ANDREWS–GORDON AND STANTON TYPE IDENTITIES: BIJECTIVE AND BAILEY LEMMA APPROACHES

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Abstract. In 2018, Stanton proved two types of generalisations of the celebrated Andrews–Gordon and Bressoud identities (in their q-series version): one with a similar shape to the original identities, and one involving binomial coefficients. In this paper, we give new proofs of these identities. For the non-binomial identities, we give bijective proofs using the original Andrews–Gordon and Bressoud identities as key ingredients. These proofs are based on particle motion introduced by Warnaar and extended by the first and third authors and Konan. For the binomial identities, we use the Bailey lemma and key lemmas of McLaughlin and Lovejoy, and the order in which we apply the different lemmas plays a central role in the result. We also give an alternative proof of the non-binomial identities using the Bailey lattice. With each of these proofs, new Stanton-type generalisations of classical identities arise naturally, such as generalisations of Kurşungöz's analogue of Bressoud's identity with opposite parity conditions, and of the Bressoud–Göllnitz–Gordon identities.

1. Introduction

A *partition* λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ such that $\sum_{i=1}^{\ell} \lambda_i = n$. For each partition λ, we associate the *frequency sequence* $f = (f_1, f_2, \cdots)$, where f_j denotes the number of times the part j appears in λ. Whenever convenient, we identify a partition with its corresponding frequency sequence.

Many classical partition identities are elegantly expressed using q-series. Throughout this paper, we use standard q-series notation which can be found in [GR04]:

$$(a)_\infty = (a;q)_\infty := \prod_{j\geq 0}(1-aq^j) \quad \text{and} \quad (a)_k = (a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where $k \in \mathbb{Z}$, and

$$(a_1,\ldots,a_m)_k = (a_1,\ldots,a_m;q)_k := (a_1)_k \cdots (a_m)_k,$$

where k is an integer or infinity, and as usual $|q| < 1$ to ensure convergence of infinite products.

One of the most famous results in the theory of partitions is the Rogers–Ramanujan identities [RR19]: for $a \in \{0,1\}$,

$$\sum_{n\geq 0} \frac{q^{n^2+(1-a)n}}{(q)_n} = \frac{1}{(q^{2-a}, q^{3+a}; q^5)_\infty}. \tag{1.1}$$

The Rogers–Ramanujan identities have elegant interpretations in terms of partitions. For $a \in \{0,1\}$, they state that the number of partitions λ of n with the difference condition $\lambda_i - \lambda_{i+1} \geq 2$ for all i, where the part 1 appears at most a times, is equal to the number of partitions of n into parts congruent to $\pm(2-a)$ modulo 5.

Gordon later extended the Rogers–Ramanujan identities and proved what is known as Gordon's partition theorem [Gor61]. For integers $k \geq 1$ and $0 \leq r \leq k$, it states that the number of partitions λ of n with the difference condition $\lambda_i - \lambda_{i+k} \geq 2$ for all i, and where the part 1 appears at most $(k-r)$ times, is equal to the number of partitions of n into parts not congruent to $0, \pm(k-r+1)$ modulo $2k+3$. The partitions satisfying the difference condition in Gordon's theorem can also be described easily in terms of frequency sequences as:

$$\text{frequency sequences } (f_i)_{i\geq 1} \text{ of } n \text{ such that } f_i + f_{i+1} \leq k \text{ for all } i, \text{ and } f_1 \leq k-r. \tag{1.2}$$

Throughout this paper, we describe such partitions with difference conditions in terms of frequency sequences.

Andrews [And74] subsequently proved the following analytic analogue of Gordon's theorem, now referred to as the Andrews–Gordon identities.

Theorem 1.1 (Andrews–Gordon identities [And74]). *Let $k \geq 1$ and $0 \leq r \leq k$ be two integers. We have*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \frac{(q^{2k+3}, q^{k+1-r}, q^{k+2+r}; q^{2k+3})_\infty}{(q)_\infty}. \tag{1.3}$$

The case $k = 1$ of (1.3) corresponds to the Rogers–Ramanujan identities (1.1). The Andrews–Gordon identities (1.3) are also proved in [Bre80] in pair with a similar formula [Bre80, (3.3)], valid for all integers $k \geq 1$ and $0 \leq j \leq k$ (we changed notation compared to Bressoud's paper):

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+3}, q^{k+2-j+2s}, q^{k+1+j-2s}; q^{2k+3})_\infty}{(q)_\infty}. \tag{1.4}$$

Note that there is a small typo in Bressoud's paper: in his formula [Bre80, (3.3)], $\pm(k - r + i)$ (in his notation) has to be changed to $\pm(k-r+i+1)$. The identity (1.4) is derived combinatorially from (1.3) in [DJK24], while it is used in [ADJM23] to solve a combinatorial conjecture of Afsharijoo arising from commutative algebra.

Bressoud also proved an even moduli counterpart of the Andrews–Gordon identities.

