ANOTHER PROOF OF BAILEY'S $_6\psi_6$ SUMMATION

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Abstract. Adapting a method used by Cauchy, Bailey, Slater, and more recently, the second author, we give a new proof of Bailey's celebrated $_6\psi_6$ summation formula.

1. Introduction

In [15], one of the authors presented a new proof of Ramanujan's $_1\psi_1$ summation formula (cf. [10, Appendix (II.29)]),

$$ {}_1\psi_1\left[\begin{matrix}a\\ b\end{matrix};q,z\right]=\frac{(q,b/a,az,q/az)_\infty}{(b,q/a,z,b/az)_\infty} \tag{1.1}$$

(the notation is defined at the end of this introduction), valid for $|q|<1$ and $|b/a|<|z|<1$. This proof used a standard method for obtaining a bilateral identity from a unilateral terminating identity, a method already utilized by Cauchy [8] in his second proof of Jacobi's [13] famous triple product identity (see (2.3)), a special case of Ramanujan's formula (1.1).

The same method (which is referred to as "Cauchy's method" in the sequel) had also been exploited by Bailey [5, Secs. 3 and 6], [6], and Slater [17, Sec. 6.2].

It was conjectured in [15, Remark 3.2] that *any* bilateral sum can be obtained from an appropriately chosen terminating identity by Cauchy's method (without appealing to analytic continuation). However, at the same place it was also pointed out that it was already not known whether Bailey's [5, Eq. (4.7)] very-well-poised $_6\psi_6$ summation (cf. [10, Appendix (II.33)]),

$$ {}_6\psi_6\left[\begin{matrix}q\sqrt{a},-q\sqrt{a},b,c,d,e\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e\end{matrix};q,\frac{qa^2}{bcde}\right] = \frac{(q,aq,q/a,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de)_\infty}{(q/b,q/c,q/d,q/e,aq/b,aq/c,aq/d,aq/e,a^2q/bcde)_\infty} \tag{1.2}$$

(again, see the end of this introduction for the notation), where $|q|<1$ and $|a^2q/bcde|<1$, would follow from such an identity. It is maybe interesting to mention that Bailey's $_6\psi_6$ summation (1.2), although it contains more parameters than Ramanujan's $_1\psi_1$ summation (1.1), does *not* include the latter as a special case.

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While the conjecture of [15, Remark 3.2] remains open, this paper features a new derivation of Bailey's ${}_6\psi_6$ summation formula using a variant of Cauchy's method. After explaining some notation in the end of this introduction, the proof from [15, Sec. 3] of Ramanujan's ${}_1\psi_1$ summation is being reviewed in Section 2, for illustration. The starting point for the derivation of Bailey's ${}_6\psi_6$ summation in Section 3 is Bailey's terminating very-well-poised ${}_{10}\phi_9$ transformation (3.2). After appropriately applying Cauchy's method to both sides of this identity, a specific transformation of ${}_6\psi_6$ series is obtained. The resulting transformation is then iterated, after which a suitable specialization yields the desired ${}_6\psi_6$ summation.

Other proofs of Bailey's very-well-poised ${}_6\psi_6$ summation have been given by Bailey [5], Slater and Lakin [18], Andrews [1], Askey and Ismail [3], Askey [2], and the second author [14].

The authors plan to give an account of Cauchy's method applied to terminating *quadratic*, *cubic* and *quartic* identities [10, Sec. 3.8] elsewhere.

Notation: It is appropriate to recall some standard notation for *q-series* and *basic hypergeometric series*.

Let q be a fixed complex parameter (the "base") with $0 < |q| < 1$. The *q-shifted factorial* is defined for any complex parameter a by

$$(a)_\infty \equiv (a;q)_\infty := \prod_{j\geq 0}(1-aq^j)$$

and

$$(a)_k \equiv (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer. Since the same base q is used throughout this paper, it may be readily omitted (in notation) which will not lead to any confusion. For brevity, write

$$(a_1,\cdots,a_m)_k := (a_1)_k\cdots(a_m)_k,$$

where k is an integer or infinity. Further, recall the definition of *basic hypergeometric series*,

$${}_s\phi_{s-1}\left[\begin{matrix}a_1,\dots,a_s\\ b_1,\dots,b_{s-1}\end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_s)_k}{(q,b_1,\dots,b_{s-1})_k}z^k,$$

and of *bilateral basic hypergeometric series*,

$${}_s\psi_s\left[\begin{matrix}a_1,\dots,a_s\\ b_1,\dots,b_s\end{matrix};q,z\right] := \sum_{k=-\infty}^{\infty}\frac{(a_1,\dots,a_s)_k}{(b_1,\dots,b_s)_k}z^k.$$

