsilon\|_{L^3}\,\|\nabla v_\varepsilon\|_{L^2}\,\|\varphi_\varepsilon\|_{L^6}\\ &\le C\int_s^t \|v_\varepsilon\|_{L^2}^{\frac12}\,\|\nabla v_\varepsilon\|_{L^2}^{\frac32}\,\|\varphi_\varepsilon\|_{H^1}\\ &\le C(t-s)^{\frac14}\,\|v_\varepsilon\|_{L^\infty(0,T;L^2)}^{\frac12}\,\|v_\varepsilon\|_{L^2(0,T;H^1)}^{\frac32}\,\|\varphi\|_{H^2}\\ &\le C(t-s)^{\frac14}\,\|\varphi\|_{H^2}\end{aligned}$$

It remains to estimate the term with the time-derivative. Notice that since φ does not depend on time, we have that $\partial_t \varphi = \partial_t \operatorname{curl} \psi_\varepsilon = 0$. Several integrations by parts give us

$$\begin{aligned}
\int_s^t \int_{\mathbb{R}^3} v_\varepsilon \cdot \partial_\tau \varphi_\varepsilon &= \int_s^t \int_{\mathbb{R}^3} v_\varepsilon \cdot \operatorname{curl} \partial_\tau (\eta_\varepsilon \psi_\varepsilon) \\
&= \int_s^t \int_{\mathbb{R}^3} \operatorname{curl} v_\varepsilon \cdot \partial_\tau (\eta_\varepsilon \psi_\varepsilon) \\
&= \int_s^t \int_{\mathbb{R}^3} \operatorname{curl} v_\varepsilon \cdot (\partial_\tau \eta_\varepsilon \psi_\varepsilon) + \int_s^t \int_{\mathbb{R}^3} \operatorname{curl} v_\varepsilon \cdot (\eta_\varepsilon \partial_\tau \psi_\varepsilon) \\
&= \int_s^t \int_{\mathbb{R}^3} \operatorname{curl} v_\varepsilon \cdot (\partial_\tau \eta_\varepsilon \psi_\varepsilon) + \int_s^t \int_{\mathbb{R}^3} v_\varepsilon \cdot \operatorname{curl} (\eta_\varepsilon \partial_\tau \psi_\varepsilon) \\
&= \int_s^t \int_{\mathbb{R}^3} \operatorname{curl} v_\varepsilon \cdot (\partial_\tau \eta_\varepsilon \psi_\varepsilon) + \int_s^t \int_{\mathbb{R}^3} v_\varepsilon \cdot (\nabla \eta_\varepsilon \times \partial_\tau \psi_\varepsilon)
\end{aligned}$$

We estimate the two terms in the right-hand side of the equality above by using Lemmas 1 and 2 and recalling that $\varepsilon^{\frac{3}{2}} |h'_\varepsilon|$ is bounded in $L^\infty(0,T)$ independently of ε:

$$\begin{aligned}
\left| \int_s^t \int_{\mathbb{R}^3} \operatorname{curl} v_\varepsilon \cdot (\partial_\tau \eta_\varepsilon \psi_\varepsilon) \right| &\leq \int_s^t \frac{|h'_\varepsilon|}{\varepsilon} \left\| \operatorname{curl} v_\varepsilon \right\|_{L^2} \left\| \nabla \eta \left(\frac{x - h_\varepsilon}{\varepsilon} \right) \right\|_{L^2} \left\| \psi_\varepsilon \right\|_{L^\infty(B(h_\varepsilon, 2\varepsilon))} \\
&\leq C \int_s^t \varepsilon^{\frac{3}{2}} |h'_\varepsilon| \left\| \operatorname{curl} v_\varepsilon \right\|_{L^2} \left\| \varphi \right\|_{H^2} \\
&\leq C (t-s)^{\frac{1}{2}} \left\| v_\varepsilon \right\|_{L^2(0,T;H^1)} \left\| \varphi \right\|_{H^2} \\
&\leq C (t-s)^{\frac{1}{2}} \left\| \varphi \right\|_{H^2}
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_s^t \int_{\mathbb{R}^3} v_\varepsilon \cdot (\nabla \eta_\varepsilon \times \partial_\tau \psi_\varepsilon) \right| &\leq \int_s^t \left\| v_\varepsilon \right\|_{L^6} \left\| \nabla \eta_\varepsilon \right\|_{L^{\frac{6}{5}}} \left\| \partial_\tau \psi_\varepsilon \right\|_{L^\infty(B(h_\varepsilon, 2\varepsilon))} \\
&\leq C \int_s^t \varepsilon^{\frac{3}{2}} |h'_\varepsilon| \left\| v_\varepsilon \right\|_{L^6} \left\| \varphi \right\|_{H^2} \\
&\leq C (t-s)^{\frac{1}{2}} \left\| v_\varepsilon \right\|_{L^2(0,T;H^1)} \left\| \varphi \right\|_{H^2} \\
&\leq C (t-s)^{\frac{1}{2}} \left\| \varphi \right\|_{H^2}
\end{aligned}$$