Theorem 1.2 (Bressoud's identities, [Bre80, (3.4)])**.** *Let r and k be integers with $k \geq 1$ and $0 \leq r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(q^{2k+2}, q^{k+1-r}, q^{k+1+r}; q^{2k+2})_\infty}{(q)_\infty}. \tag{1.5}$$

From the form of the identities (1.5), the partitions with a congruence condition are readily obtained from the product side. In [Bre79], Bressoud introduced the corresponding partitions satisfying a difference condition along with an additional restriction. In terms of frequency sequences, the partition identity corresponding to (1.5) can be formulated as follows. The number of partitions of n into parts not congruent to $0, \pm(k - r + 1)$ modulo $2k + 2$, is equal to the number of frequency sequences $(f_i)_{i\geq 1}$ of n such that $f_i + f_{i+1} \leq k$ for all i, $f_1 \leq k - r$, and whenever $f_i + f_{i+1} = k$ for some i, the parity condition $if_i + (i + 1)f_{i+1} \equiv k - r \pmod 2$ holds.

Bressoud [Bre80, (3.5)] also proved a similar formula, which was recently derived combinatorially from (1.5) in [DJK24]:

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+1+j-2s}, q^{k+1-j+2s}; q^{2k+2})_\infty}{(q)_\infty}. \tag{1.6}$$

Kurşungöz [Kur16] considered partitions with the same frequency conditions as in Bressoud's identity, but with the opposite parity condition $if_i + (i + 1)f_{i+1} \equiv k - r + 1 \pmod 2$. He showed that their generating function is equal to

$$\frac{1}{(1+q)(q)_\infty}\left((q^{2k+2}, q^{k+2-r}, q^{k+r}; q^{2k+2})_\infty + q(q^{2k+2}, q^{k-r}, q^{k+2+r}; q^{2k+2})_\infty\right).$$

In [DJK24], a multisum formula was given for the same generating function, and the identity of Kurşungöz was proved combinatorially using this multisum and Bressoud's identity, leading to the following identities:

Theorem 1.3 (Kurşungöz identities, [Kur16] and [DJK24])**.** *Let r and k be integers with $k \geq 1$ and $0 \leq r \leq k$. Then*

$$(1+q) \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_{k-1}+2s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \\ = \frac{1}{(q)_\infty}\left((q^{2k+2}, q^{k+r}, q^{k-r+2}; q^{2k+2})_\infty + q(q^{2k+2}, q^{k+2+r}, q^{k-r}; q^{2k+2})_\infty\right). \tag{1.7}$$

Let r and k be integers with $k \geq 1$ and $0 \leq j \leq k$. Then

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k-j+2s}, q^{k+2+j-2s}; q^{2k+2})_\infty}{(q)_\infty}. \tag{1.8}$$

Stanton [Stan18] recently generalised both the Andrews–Gordon identities (1.3) and the related identities (1.4) in two ways.

Theorem 1.4. [Stan18, Theorem 3.1] *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k} \cdot \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} \\ = \sum_{s=0}^{j} \binom{j}{s} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

Moreover, the j factors q^{-s_1} and $q^{-s_i}(1 + q^{s_{i-1}+s_i})$, $2 \leq i \leq j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1 + q^{s_{i-1}+s_i}) : 2 \leq i \leq k - r\}$.

Theorem 1.5. [Stan18, Theorem 3.2] *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}. \tag{1.9}$$

We refer to Theorem 1.4 and 1.5 as the binomial extension and non-binomial extension of the Andrews–Gordon identities, respectively. As remarked by Stanton, for $j = 0$ both identities reduce to (1.3), while for $r = 0$ Theorem 1.5 yields (1.4).

Furthermore, he also presented even moduli versions, generalising both of Bressoud's identities (1.5) and (1.6) in a binomial and a non-binomial extension as well.

Theorem 1.6. [Stan18, Theorem 4.1] *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k} \cdot \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \sum_{s=0}^{j} \binom{j}{s} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}.$$

Moreover, the j factors q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \leq i \leq j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \leq i \leq k-r\}$.

Theorem 1.7. [Stan18, Theorem 4.2] *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}. \tag{1.10}$$

Again, for $j = 0$ these two identities give (1.5), while for $r = 0$ Theorem 1.7 yields Bressoud's formula (1.6). Stanton proved his results by using some (new properties of) Laurent polynomials $H_{2n}(z, a|q)$ and some iterative process.

In the first part of this paper, we give alternative proofs of Stanton's results using the Bailey pair machinery.

The Bailey lemma [Bai49], whose iterative nature was highlighted by Andrews [And84, And86, AAR99] through the Bailey chain, provides an efficient framework to prove q-series identities. For more details on the Bailey lemma and its extensions, see Section 2. Using Bailey-type techniques, we prove a binomial and a non-binomial generalisation of Theorem 1.3, in the same style as Stanton's results.