See Gasper and Rahman's text [10] for a comprehensive study on the theory of basic hypergeometric series. In particular, the computations in this paper rely on some elementary identities for q-shifted factorials, listed in [10, Appendix I].

2. Cauchy's method and Ramanujan's ${}_1\psi_1$ summation

For illustration of "Cauchy's method", it is convenient to review the second author's proof [15, Sec. 3] of Ramanujan's ${}_1\psi_1$ summation. A closely related analysis is being applied in Section 3 to prove Bailey's very-well-poised ${}_6\psi_6$ summation (1.2).

In Jackson's [11] q-Pfaff–Saalschütz summation (cf. [10, Appendix (II.12)]),

$${}_3\phi_2\left[\begin{matrix}a,b,q^{-n}\\ c,abq^{1-n}/c\end{matrix};q,q\right] = \frac{(c/a,c/b)_n}{(c,c/ab)_n}, \tag{2.1}$$

first replace n by $2n$ and then shift the index of summation by n such that the new sum runs from $-n$ to n:

$$\frac{(c/a,c/b)_{2n}}{(c,c/ab)_{2n}}=\sum_{k=0}^{2n}\frac{(a,b,q^{-2n})_k}{(q,c,abq^{1-2n}/c)_k}q^k$$
$$=\frac{(a,b,q^{-2n})_n}{(q,c,abq^{1-2n}/c)_n}q^n\sum_{k=-n}^{n}\frac{(aq^n,bq^n,q^{-n})_k}{(q^{1+n},cq^n,abq^{1-n}/c)_k}q^k.$$

Next, replace a by aq^{-n}, and c by cq^{-n}, respectively, to get

$$\sum_{k=-n}^{n}\frac{(a,bq^n,q^{-n})_k}{(q^{1+n},c,abq^{1-n}/c)_k}q^k=\frac{(c/a,cq^{-n}/b)_{2n}}{(cq^{-n},c/ab)_{2n}}\frac{(q,cq^{-n},abq^{1-2n}/c)_n}{(aq^{-n},b,q^{-2n})_n}q^{-n}$$
$$=\frac{(c/a)_{2n}}{(q)_{2n}}\frac{(q,q,c/b,bq/c)_n}{(c,q/a,b,c/ab)_n}.$$

Now, one may let $n\to\infty$, assuming $|c/ab|<1$ and $|b|<1$, while appealing to Tannery's theorem [7] for being allowed to interchange limit and summation. This gives

$$\sum_{k=-\infty}^{\infty}\frac{(a)_k}{(c)_k}\left(\frac{c}{ab}\right)^k=\frac{(q,c/a,c/b,bq/c)_\infty}{(c,q/a,b,c/ab)_\infty},$$

where $|c/a|<|c/ab|<1$. Finally, replacing b by c/az and then c by b gives (1.1).

Remark 2.1. Cauchy [8] had applied the above method to a special case of (2.1). (After all, (2.1) was not available to him yet.) His starting point was the terminating q-binomial theorem,

$$_1\phi_0\left[\begin{matrix}q^{-n}\\ -\end{matrix};q,zq^n\right]=(z)_n, \tag{2.2}$$

an identity already known to Euler [9], which can be derived from (2.1) by first doing the substitution $b\mapsto c/z$, then $c\mapsto b$, and then letting $a\to\infty$ and $b\to 0$. As a result of "bilateralizing" (2.2) by the above procedure, Cauchy recovered Jacobi's [13] well-known triple product identity,

$$\sum_{k=-\infty}^{\infty}(-1)^kq^{\binom{k}{2}}z^k=(q,z,q/z)_\infty. \tag{2.3}$$

Note that (2.3) can be obtained from (1.1) by first replacing z by z/a, and then letting $a\to\infty$ and $b\to 0$.