where we used the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

Gathering the two estimates above, we infer that

$$\left| \int_s^t \int_{\mathbb{R}^3} v_\varepsilon \cdot \partial_\tau \varphi_\varepsilon \right| \leq C (t-s)^{\frac{1}{2}} \left\| \varphi \right\|_{H^2}.$$

Putting together all the estimates above yields the following bound for Ξ_ε:

$$\begin{aligned}
|\langle \Xi_\varepsilon(t) - \Xi_\varepsilon(s), \varphi \rangle| &\leq C\nu (t-s)^{\frac{1}{2}} \|\varphi\|_{H^2} + C(t-s)^{\frac{1}{4}} \|\varphi\|_{H^2} + C(t-s)^{\frac{1}{2}} \left\| \varphi \right\|_{H^2} \\
&\leq C(t-s)^{\frac{1}{4}} \left\| \varphi \right\|_{H^2}
\end{aligned}$$

where the constant C above depends on T and ν.

By density of $C^\infty_{0,\sigma}$ in H^2_σ, we then obtain that $\Xi_\varepsilon(t)$ is equicontinuous in time with value in H^{-2}_σ

$$\left\| \Xi_\varepsilon(t) - \Xi_\varepsilon(s) \right\|_{H^{-2}} \leq C(t-s)^{\frac{1}{4}}.$$

On the other hand, $\Xi_\varepsilon(t)$ is also bounded in H^{-2}_σ, see relation (23). So the compact embedding $H^{-2} \hookrightarrow H^{-3}_{\text{loc}}$ and the Arzelà-Ascoli theorem enable us to extract a subsequence Ξ_{ε_k} of Ξ_ε converging to some Ξ strongly in H^{-3}_{loc}:

$$\Xi_{\varepsilon_k} \to \Xi \quad \text{in } C^0(0,T;H^{-3}_{\text{loc}}). \tag{25}$$

We now use Lemmas 1 and 2 to estimate

$$\begin{aligned}
|\langle \Xi_\varepsilon(t) - v_\varepsilon(t), \varphi\rangle| &= |\int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi_\varepsilon(t,x) - \int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi(t,x)| \\
&= |\int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot(\eta_\varepsilon\varphi + \nabla\eta_\varepsilon\times\psi_\varepsilon) - \int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot\varphi(t,x)| \\
&= |\int_{\mathbb{R}^3} (\eta_\varepsilon - 1)v_\varepsilon(t,x)\cdot\varphi + \int_{\mathbb{R}^3} v_\varepsilon\cdot(\nabla\eta_\varepsilon\times\psi_\varepsilon)| \\
&\le \|v_\varepsilon\|_{L^2}\|\eta_\varepsilon - 1\|_{L^2}\|\varphi\|_{L^\infty} + \|v_\varepsilon\|_{L^2}\|\nabla\eta_\varepsilon\|_{L^2}\|\psi_\varepsilon\|_{L^\infty(B(h_\varepsilon,2\varepsilon))} \\
&\le C\left(\varepsilon^{\frac{3}{2}}\|v_\varepsilon\|_{L^2}\|\varphi\|_{L^\infty} + \varepsilon^{\frac{3}{2}}\|v_\varepsilon\|_{L^2}\|\varphi\|_{H^2}\right) \\
&\le C\varepsilon^{\frac{3}{2}}\|\varphi\|_{H^2}\|v_\varepsilon\|_{L^2}.
\end{aligned}$$

Using again the density of $C^\infty_{0,\sigma}$ in H^2_σ, the above estimate implies that

$$\|\Xi_\varepsilon(t) - v_\varepsilon(t)\|_{H^{-2}} \le C\varepsilon^{\frac{3}{2}}\|v_\varepsilon\|_{L^2}.$$