Theorem 1.8 (Binomial extension of the Kurşungöz identities)**.** *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_{k-1}+2s_k} \cdot \frac{q^{-s_1-\cdots-s_j}(1+q^{s_1+s_2})(1+q^{s_2+s_3})\cdots(1+q^{s_{j-1}+s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{1}{1+q}\left(\sum_{s=0}^{j}\binom{j}{s}\left(\frac{(q^{2k+2}, q^{k+2-r+j-2s}, q^{k+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty} + q\frac{(q^{2k+2}, q^{k-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}\right)\right).$$

Moreover, the j factors q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \leq i \leq j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \leq i \leq k-r\}$.

Theorem 1.9 (Non-binomial extension of the Kurşungöz identities)**.** *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_{k-1}+2s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} \tag{1.11}$$
$$= \frac{1}{1+q}\sum_{s=0}^{j}\left(\frac{(q^{2k+2}, q^{k+2-r+j-2s}, q^{k+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty} + q\frac{(q^{2k+2}, q^{k-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}\right).$$

Again, for $j = 0$ both theorems give (1.7), while for $r = 0$ Theorem 1.9 yields (1.8).

Furthermore, we explore analogues of these constructions for the Bressoud–Göllnitz–Gordon identities. In [Bre80, (3.6)–(3.9)], Bressoud proved four extensions of the famous Göllnitz–Gordon identities, introduced in [Gor65] and [Gol67] independently,

$$\sum_{n \geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q, q^4, q^7; q^8)_\infty}, \quad \text{and} \quad \sum_{n \geq 0} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3, q^4, q^5; q^8)_\infty}, \tag{1.12}$$

which are modulo 8 Rogers–Ramanujan-type identities. Among them, the identities [Bre80, (3.6)] state the following.

Theorem 1.10 (Bressoud–Göllnitz–Gordon)**.** *For all integers $k \geq 1$ and $0 \leq j \leq k$,*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{2(s_1^2+\cdots+s_k^2-s_1-\cdots-s_j)}(-q^{1+2s_k};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\dots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{s=0}^{j}(q^{4k+4},q^{2k+1-2j+2s},q^{2k+3+2j-2s};q^{4k+4})_\infty. \tag{1.13}$$

Note that taking $k=1$ and $j=0$, this formula becomes

$$\sum_{s_1\geq 0}\frac{q^{2s_1^2}(-q^{1+2s_1};q^2)_\infty}{(q^2;q^2)_{s_1}} = \frac{(q^2;q^4)_\infty}{(q)_\infty}(q^8,q^3,q^5;q^8)_\infty,$$

therefore the product side is the same as in the first Göllnitz–Gordon identity in (1.12). Using the infinite q-binomial theorem [AAR99, Theorem 10.2.1], the left-hand side is

$$\sum_{m,\ell\geq 0}\frac{q^{2m^2+\ell^2+2m\ell}}{(q^2;q^2)_m(q^2;q^2)_\ell} = \sum_{m,\ell\geq 0}\frac{q^{(m+\ell)^2}}{(q^2;q^2)_{m+\ell}}\cdot q^{m^2}\begin{bmatrix}m+\ell\\ m\end{bmatrix}_{q^2} = \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\sum_{m=0}^{n}q^{m^2}\begin{bmatrix}n\\ m\end{bmatrix}_{q^2},$$

which, by using the finite q-binomial theorem [AAR99, Corollary 10.2.2. (c)], is the left-hand side of the first identity in (1.12). Similarly, when $k=j=1$, Formula (1.13) is equivalent to the sum of both Göllnitz–Gordon identities in (1.12).

In [DJK25], m-versions of the formulas [Bre80, (3.6)–(3.9)] are proved using the Bailey lattice, and two more identities of the same kind are discovered in passing. A natural question is whether it is possible to prove "Stanton type" formulas generalising these Bressoud–Göllnitz–Gordon identities. We answer positively this question by providing binomial and non-binomial extensions of the Bresssoud–Göllnitz–Gordon identities.

Theorem 1.11 (Binomial extension of the Bresssoud–Göllnitz–Gordon identities)**.** *Let $j,r\geq 0$ and $k\geq 1$ be integers such that $j+r\leq k$. Then*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{2(s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k)}q^{-2s_1}(q^{2s_1}+q^{-2s_2})\cdots(q^{2s_{j-1}}+q^{-2s_j})(-q^{1+2s_k};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}$$
$$= \frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{s=0}^{j}\binom{j}{s}\Big((q^{4k+4},q^{2k+3-2r+2j-4s},q^{2k+1+2r-2j+4s};q^{4k+4})_\infty + q(q^{4k+4},q^{2k+1-2r+2j-4s},q^{2k+3+2r-2j+4s};q^{4k+4})_\infty\Big).$$

Moreover, $q^{-2s_1}(q^{2s_1}+q^{-2s_2})\cdots(q^{2s_{j-1}}+q^{-2s_j})$ can be replaced by any product of j factors taken from $\{q^{-2s_1}\}\cup\{(q^{2s_{i-1}}+q^{-2s_i}):2\leq i\leq k-r\}$.