3. Proof of Bailey's $_6\psi_6$ summation formula

In order to prove Bailey's $_6\psi_6$ summation formula (1.2) by Cauchy's method, one should start with a known terminating identity which contains enough parameters. If one considers Jackson's [12] summation formula [10, Appendix (II.22)],

$$_8\phi_7\left[\begin{matrix}a,\ q\sqrt{a},-q\sqrt{a},b,c,d,a^2q^{n+1}/bcd,q^{-n}\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,bcdq^{-n}/a,aq^{n+1}\end{matrix};q,q\right]$$
$$=\frac{(aq,aq/bc,aq/bd,aq/cd)_n}{(aq/b,aq/c,aq/d,aq/bcd)_n}, \tag{3.1}$$

it becomes apparent that at least one parameter is missing here for the given purpose. As a next step, one might consider Watson's transformation formula of a terminating ${}_8\phi_7$ into a multiple of a ${}_4\phi_3$ [10, Appendix (III.18)], which involves an additional parameter, and apply Cauchy's method (to both sides of the transformation). In fact, this was undertaken by Bailey [6] who obtained by this procedure, and some symmetry argument, a transformation for ${}_2\psi_2$ series (see also [10, Ex. 5.11]).

The next level object in the hierarchy of identities for (very-well-poised) basic hypergeometric series is Bailey's [4] transformation formula [10, Appendix (III.28)],

$$\begin{aligned}
&{}_{10}\phi_9\left[\begin{matrix}a,\ q\sqrt{a},-q\sqrt{a},b,c,d,e,f,\lambda aq^{n+1}/ef,q^{-n}\\ \sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1}\end{matrix};q,q\right]\\
&\qquad=\frac{(aq,aq/ef,\lambda q/e,\lambda q/f)_n}{(aq/e,aq/f,\lambda q/ef,\lambda q)_n}\\
&\qquad\times{}_{10}\phi_9\left[\begin{matrix}\lambda,\ q\sqrt{\lambda},-q\sqrt{\lambda},\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{n+1}/ef,q^{-n}\\ \sqrt{\lambda},-\sqrt{\lambda},aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-n}/a,\lambda q^{n+1}\end{matrix};q,q\right],
\end{aligned}\tag{3.2}$$

where $\lambda=qa^2/bcd$. A standard proof of (3.2) involves two applications of (3.1), together with an interchange of summations, see [10, Sec. 2.9]. Jackson's summation (3.1) itself can be proved in various ways, see e. g. Slater [17, Sec. 3.3.1], or Gasper and Rahman [10, Sec. 2.6].

Note that for b, c or $d\to\infty$, (3.2) reduces to Watson's transformation formula. By a further specialization one obtains Jackson's ${}_8\phi_7$ summation formula which may also be derived directly from (3.2) by letting $b=aq/c$ (thus $\lambda=a/d$).

To prove (1.2), start from (3.2). First, replace n by $2n$, shift the index of summation by n on both sides, and obtain