So $\Xi_\varepsilon - v_\varepsilon \to 0$ in $L^\infty(0,T;H^{-2})$. In particular $\Xi_\varepsilon - v_\varepsilon \to 0$ in $L^\infty(0,T;H^{-3}_{\text{loc}})$. Recalling (25) and relabelling $\Xi = v$ we infer that

$$v_{\varepsilon_k} \to v \quad \text{in } L^\infty(0,T;H^{-3}_{\text{loc}}). \tag{26}$$

Let $f \in C^\infty_0(\mathbb{R}^3)$. We have the interpolation inequality

$$\|f(v_{\varepsilon_k} - v)\|_{L^2} \le C\|f(v_{\varepsilon_k} - v)\|^{\frac{1}{4}}_{H^{-3}}\|f(v_{\varepsilon_k} - v)\|^{\frac{3}{4}}_{H^1}$$

so

$$\|f(v_{\varepsilon_k} - v)\|_{L^{\frac{8}{3}}(0,T;L^2)} \le C\|f(v_{\varepsilon_k} - v)\|^{\frac{1}{4}}_{L^\infty(0,T;H^{-3})}\|f(v_{\varepsilon_k} - v)\|^{\frac{3}{4}}_{L^2(0,T;H^1)}.$$

Given relation (26) and the boundedness of v_ε in $L^2(0,T;H^1)$ we observe that the right-hand side above goes to 0 as $\varepsilon_k \to 0$. We deduce that

$$v_{\varepsilon_k} \to v \quad \text{strongly in } L^{\frac{8}{3}}(0,T;L^2_{\text{loc}}).$$

The embedding $L^{\frac{8}{3}}(0,T;L^2_{\text{loc}}) \subset L^2(0,T;L^2_{\text{loc}})$ completes the proof of Proposition 1.

5. Passing to the limit

In this section we are going to complete the proof of Theorem 3 by passing to the limit with compactness methods.

Let $T > 0$ be finite and fixed. We will pass to the limit only on the time interval $[0,T]$. A diagonal extraction allows us to find a subsequence which converges to the expected limit for all $t \ge 0$.

Thanks to the assumptions on v_ε, we know that

$$v_\varepsilon \text{ is bounded in } L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H^1\right).$$

This implies that there exists some $v \in L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H^1\right)$ and some subsequence v_{ε_k} of v_ε such that

$$\begin{aligned} v_{\varepsilon_k} &\rightharpoonup v \quad \text{weak}* \text{ in } L^\infty(0,T;L^2), \\ v_{\varepsilon_k} &\rightharpoonup v \quad \text{weakly in } L^2(0,T;H^1). \end{aligned} \tag{27}$$

Moreover, using Proposition 1, we can further assume that

$$v_{\varepsilon_k} \to v \quad \text{strongly in } L^2(0,T;L^2_{\text{loc}}).$$

The main goal of this Section is to prove that the limit v is the solution of the Navier-Stokes equations in $\mathbb{R}^3$ with initial data $v_0(x)$.

Let $\varphi \in C_0^\infty([0,T) \times \mathbb{R}^3)$ be a divergence-free vector field. We construct the family of vector fields φ_{ε_k} as in Section 3 (see relation (20)). These vector fields are compactly supported in the exterior of the ball $B(h_{\varepsilon_k}(t), \varepsilon_k)$, so they can be used as test functions in (13). Multiplying (13) by v_{ε_k} and integrating in space and time yields

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \partial_t \varphi_{\varepsilon_k} + \nu \int_0^T \int_{\mathbb{R}^3} \nabla v_{\varepsilon_k} : \nabla \varphi_{\varepsilon_k} + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot \varphi_{\varepsilon_k} \\ = \int_{\mathbb{R}^3} v_{\varepsilon_k}(0) \cdot \varphi_{\varepsilon_k}(0). \end{aligned} \tag{28}$$

We will pass to the limit $\varepsilon_k \to 0$ in each of the term in the equation above. First, from Lemma 3, we have that

$$\varphi_{\varepsilon_k}(0) \to \varphi(0) \text{ strongly in } L^2(\mathbb{R}^3).$$