Theorem 1.12 (Non-binomial extension of the Bresssoud–Göllnitz–Gordon identities)**.** *Let $j,r\geq 0$ and $k\geq 1$ be integers such that $j+r\leq k$. Then*

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{2(s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k)}(-q^{1+2s_k};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\dots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{s=0}^{j}\Big((q^{4k+4},q^{2k+3-2r+2j-4s},q^{2k+1+2r-2j+4s};q^{4k+4})_\infty + q(q^{4k+4},q^{2k+1-2r+2j-4s},q^{2k+3+2r-2j+4s};q^{4k+4})_\infty\Big). \tag{1.14}$$

If we take $r=0$ in (1.14), some elementary manipulation yields (1.13): it is clear for the left-hand sides, while splitting the right-hand side of (1.14) (with $r=0$) into two sums and replacing s by $j-s$ in the second one, we see that both sums are the same, therefore one can factorise by $1+q$. The remaining sum is the one on the left-hand side of (1.13) in which even and odd values of s have been split. On the other hand, setting $j = 0$ in (1.14) gives

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{2(s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k)}(-q^{1+2s_k};q^2)_\infty}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty} \times \left((q^{4k+4}, q^{2k+3-2r}, q^{2k+1+2r}, q^{4k+4})_\infty + q(q^{4k+4}, q^{2k+1-2r}, q^{2k+3+2r}, q^{4k+4})_\infty \right), \tag{1.15}$$

which seems to be new. Note that taking $k = 1$ in (1.14), we have three possible choices for our integral parameters r, j, namely $(0,0)$, $(1,0)$ and $(0,1)$. As explained below (1.13), the cases $r = 0$ and $j = 0$, $j = 1$ yield the Göllnitz–Gordon identities (1.12). For $r = 1$ and $j = 0$, the formula is the one obtained by taking $k = r = 1$ in (1.15), and we see similarly that it is equivalent to the first Göllnitz–Gordon identity plus q times the second one, divided by $1 + q$.

Moreover, we prove two additional general identities similar to Theorem 1.12 (see Theorems 4.3 and 4.4), which extend Slater's identities [Sl52, (8), (12), (13)].

In the second part of this paper, we turn to a combinatorial perspective on Stanton's identities. While Bailey pairs are a remarkably powerful and flexible tool for proving partition identities, they do not provide a combinatorial interpretation of these identities. Combinatorial approaches to partition identities are generally more difficult. A famous example is the lack of a simple bijection for the combinatorial version of the Rogers–Ramanujan identities. Several efforts have been made to understand there identities from a combinatorial perspective, including work on bijective proofs (see, e.g., [BP06], [BZ82], [Cor17], [GM81]).

It is always simple to interpret the product side of the identities as partitions with congruence conditions. In the case of Rogers–Ramanujan, it is also relatively simple to see that the sum side is the generating function for partitions where parts differ by at least two, or equivalently $f_i + f_{i+1} \leq 1$. However, in the Andrews–Gordon identities and the other identities we consider here, it is not at all obvious combinatorially that the sum side is the generating function for partitions with difference conditions. In the case of Andrews–Gordon, a more natural interpretation has been given by Andrews in terms of Durfee squares [And79].

To prove combinatorially that the sum side of the Andrews–Gordon identities is the generating function for partitions with difference conditions, Warnaar [War97] introduced a bijection based on particle motions. He used it to derive a finitisation of the Andrews–Gordon identities. It was formulated as a one-dimensional lattice-gas of fermionic particles using slightly different notation. He then used this finisation to prove the identities. Inspired by Warnaar's work, the authors of [DJK24], including the first and third authors, generalised his approach by adding parts equal to 0 to the reasoning and introducing an explicit bijection Λ and its inverse Γ, which provided a combinatorial framework for proving several partition identities. Rather than using a finitisation, they took a different approach: by combining the infinite version with the classical Andrews–Gordon and Bressoud identities, they established several identities, among which (1.4), (1.6), (1.7), and (1.8).

Although Stanton proved Theorems 1.4–1.7 in [Stan18], his proofs did not provide combinatorial interpretations for the sum sides. After reading [DJK24], he asked the authors the following problem, hoping that their techniques could be applied to his identities.

Problem 1.13 (Stanton)**.** Give partition-theoretic interpretations and combinatorial proofs of Theorems 1.4–1.7.

Indeed, the ideas of [War97] and [DJK24] can also be used to give combinatorial interpretations and proofs of the identities in Theorems 1.5, 1.7, and 1.9. We give a suitable interpretation of the sum sides of the identities using particle motion and a combinatorial reasoning on parts equal to 0. Then we give combinatorial proofs of these identities, using this interpretation and the Andrews–Gordon and Bressoud identities.