$$\begin{aligned}
&\frac{(1-aq^{2n})}{(1-a)}\frac{(a,b,c,d,e,f,\lambda aq^{2n+1}/ef,q^{-2n})_n}{(q,aq/b,aq/c,aq/d,aq/e,aq/f,efq^{-2n}/\lambda,aq^{2n+1})_n}q^n\\
&\quad\times\sum_{k=-n}^{n}\frac{(1-aq^{2n+2k})}{(1-aq^{2n})}\frac{(aq^n,bq^n,cq^n,dq^n)_k}{(q^{1+n},aq^{1+n}/b,aq^{1+n}/c,aq^{1+n}/d)_k}\\
&\qquad\times\frac{(eq^n,fq^n,\lambda aq^{3n+1}/ef,q^{-n})_k}{(aq^{1+n}/e,aq^{1+n}/f,efq^{-n}/\lambda,aq^{3n+1})_k}q^k\\
&\quad=\frac{(aq,aq/ef,\lambda q/e,\lambda q/f)_{2n}}{(aq/e,aq/f,\lambda q/ef,\lambda q)_{2n}}\\
&\quad\times\frac{(1-\lambda q^{2n})}{(1-\lambda)}\frac{(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f,\lambda aq^{2n+1}/ef,q^{-2n})_n}{(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f,efq^{-2n}/a,\lambda q^{2n+1})_n}q^n\\
&\quad\times\sum_{k=-n}^{n}\frac{(1-\lambda q^{2n+2k})}{(1-\lambda q^{2n})}\frac{(\lambda q^n,\lambda bq^n/a,\lambda cq^n/a,\lambda dq^n/a)_k}{(q^{1+n},aq^{1+n}/b,aq^{1+n}/c,aq^{1+n}/d)_k}\\
&\qquad\times\frac{(eq^n,fq^n,\lambda aq^{3n+1}/ef,q^{-n})_k}{(\lambda q^{1+n}/e,\lambda q^{1+n}/f,efq^{-n}/a,\lambda q^{3n+1})_k}q^k.
\end{aligned}$$

Next, replace a, c, d, e and f by aq^{-2n}, cq^{-n}, dq^{-n}, eq^{-n} and fq^{-n}, respectively. This gives

$$\sum_{k=-n}^{n}\frac{(1-aq^{2k})}{(1-a)}\frac{(aq^{-n},bq^n,c,d,e,f,\lambda aq^{n+1}/ef,q^{-n})_k}{(q^{1+n},aq^{1-n}/b,aq/c,aq/d,aq/e,aq/f,efq^{-n}/\lambda,aq^{n+1})_k}q^k$$

$$
\begin{aligned}
&= \frac{(1-aq^{-2n})}{(1-a)} \frac{(aq^{1-n}/e, aq^{1-n}/f, efq^{-2n}/\lambda, aq)_n}{(aq^{-2n}, b, cq^{-n}, dq^{-n})_n} \\
&\quad \times \frac{(aq^{1-2n}, aq/ef, \lambda q^{1-n}/e, \lambda q^{1-n}/f)_{2n}}{(aq^{1-n}/e, aq^{1-n}/f, \lambda q/ef, \lambda q^{1-2n})_{2n}} \\
&\quad \times \frac{(1-\lambda)}{(1-\lambda q^{-2n})} \frac{(\lambda q^{-2n}, \lambda b/a, \lambda cq^{-n}/a, \lambda dq^{-n}/a)_n}{(\lambda q^{1-n}/e, \lambda q^{1-n}/f, efq^{-2n}/a, \lambda q)_n} \\
&\times \sum_{k=-n}^{n} \frac{(1-\lambda q^{2k})}{(1-\lambda)} \frac{(\lambda q^{-n}, \lambda bq^n/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n})_k}{(q^{1+n}, aq^{1-n}/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-n}/a, \lambda q^{n+1})_k} q^k \\
&= \frac{(\lambda q/e, \lambda q/f, aq, \lambda b/a, aq/\lambda c, aq/\lambda d, q/a, aq/ef)_n}{(aq/e, aq/f, b, \lambda q, q/c, q/d, q/\lambda, \lambda q/ef)_n} \\
&\times \sum_{k=-n}^{n} \frac{(1-\lambda q^{2k})}{(1-\lambda)} \frac{(\lambda q^{-n}, \lambda bq^n/a, \lambda c/a, \lambda d/a, e, f, \lambda aq^{n+1}/ef, q^{-n})_k}{(q^{1+n}, aq^{1-n}/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-n}/a, \lambda q^{n+1})_k} q^k.
\end{aligned}
$$