We also know by hypothesis that $v_\varepsilon(0,x)$ converges weakly to $v_0(x)$ in $L^2(\mathbb{R}^3)$. We infer that

$$\int_{\mathbb{R}^3} v_{\varepsilon_k}(0) \cdot \varphi_{\varepsilon_k}(0) \overset{\varepsilon_k \to 0}{\longrightarrow} \int_{\mathbb{R}^3} v(0) \cdot \varphi(0). \tag{29}$$

Next, we also know from Lemma 3 that

$$\nabla \varphi_{\varepsilon_k} \to \nabla \varphi \text{ strongly in } L^\infty(0,T;L^2).$$

Recalling that $\nabla v_{\varepsilon_k} \rightharpoonup \nabla v$ weakly in $L^2([0,T] \times \mathbb{R}^3)$, see relation (27), we deduce that

$$\int_0^T \int_{\mathbb{R}^3} \nabla v_{\varepsilon_k} : \nabla \varphi_{\varepsilon_k} \overset{\varepsilon_k \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^3} \nabla v : \nabla \varphi. \tag{30}$$

We decompose the non-linear term in the left-hand of (28) as follows:

$$\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot \varphi_{\varepsilon_k} = \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot \varphi + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot (\varphi_{\varepsilon_k} - \varphi).$$

To treat the first term on the right-hand side, we know that φ is compactly supported, that $\nabla v_{\varepsilon_k} \rightharpoonup \nabla v$ weakly in $L^2([0,T] \times \mathbb{R}^3)$ and that $v_{\varepsilon_k} \to v$ strongly in $L^2(0,T;L^2_{\text{loc}})$. These observations enable us to pass to the limit:

$$\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot \varphi \overset{\varepsilon_k \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^3} v \cdot \nabla v \cdot \varphi.$$

For the second term, we make an integration by parts to get that

$$\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot (\varphi_{\varepsilon_k} - \varphi) = -\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \otimes v_{\varepsilon_k} : \nabla(\varphi_{\varepsilon_k} - \varphi).$$

By the Hölder inequality, the Gagliardo-Nirenberg inequality $\|v_{\varepsilon_k}\|_{L^4} \leq C \|v_{\varepsilon_k}\|_{L^2}^{\frac{1}{4}} \|\nabla v_{\varepsilon_k}\|_{L^2}^{\frac{3}{4}}$ and the strong convergence of φ_{ε_k} in $L^\infty(0,T;H^1)$ stated in Lemma 3, we obtain that

$$\begin{aligned}
|-\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \otimes v_{\varepsilon_k} : \nabla(\varphi_{\varepsilon_k} - \varphi)| &\leq \int_0^T \|v_{\varepsilon_k}\|_{L^4}^2 \|\nabla(\varphi_{\varepsilon_k} - \varphi)\|_{L^2} \\
&\leq C \int_0^T \|v_{\varepsilon_k}\|_{L^2}^{\frac{1}{2}} \|\nabla v_{\varepsilon_k}\|_{L^2}^{\frac{3}{2}} \|\varphi_{\varepsilon_k} - \varphi\|_{H^1} \\
&\leq C T^{\frac{1}{4}} \|v_{\varepsilon_k}\|_{L^\infty(0,T;L^2)}^{\frac{1}{2}} \|v_{\varepsilon_k}\|_{L^2(0,T;H^1)}^{\frac{3}{2}} \|\varphi_{\varepsilon_k} - \varphi\|_{L^\infty(0,T;H^1)} \\
&\overset{\varepsilon_k \to 0}{\longrightarrow} 0
\end{aligned}$$

where we also used the boundedness of v_{ε_k} in $L^\infty\left(0,T;L^2\right) \cap L^2\left(0,T;H^1\right)$.

Combining the relations above, we deduce that

$$\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \nabla v_{\varepsilon_k} \cdot \varphi_{\varepsilon_k} \overset{\varepsilon_k \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^3} v \cdot \nabla v \cdot \varphi. \tag{31}$$

Now, it remains to pass to the limit in the first term on the left-hand side of (28). Integrating by parts twice allows us to decompose this term into three parts as follows:

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \partial_t \varphi_{\varepsilon_k} &= \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \operatorname{curl} \partial_t \left(\eta_{\varepsilon_k} \psi_{\varepsilon_k}\right) \\
&= \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot \partial_t \left(\eta_{\varepsilon_k} \psi_{\varepsilon_k}\right) \\
&= \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot (\partial_t \eta_{\varepsilon_k} \psi_{\varepsilon_k}) + \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot (\eta_{\varepsilon_k} \partial_t \psi_{\varepsilon_k}) \\
&= \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot (\partial_t \eta_{\varepsilon_k} \psi_{\varepsilon_k}) + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \operatorname{curl} (\eta_{\varepsilon_k} \partial_t \psi_{\varepsilon_k}) \\
&= \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot (\partial_t \eta_{\varepsilon_k} \psi_{\varepsilon_k}) + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\nabla \eta_{\varepsilon_k} \times \partial_t \psi_{\varepsilon_k}) \\
&\qquad + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\eta_{\varepsilon_k} \operatorname{curl} \partial_t \psi_{\varepsilon_k}) \\
&= \int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot (\partial_t \eta_{\varepsilon_k} \psi_{\varepsilon_k}) + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\nabla \eta_{\varepsilon_k} \times \partial_t \psi_{\varepsilon_k}) \\
&\qquad + \int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\eta_{\varepsilon_k} \partial_t \varphi)
\end{aligned}$$

where we used the fact that $\operatorname{curl} \psi_{\varepsilon_k} = \varphi$ (see Lemma 1).

We will treat the three terms in the right-hand side of the relation above. For the first term, we use the Hölder inequality twice, the definition of η_ε (see relation (19)) and

Lemma 1 to bound

$$
\begin{aligned}
&|\int_0^T \int_{\mathbb{R}^3} \operatorname{curl} v_{\varepsilon_k} \cdot (\partial_t \eta_{\varepsilon_k} \psi_{\varepsilon_k})| \\
&\qquad \leq \int_0^T \frac{|h'_{\varepsilon_k}(t)|}{\varepsilon_k} \|\operatorname{curl} v_{\varepsilon_k}\|_{L^2} \left\| \nabla \eta \left(\frac{x - h_{\varepsilon_k}(t)}{\varepsilon_k} \right) \right\|_{L^2} \|\psi_{\varepsilon_k}\|_{L^\infty(B(h_{\varepsilon_k}(t), 2\varepsilon_k))} \\
&\qquad \leq C \int_0^T \varepsilon_k^{3/2} |h'_{\varepsilon_k}(t)| \, \|\operatorname{curl} v_{\varepsilon_k}\|_{L^2} \, \|\varphi\|_{H^2} \\
&\qquad \leq C T^{\frac{1}{2}} \varepsilon_k^{3/2} \|h'_{\varepsilon_k}(t)\|_{L^\infty(0,T)} \, \|v_{\varepsilon_k}\|_{L^2(0,T;H^1)} \, \|\varphi\|_{L^\infty(0,T;H^2)} \\
&\qquad \overset{\varepsilon_k \to 0}{\longrightarrow} 0.
\end{aligned}
$$

where we used the hypothesis $\varepsilon_k^{3/2} h'_{\varepsilon_k}(t) \to 0$ in $L^\infty_{\text{loc}}(\mathbb{R}_+)$ when $\varepsilon_k \to 0$.

To bound the second term, we use again the Hölder inequality, Lemmas 1 and 2 and the hypothesis on $\varepsilon_k^{3/2} h'_{\varepsilon_k}$:

$$
\begin{aligned}
|\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\nabla \eta_{\varepsilon_k} \times \partial_t \psi_{\varepsilon_k})| &\leq \int_0^T \|v_{\varepsilon_k}\|_{L^6} \, \|\nabla \eta_{\varepsilon_k}\|_{L^{\frac{6}{5}}} \, \|\partial_t \psi_{\varepsilon_k}\|_{L^\infty(B(h_{\varepsilon_k}(t), 2\varepsilon_k))} \\
&\leq C \int_0^T \varepsilon_k^{3/2} \, \|v_{\varepsilon_k}\|_{L^6} \left(\varepsilon_k \, \|\partial_t \varphi\|_{H^2} + |h'_{\varepsilon_k}(t)| \, \|\varphi\|_{H^2} \right) \\
&\leq C \int_0^T \|v_{\varepsilon_k}\|_{H^1} \left(\varepsilon_k^{5/2} \, \|\partial_t \varphi\|_{H^2} + \varepsilon_k^{3/2} |h'_{\varepsilon_k}(t)| \, \|\varphi\|_{H^2} \right) \\
&\leq C T^{\frac{1}{2}} \, \|v_{\varepsilon_k}\|_{L^2(0,T;H^1)} \left(\varepsilon_k^{5/2} \, \|\partial_t \varphi\|_{L^\infty(0,T;H^2)} \right. \\
&\qquad \left. + \varepsilon_k^{3/2} \|h'_{\varepsilon_k}\|_{L^\infty(0,T)} \, \|\varphi\|_{L^\infty(0,T;H^2)} \right) \\
&\overset{\varepsilon_k \to 0}{\longrightarrow} 0.
\end{aligned}
$$