Following [DJK24], we allow partitions to have non-negative parts, not just positive ones. That is, a *partition* λ of n is a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ whose sum is n. The difference from the previous definition is that 0 is now allowed as a part, and is taken into account in the length of the partition. Accordingly, the associated frequency sequence becomes $f = (f_0, f_1, \cdots)$, rather than $(f_1, f_2, \cdots)$. With this extension, the authors in [DJK24] provided combinatorial proofs of (1.4), (1.6), and (1.8), and gave combinatorial interpretations in terms of partitions with difference conditions. For example, they proved that the combinatorial model for (1.4) is given by

(1.16) frequency sequences $(f_i)_{i\geq 0}$ of n such that $f_i + f_{i+1} \leq k$ for all i, and $f_0 \leq j$.

Following this framework, we use particle motion and Theorems 1.1 and 1.2 to give combinatorial proofs of Theorems 1.5, 1.7, and 1.9, and we also provide their following partition-theoretic interpretations.

Theorem 1.14. *The left-hand side of* (1.9) *in Theorem 1.5 is the generating function for the frequency sequences* $(f_i)_{i\geq 0}$ *such that*

- $f_i + f_{i+1} \leq k$ *for all* i*, and*
- $f_0 \leq j - \max\{f_0 + f_1 - (k - r), 0\}$.

Theorem 1.15. *The left-hand side of* (1.10) *in Theorem 1.7 is the generating function for the frequency sequences* $(f_i)_{i\geq 0}$ *such that*

- $f_i + f_{i+1} \leq k$ *for all* i,
- $f_0 \leq j - \max\{f_0 + f_1 - (k-r), 0\}$, *and*
- *if* $f_i + f_{i+1} = k$ *for some* i, *then* $if_i + (i+1)f_{i+1} \equiv k + r - j \pmod 2$.

In addition, we obtain new identities of Kurşungöz type using the combinatorial model. Interestingly, these formulas coincide with those previously obtained using the Bailey pairs, thus offering a purely combinatorial approach.

Theorem 1.16. *The left-hand side of* (1.11) *in Theorem 1.9 is the generating function for the frequency sequences* $(f_i)_{i\geq 0}$ *such that*

- $f_i + f_{i+1} \leq k$ *for all* i,
- $f_0 \leq j - \max\{f_0 + f_1 - (k-r), 0\}$, *and*
- *if* $f_i + f_{i+1} = k$ *for some* i, *then* $if_i + (i+1)f_{i+1} \equiv k + r - j + 1 \pmod 2$.

The combinatorial interpretation in Theorem 1.14 is new. When $j = 0$, the second condition becomes $f_0 = 0$ and $f_1 \leq k - r$, which yields the interpretation (1.2), and when $r = 0$, it satisfies $f_0 \leq j$, yielding (1.16). Furthermore, the same model extends naturally to the Bressoud (Theorem 1.15) and Kurşungöz (Theorem 1.16) cases, which shows its flexibility and confirms that it provides a natural combinatorial interpretation.

This paper is organised as follows. In Section 2, we recall some basic results and tools from the Bailey machinery that we will use later in our proofs. In Section 3, we use combinations of the simplest of these tools (the Bailey lemma and key lemmas of Lovejoy and McLaughlin) to prove all the binomial extensions in Theorems 1.4, 1.6, 1.8 and 1.11. In Section 4, some combinations of the Bailey lemma and lattice are used to provide proofs of all the non-binomial extensions in Theorems 1.5, 1.7, 1.9, 1.12, 4.3 and 4.4. In Section 5, we recall the particle motion bijection to prepare the combinatorial proofs of the theorems in the next sections. In Section 6, using this bijection and the Andrews–Gordon identities, a combinatorial proof of Theorem 1.5 is provided. In Section 7, we use the bijection and the Bressoud identities to give combinatorial proofs for Theorems 1.7 and 1.9. In Section 8, we conclude the paper by a list of remarks and questions.

2. Bailey pairs

Fix complex numbers a and q. Recall [Bai49] that a *Bailey pair* $((\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0})$ ((α_n, β_n) for short) relative to a is a pair of sequences satisfying:

$$\beta_n = \sum_{\ell=0}^{n} \frac{\alpha_\ell}{(q)_{n-\ell}(aq)_{n+\ell}} \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

By convention, we set $\alpha_\ell = 0$ for all $\ell < 0$.

Given a Bailey pair, the Bailey lemma [Bai49] allows one to produce infinitely new Bailey pairs. Bailey [Bai49] originally applied it without iterating it, and Andrews [And84] generalised Bailey's approach to exhibit its iterative nature with the concept of Bailey chain.

Theorem 2.1 (Bailey lemma, Andrews' version)**.** *If* (α_n, β_n) *is a Bailey pair relative to* a, *then so is* (α'_n, β'_n), *where*

$$\alpha'_n = \frac{(\rho,\sigma)_n (aq/\rho\sigma)^n}{(aq/\rho, aq/\sigma)_n}\, \alpha_n,$$

and

$$\beta'_n = \sum_{\ell=0}^{n} \frac{(\rho,\sigma)_\ell (aq/\rho\sigma)_{n-\ell} (aq/\rho\sigma)^\ell}{(q)_{n-\ell}(aq/\rho, aq/\sigma)_n}\, \beta_\ell.$$

In this paper, we use the following two particular cases.