Now, one may let $n \to \infty$, assuming $|qa^2/cdef| < 1$ while appealing to Tannery's theorem, which yields the following transformation formula:

$$
\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{qa^2}{cdef} \right] \\
&\qquad = \frac{(aq, q/a, aq/ef, aq/cd, \lambda q/e, \lambda q/f, aq/\lambda c, aq/\lambda d)_\infty}{(aq/e, aq/f, q/c, q/d, \lambda q, q/\lambda, \lambda q/ef, b)_\infty} \\
&\qquad\qquad \times {}_6\psi_6 \left[\begin{matrix} q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda c/a, \lambda d/a, e, f \\ \sqrt{\lambda}, -\sqrt{\lambda}, aq/c, aq/d, \lambda q/e, \lambda q/f \end{matrix} ; q, \frac{qa^2}{cdef} \right], \qquad (3.3)
\end{aligned}
$$

where $\lambda = qa^2/bcd$. In this identity, the right-hand side involves one more parameter (namely b) than the left-hand side. Note that when $b = aq/cd$ (whence $\lambda = a$), this identity is trivial. It should also be pointed out that (3.3) is *not* Slater's [16] transformation [10, Eq. (5.5.3)], the latter involving more symmetric parameters.

A possibility, of course, would be to directly specialize the extra parameter b in (3.3) such that the ${}_6\psi_6$ on the right-hand side reduces to a ${}_6\phi_5$ series, which can be summed by the $n \to \infty$ case of (3.1). However, this was not the idea of the above derivaton of the ${}_6\psi_6$ transformation in (3.3). Indeed, several proofs of Bailey's ${}_6\psi_6$ summation which make use of the nonterminating ${}_6\phi_5$ summation already exist, see e. g. Slater and Lakin [18], Andrews [1], Askey and Ismail [3], and the second author [14].

The clue is to *iterate* formula (3.3), more precisely, to apply the same transformation to the ${}_6\psi_6$ on the right-hand side of (3.3) again, with the parameters a, c, d and e, respectively, being replaced by λ, $\lambda c/a$, e and $\lambda d/a$. By this iteration an additional free parameter[1], say b', is obtained on the right-hand side. The result is the following.

$$
\begin{aligned}
&{}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, -q\sqrt{a}, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f \end{matrix} ; q, \frac{qa^2}{cdef} \right] \\
&\qquad = \frac{(aq, q/a, aq/ef, aq/cd, \lambda q/e, aq/\lambda d)_\infty}{(aq/e, aq/f, q/c, q/d, \lambda q/ef, b)_\infty}
\end{aligned}
$$

[1]On the contrary, iteration of [10, Eq. (5.5.3)] would only yield an additional *redundant* parameter and essentially reduce to the same identity.

$$\times \frac{(aq/df, aq/ec, a\lambda' q/\lambda d, \lambda' q/f, aq/\lambda' c, \lambda q/\lambda' e)_\infty}{(aq/d, q/e, \lambda' q, q/\lambda', a\lambda' q/\lambda df, b')_\infty}$$
$$\times {}_6\psi_6\left[\begin{matrix} q\sqrt{\lambda'}, -q\sqrt{\lambda'}, \lambda' c/a, \lambda' e/\lambda, \lambda d/a, f \\ \sqrt{\lambda'}, -\sqrt{\lambda'}, aq/c, \lambda q/e, \lambda' aq/\lambda d, \lambda' q/f \end{matrix}; q, \frac{qa^2}{cdef}\right],$$

where $\lambda = qa^2/bcd$ and $\lambda' = aq\lambda/b'ce$.

Now there are two extra parameters appearing on the right-hand side. Take $\lambda = e$ (thus $b = qa^2/cde$), and $\lambda' = a/c$ (thus $b' = q$), such that the last ${}_6\psi_6$, getting terminated from above and from below, equals 1, which immediately establishes Bailey's formula (1.2) (where b has been replaced by f).

Remark 3.1. In the above analysis, after applying Cauchy's method to both sides of Bailey's ${}_{10}\phi_9$ transformation (3.2), the resulting ${}_6\psi_6$ transformation (3.3) first was iterated and then specialized. A natural question is what happens if one would start with the iterate of (3.2), listed in [10, Ex. 2.19], and then apply Cauchy's method. In fact, the whole analysis would be similar to the one above. In particular, one would also obtain a transformation of ${}_6\psi_6$ series (similar to but different from (3.3) though) with an extra free parameter on the right-hand side. Again, one can iterate the transformation to obtain a second additional parameter, and then specialize the two extra parameters such that the ${}_6\psi_6$ on the right-hand side reduces to a sum of one single term only.

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