where we also used the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

For the third term, we shall write $v_{\varepsilon_k} \eta_{\varepsilon_k} = v_{\varepsilon_k}(\eta_{\varepsilon_k} - 1) + (v_{\varepsilon_k} - v) + v$ to get that

$$
\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\eta_{\varepsilon_k} \partial_t \varphi) = \int_0^T \int_{\mathbb{R}^3} (\eta_{\varepsilon_k} - 1) v_{\varepsilon_k} \cdot \partial_t \varphi + \int_0^T \int_{\mathbb{R}^3} (v_{\varepsilon_k} - v) \cdot \partial_t \varphi + \int_0^T \int_{\mathbb{R}^3} v \cdot \partial_t \varphi
$$

Recalling that $v_{\varepsilon_k} \to v$ strongly in $L^2(0,T;L^2_{\text{loc}})$, we observe that the second term in the right-hand side of the equality above converges to 0. We estimate the first term by the Hölder inequality and by Lemma 2

$$
\begin{aligned}
|\int_0^T \int_{\mathbb{R}^3} (\eta_{\varepsilon_k} - 1) v_{\varepsilon_k} \cdot \partial_t \varphi| &\leq \int_0^T \|v_{\varepsilon_k}\|_{L^2} \, \|\eta_{\varepsilon_k} - 1\|_{L^2} \, \|\partial_t \varphi\|_{L^\infty} \\
&\leq C \varepsilon_k^{\frac{3}{2}} \int_0^T \|v_{\varepsilon_k}\|_{L^2} \, \|\partial_t \varphi\|_{L^\infty} \\
&\leq C T \varepsilon_k^{\frac{3}{2}} \, \|v_{\varepsilon_k}\|_{L^\infty(0,T;L^2)} \, \|\partial_t \varphi\|_{L^\infty(0,T;L^\infty)} \\
&\overset{\varepsilon_k \to 0}{\longrightarrow} 0.
\end{aligned}
$$

We infer that

$$
\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot (\eta_{\varepsilon_k} \partial_t \varphi) \overset{\varepsilon_k \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^3} v \cdot \partial_t \varphi,
$$

which implies that

$$\int_0^T \int_{\mathbb{R}^3} v_{\varepsilon_k} \cdot \partial_t \varphi_{\varepsilon_k} \overset{\varepsilon_k \to 0}{\longrightarrow} \int_0^T \int_{\mathbb{R}^3} v \cdot \partial_t \varphi. \tag{32}$$

Gathering (28), (29), (30), (31) and (32), we conclude that

$$-\int_0^T \int_{\mathbb{R}^3} v \cdot \partial_t \varphi + \nu \int_0^T \int_{\mathbb{R}^3} \nabla v : \nabla \varphi + \int_0^T \int_{\mathbb{R}^3} v \cdot \nabla v \cdot \varphi = \int_{\mathbb{R}^3} v(0) \cdot \varphi(0)$$

which is the weak formulation of the Navier-Stokes equations in $\mathbb{R}^3$. This completes the proof that v is a solution of the Navier-Stokes equations in $\mathbb{R}^3$ in the sense of distributions.

In order to complete the proof of Theorem 3, it remains to prove the energy inequality (15) under the additional assumption that $v_\varepsilon(0)$ converges strongly to v_0 in L^2.