Lemma 2.2 (Bailey lemma with $\rho, \sigma \to \infty$)**.** *If* (α_n, β_n) *is a Bailey pair relative to* a, *then so is* (α'_n, β'_n), *where*

$$\alpha'_n = a^n q^{n^2} \alpha_n, \quad \text{and} \quad \beta'_n = \sum_{\ell=0}^{n} \frac{a^\ell q^{\ell^2}}{(q)_{n-\ell}}\, \beta_\ell.$$

Lemma 2.3 (Bailey lemma with $\sigma \to \infty$)**.** *If* (α_n, β_n) *is a Bailey pair relative to* a, *then so is* (α'_n, β'_n), *where*

$$\alpha'_n = \frac{(-1)^n (\rho)_n (aq/\rho)^n q^{\binom{n}{2}}}{(aq/\rho)_n}\, \alpha_n, \quad \text{and} \quad \beta'_n = \sum_{\ell=0}^{n} \frac{(-1)^\ell (\rho)_\ell (aq/\rho)^\ell q^{\binom{\ell}{2}}}{(aq/\rho)_n (q)_{n-\ell}}\, \beta_\ell.$$

Despite its simple form and quite elementary proof (see [GR04, Appendix (II.12)]), the Bailey lemma can be used to prove many q-series identities. One of the most widely used Bailey pairs is the unit Bailey pair, defined in [And84, (2.12) and (2.13)].

Definition 2.4 ([And84])**.** The *unit Bailey pair* relative to a is the pair (α_n, β_n) defined by

$$\alpha_n = (-1)^n q^{\binom{n}{2}} \frac{1-aq^{2n}}{1-a}\frac{(a)_n}{(q)_n}, \quad \text{and} \quad \beta_n = \delta_{n,0}. \tag{2.2}$$

The Rogers–Ramanujan identities can be proved easily by applying Theorem 2.1 twice to the unit Bailey pair (2.2), and the $r = 0$ and $r = k$ special instances of the Andrews–Gordon identities in Theorem 1.1 follow by iterating $k + 1 \geq 2$ times this process.

But the Bailey chain is not sufficient to prove the cases $0 < r < k$ of the Andrews–Gordon identities, and the Bailey lattice was developed in [AAB87] to remedy this problem by switching the parameter a to a/q at some point before iterating the Bailey lemma.

Theorem 2.5 (Bailey lattice)**.** *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to a/q, where*

$$\alpha'_n = \frac{(\rho,\sigma)_n (a/\rho\sigma)^n}{(a/\rho, a/\sigma)_n}(1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right),$$

and

$$\beta'_n = \sum_{\ell=0}^{n} \frac{(\rho,\sigma)_\ell (a/\rho\sigma)_{n-\ell}(a/\rho\sigma)^\ell}{(q)_{n-\ell}(a/\rho, a/\sigma)_n}\beta_\ell.$$

We use the particular case when $\rho, \sigma \to \infty$.

Lemma 2.6 (Bailey lattice with $\rho, \sigma \to \infty$)**.** *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to a/q, where*

$$\alpha'_n = a^n q^{n^2-n}(1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \textit{and} \quad \beta'_n = \sum_{\ell=0}^{n} \frac{a^\ell q^{\ell^2-\ell}}{(q)_{n-\ell}}\beta_\ell.$$

Authors have been interested in ways to avoid using the Bailey lattice. For example, Andrews, Schilling and Warnaar [ASW99, Section 3] proved (1.3) using the Bailey lemma and bypassing the Bailey lattice, Bressoud, Ismail and Stanton [BIS00] used a change of base to avoid the lattice, and McLaughlin [McL18] showed that (1.3) can be proved easily by combining the Bailey Lemma with a simple lemma, which gives a Bailey pair relative to a/q given a Bailey pair relative to a.

Lemma 2.7 (McLaughlin, Key lemma 1)**.** *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to a/q, where*

$$\alpha'_n = (1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \textit{and} \quad \beta'_n = \beta_n.$$

In [DJK25], the first and third authors, together with Konan, showed that the Bailey lattice follows directly from this key lemma and the Bailey lemma. They also extended it to a bilateral version and deduced a bilateral Bailey lattice which they used to prove m-versions of the Andrews–Gordon and Bressoud identities. The following lemma, also due to McLaughlin [McL18], is similar to Lemma 2.7 (as it also transforms a into a/q) and led in [DJK25] to a different bilateral Bailey lattice.

Lemma 2.8 (McLaughlin, Key lemma 2)**.** *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to a/q, where*

$$\alpha'_n = (1-a)\left(\frac{q^n\alpha_n}{1-aq^{2n}} - \frac{q^{n-1}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \textit{and} \quad \beta'_n = q^n\beta_n.$$

On the other hand, Lovejoy [Lov04, (2.4) and (2.5)] proved a lemma which transforms a Bailey pair relative to a into a Bailey pair relative to aq.

Lemma 2.9 (Lovejoy)**.** *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to aq, where*

$$\alpha'_n = \frac{(1-aq^{2n+1})(aq/b)_n(-b)^n q^{n(n-1)/2}}{(1-aq)(bq)_n}\sum_{\ell=0}^{n}\frac{(b)_\ell}{(aq/b)_\ell}(-b)^{-\ell}q^{-\ell(\ell-1)/2}\alpha_\ell,$$

and

$$\beta'_n = \frac{1-b}{1-bq^n}\beta_n.$$

Again, we will mostly need in our calculations a particular case of this lemma (namely $b = 0$), which can be seen as the inverse of Lemma 2.7.

Lemma 2.10 (Lovejoy's lemma with $b = 0$). *If (α_n, β_n) is a Bailey pair relative to a, then (α'_n, β'_n) is a Bailey pair relative to aq, where*

$$\alpha'_n = \frac{(1-aq^{2n+1})a^n q^{n^2}}{1-aq} \sum_{\ell=0}^{n} a^{-\ell} q^{-\ell^2} \alpha_\ell, \quad \textit{and} \quad \beta'_n = \beta_n.$$

In [AAB87], the following result is obtained (in a more general form) by iterating $r+1$ times Theorem 2.1, using Theorem 2.5, and concluding with $k-r-1$ times Theorem 2.1 with a replaced by a/q.

Theorem 2.11 (Agarwal–Andrews–Bressoud, new notation). *If (α_n, β_n) is a Bailey pair relative to a, then for all integers $k \geq 1$ and $-1 \leq r \leq k$, we have:*

$$\sum_{s_1 \geq \cdots \geq s_{k+1} \geq 0} \frac{a^{s_1+\cdots+s_{k+1}} q^{s_1^2+\cdots+s_{k+1}^2-s_1-\cdots-s_{k-r}}}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}} \beta_{s_{k+1}} = \frac{1}{(aq)_\infty} \sum_{\ell \geq 0} a^{(k+1)\ell} q^{(k+1)\ell^2-(k-r)\ell} \frac{1-a^{k-r+1}q^{2\ell(k-r+1)}}{1-aq^{2\ell}} \alpha_\ell. \tag{2.3}$$

In [AAB87], Agarwal, Andrews and Bressoud prove the Andrews–Gordon identities (1.3) by applying Theorem 2.11 to the unit Bailey pair (2.2) with $a = q$ and factorising the right-hand side using the Jacobi triple product identity [GR04, Appendix, (II.28)]

$$\sum_{\ell \in \mathbb{Z}} (-1)^\ell z^\ell q^{\ell(\ell-1)/2} = (q, z, q/z; q)_\infty. \tag{2.4}$$

Moreover, it is explained in [DJK25] how (1.4) is simply a consequence of (2.3) with $a = 1$ and (2.4). In the same paper, (1.3) and (1.4) are embedded through a bilateral version of the Bailey lattice in a single generalisation, which is called "m-version of the Andrews–Gordon identities", where the parameter m is a non-negative integer. Interestingly, Stanton's non-binomial formula (1.9) provides another embedding of (1.3) and (1.4).

The structure of our proofs of Theorems 1.4–1.12 using Bailey pairs is the following. We take a well-chosen Bailey pair and apply $k+1$ times either the Bailey lemma or Bailey lattice or one of the key lemmas to obtain a new Bailey pair, and then we let n tend to infinity in the relation (2.1) corresponding to that new Bailey pair. We then simplify the right-hand side and apply the Jacobi triple product identity (2.4) to obtain the corresponding product expression. This type of techniques allows us to obtain both the binomial and non-binomial extensions of the identities under consideration.

We conclude this section by introducing two Bailey pairs, which will be used later in our proofs in addition to the unit Bailey pair. The first one is obtained from the change of base given in [BIS00, (D4)], which asserts that if (α_n, β_n) is a Bailey pair relative to a, then so is (α'_n, β'_n), where

$$\alpha'_n = \frac{1+a}{1+aq^{2n}} q^n \alpha_n(a^2, q^2) \quad \text{and} \quad \beta'_n = \sum_{\ell=0}^{n} \frac{(-a)_{2\ell}}{(q^2;q^2)_{n-\ell}} q^\ell \beta_\ell(a^2, q^2).$$

Applying this to the unit Bailey pair (2.2) gives the following Bailey pair.

Definition 2.12 ([BIS00]). The *Bailey pair (D'4)* relative to a is (α_n, β_n) where

$$\alpha_n = (-1)^n q^{n^2} \frac{1-aq^{2n}}{1-a} \frac{(a^2;q^2)_n}{(q^2;q^2)_n}, \quad \text{and} \quad \beta_n = \frac{1}{(q^2;q^2)_n}. \tag{2.5}$$

Similarly, the change of base from [BIS00, (D1)] states that if (α_n, β_n) is a Bailey pair relative to a, then so is (α'_n, β'_n), where

$$\alpha'_n = \alpha_n(a^2, q^2) \quad \text{and} \quad \beta'_n = \sum_{\ell=0}^{n} \frac{(-aq)_{2\ell}}{(q^2;q^2)_{n-\ell}} q^{n-\ell} \beta_\ell(a^2, q^2).$$

Applying this to the unit Bailey pair (2.2) gives the following Bailey pair.