Let us observe first that $v \in C^0_w(\mathbb{R}_+; L^2(\mathbb{R}^3))$. This follows from the fact that $v \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$ is a solution in the sense of distributions of the Navier-Stokes equations in $\mathbb{R}^3$. The argument is classical, but let us recall it for the benefit of the reader. We apply the Leray projector $\mathbb{P}$ in $\mathbb{R}^3$ to the Navier-Stokes equations verified by v to obtain that

$$\partial_t v - \nu \Delta v + \mathbb{P} \operatorname{div}(v \otimes v) = 0.$$

Because $v \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$ we have that $v \otimes v \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3)) \subset L^\infty_{\text{loc}}(\mathbb{R}_+; H^{-2}(\mathbb{R}^3))$. We also have that $\Delta v \in L^\infty_{\text{loc}}(\mathbb{R}_+; H^{-2}(\mathbb{R}^3))$, so $\partial_t v \in L^\infty_{\text{loc}}(\mathbb{R}_+; H^{-2}(\mathbb{R}^3))$. We infer that v is Lipschitz in time with values in $H^{-2}(\mathbb{R}^3)$; in particular it is strongly continuous in time with values in $H^{-2}(\mathbb{R}^3)$. This strong continuity in time together with the boundedness of the L^2 norm implies the weak continuity of v in time with values in L^2. In particular, we have that $v(t)$ is well-defined and belongs to $L^2(\mathbb{R}^3)$ for all times $t \geq 0$ (and not only for almost all times).

Let us observe now that for all $t \geq 0$ we have that $v_{\varepsilon_k}(t) \rightharpoonup v(t)$ weakly in $L^2(\mathbb{R}^3)$. Indeed, we know from relation (26) that $v_{\varepsilon_k}(t) \to v(t)$ strongly in $H^{-3}_{\text{loc}}(\mathbb{R}^3)$, so $\langle v_{\varepsilon_k}(t), \varphi \rangle \to \langle v(t), \varphi \rangle$ for all test functions φ. The boundedness of $v_{\varepsilon_k}(t)$ in L^2 and the density of the test functions in L^2 imply that $\langle v_{\varepsilon_k}(t), \varphi \rangle \to \langle v(t), \varphi \rangle$ for all $\varphi \in L^2$, that is $v_{\varepsilon_k}(t) \rightharpoonup v(t)$ weakly in $L^2(\mathbb{R}^3)$.

Let us denote by χ_A the characteristic function of the set A. We prove now that for all $t \geq 0$

$$v_{\varepsilon_k}(t) \chi_{B(h_{\varepsilon_k}(t), \varepsilon_k)} \rightharpoonup 0$$

weakly in L^2. Indeed, let $g \in L^2$. Then

$$|\langle v_{\varepsilon_k}(t) \chi_{B(h_{\varepsilon_k}(t), \varepsilon_k)}, g \rangle| = |\int_{B(h_{\varepsilon_k}(t), \varepsilon_k)} v_{\varepsilon_k}(t) g| \leq \|v_{\varepsilon_k}(t)\|_{L^2} \|g\|_{L^2(B(h_{\varepsilon_k}(t), \varepsilon_k))} \overset{\varepsilon_k \to 0}{\longrightarrow} 0$$

because $\|v_{\varepsilon_k}(t)\|_{L^2}$ is bounded and $\|g\|_{L^2(B(h_{\varepsilon_k}(t), \varepsilon_k))}$ goes to 0 as $\varepsilon_k \to 0$.

We infer that

$$v_{\varepsilon_k}(t) \chi_{\mathbb{R}^3 \setminus B(h_{\varepsilon_k}(t), \varepsilon_k)} \rightharpoonup v(t) \quad \text{weakly in } L^2(\mathbb{R}^3).$$

By the weak lower semi-continuity of the L^2 norm we infer that

$$\|v(t)\|_{L^2(\mathbb{R}^3)} \leq \liminf_{\varepsilon_k \to 0} \|v_{\varepsilon_k}(t)\|_{L^2(\mathbb{R}^3 \setminus B(h_{\varepsilon_k}(t), \varepsilon_k))}. \tag{33}$$

Similarly, from the weak convergence

$$D(v_{\varepsilon_k}) \rightharpoonup D(v) \quad \text{weakly in } L^2((0,t) \times \mathbb{R}^3)$$

we infer that

$$\chi_{\mathbb{R}^3 \setminus B(h_{\varepsilon_k}(t), \varepsilon_k)} D(v_{\varepsilon_k}) \rightharpoonup D(v) \quad \text{weakly in } L^2((0,t) \times \mathbb{R}^3)$$

so by lower semi-continuity

$$\|D(v)\|^2_{L^2((0,t)\times\mathbb{R}^3)} \leq \liminf_{\varepsilon_k\to 0} \int_0^t \|D(v_{\varepsilon_k})\|^2_{L^2(\mathbb{R}^3\setminus B(h_{\varepsilon_k}(t),\varepsilon_k))}. \tag{34}$$

We also observe at this point that the strong L^2 convergence of $v_\varepsilon(0)$ towards v_0 gives that

$$\|v_0\|_{L^2} = \liminf_{\varepsilon_k\to 0} \|v_{\varepsilon_k}(0)\|_{L^2}. \tag{35}$$

Finally, taking the $\liminf\limits_{\varepsilon_k\to 0}$ in (14) and using (33), (34) and (35) implies the required energy inequality (15). This completes the proof of Theorem 3.

We end this paper with a final remark about the weak time continuity assumed in Theorem 3: $v_\varepsilon \in C^0_w(\mathbb{R}_+; L^2(\mathbb{R}^3))$. We know that $\widetilde{u}_\varepsilon \in C^0_w(\mathbb{R}_+; L^2(\mathbb{R}^3))$ so we made this hypothesis for the sake of simplicity, but it is in fact not necessary to make such an assumption. Indeed, we used it to make sense of the various terms of the form $\int_{\mathbb{R}^3} v_\varepsilon(t,x)\cdot \varphi_\varepsilon(t,x)\,dx$, see for instance on page 11. But the Navier-Stokes equation itself implies a time-continuity property allowing to make sense of such terms. More precisely, let us make a change of variables to go to a fixed domain. The vector field $\widetilde{v}_\varepsilon(t,x) = v_\varepsilon(t, x - h_\varepsilon(t))$ verifies the following PDE:

$$\partial_t \widetilde{v}_\varepsilon + h'_\varepsilon \cdot \nabla \widetilde{v}_\varepsilon - \nu\Delta\widetilde{v}_\varepsilon + \widetilde{v}_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon = -\nabla\widetilde{\pi}_\varepsilon \quad \text{for } |x|>\varepsilon \tag{36}$$

where $\widetilde{\pi}_\varepsilon(t,x) = \pi_\varepsilon(t, x - h_\varepsilon(t))$. Since $v_\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3))$ we also have that $\widetilde{v}_\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3))$. Recalling that h_ε is Lipschitz, we infer by classical estimates that $h'_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon - \nu\Delta\widetilde{v}_\varepsilon + \widetilde{v}_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon \in L^{\frac{4}{3}}_{\text{loc}}(\mathbb{R}_+; H^{-1}(\mathbb{R}^3))$. So, if we choose some $\Phi \in C^\infty_{0,\sigma}(|x|>\varepsilon)$ and we multiply (36) by Φ, the pressure goes away and we obtain that

$$|\langle \partial_t\widetilde{v}_\varepsilon, \Phi\rangle| = |\langle h'_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon - \nu\Delta\widetilde{v}_\varepsilon + \widetilde{v}_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon, \Phi\rangle| \leq \|h'_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon - \nu\Delta\widetilde{v}_\varepsilon + \widetilde{v}_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon\|_{H^{-1}}\|\Phi\|_{H^1}.$$

If we denote by X the dual of $C^\infty_{0,\sigma}(|x|>\varepsilon)$ for the H^1 norm, the relation above implies that

$$\|\partial_t\widetilde{v}_\varepsilon\|_X \leq \|h'_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon - \nu\Delta\widetilde{v}_\varepsilon + \widetilde{v}_\varepsilon\cdot\nabla\widetilde{v}_\varepsilon\|_{H^{-1}}$$

so $\partial_t\widetilde{v}_\varepsilon \in L^{\frac{4}{3}}_{\text{loc}}(\mathbb{R}_+; X)$. In particular $\widetilde{v}_\varepsilon \in C^0(\mathbb{R}_+; X)$. We infer that $\int \widetilde{v}_\varepsilon(t,\cdot)\cdot\Phi$ is well-defined for all $t\geq 0$ and $\Phi \in C^\infty_{0,\sigma}(|x|>\varepsilon)$. Going back to the original variables, we infer that if $\varphi_\varepsilon \in C^\infty_{0,\sigma}(|x-h_\varepsilon(t)|>\varepsilon)$ then $\int v_\varepsilon(t,\cdot)\cdot\varphi_\varepsilon$ is well-defined.

Acknowledgments

J.H. and D.I. have been partially funded by the ANR project Dyficolti ANR-13-BS01-0003-01. D.I. has been partially funded by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

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