Definition 2.13 ([BIS00]). The *Bailey pair (D'1)* relative to a is (α_n, β_n) where

$$\alpha_n = (-1)^n q^{n^2-n} \frac{1-a^2q^{4n}}{1-a^2} \frac{(a^2;q^2)_n}{(q^2;q^2)_n}, \quad \text{and} \quad \beta_n = \frac{q^n}{(q^2;q^2)_n}. \tag{2.6}$$

3. Proofs of binomial extensions

3.1. **Preliminary results.** In this section, we prove preliminary results that will be useful in the proofs of Theorems 1.4, 1.6, 1.8, and 1.11. We first combine several simple lemmas mentioned in Section 2 to prove a slightly more involved one.

Lemma 3.1. *If (α_n, β_n) is a Bailey pair relative to a, then so is (α_n', β_n'), where*

$$\alpha_n' = a^n q^{n^2-n}\left((1+q^{2n})\alpha_n + (1-aq^{2n})(1-a^{-1})\sum_{\ell=0}^{n-1}\alpha_\ell\right),$$

and

$$\beta_n' = \sum_{\ell=0}^{n} \frac{a^\ell q^{\ell^2}}{(q)_{n-\ell}}(q^n + q^{-\ell})\beta_\ell.$$

Proof. Let (α_n, β_n) be a Bailey pair relative to a. By Lemma 2.7, $(\alpha_n^{(1)}, \beta_n^{(1)})$ is a Bailey pair relative to a/q, where

$$\alpha_n^{(1)} = (1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \beta_n^{(1)} = \beta_n.$$

Then by Lemma 2.2 with a replaced by a/q, $(\alpha_n^{(2)}, \beta_n^{(2)})$ is a Bailey pair relative to a/q, where

$$\alpha_n^{(2)} = a^n q^{n^2-n}(1-a)\left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \beta_n^{(2)} = \sum_{\ell=0}^{n}\frac{a^\ell q^{\ell^2-\ell}}{(q)_{n-\ell}}\beta_\ell.$$

Finally, by Lemma 2.10 with a replaced by a/q, $(\alpha_n^{(3)}, \beta_n^{(3)})$ is a Bailey pair relative to a, where

$$\alpha_n^{(3)} = a^n q^{n^2-n}\left(\alpha_n + (1-aq^{2n})\sum_{\ell=0}^{n-1}\alpha_\ell\right), \quad \beta_n^{(3)} = \sum_{\ell=0}^{n}\frac{a^\ell q^{\ell^2-\ell}}{(q)_{n-\ell}}\beta_\ell. \tag{3.1}$$

Now start again with (α_n, β_n) Bailey pair relative to a. By Lemma 2.2, so is $(\tilde{\alpha}_n^{(1)}, \tilde{\beta}_n^{(1)})$, where

$$\tilde{\alpha}_n^{(1)} = a^n q^{n^2}\alpha_n, \quad \tilde{\beta}_n^{(1)} = \sum_{\ell=0}^{n}\frac{a^\ell q^{\ell^2}}{(q)_{n-\ell}}\beta_\ell.$$

By Lemma 2.8, $(\tilde{\alpha}_n^{(2)}, \tilde{\beta}_n^{(2)})$ is a Bailey pair relative to a/q, where

$$\tilde{\alpha}_n^{(2)} = (1-a)\left(\frac{a^n q^{n^2+n}\alpha_n}{1-aq^{2n}} - \frac{a^{n-1}q^{n^2-n}\alpha_{n-1}}{1-aq^{2n-2}}\right), \quad \tilde{\beta}_n^{(2)} = q^n\sum_{\ell=0}^{n}\frac{a^\ell q^{\ell^2}}{(q)_{n-\ell}}\beta_\ell.$$

Finally, by Lemma 2.10 with a replaced by a/q, $(\tilde{\alpha}_n^{(3)}, \tilde{\beta}_n^{(3)})$ is a Bailey pair relative to a, where

$$\tilde{\alpha}_n^{(3)} = a^n q^{n^2-n}\left(q^{2n}\alpha_n - a^{-1}(1-aq^{2n})\sum_{\ell=0}^{n-1}\alpha_\ell\right), \quad \tilde{\beta}_n^{(3)} = q^n\sum_{\ell=0}^{n}\frac{a^\ell q^{\ell^2}}{(q)_{n-\ell}}\beta_\ell. \tag{3.2}$$

Adding (3.1) and (3.2) gives the desired result, by linearity of the Bailey pair relation (2.1). □

When $a = 1$, Lemma 3.1 simplifies and gives the following, which we will use in our proofs.

Lemma 3.2. *If (α_n, β_n) is a Bailey pair relative to 1, then so is (α_n', β_n'), where*

$$\alpha_n' = q^{n^2-n}(1+q^{2n})\alpha_n, \quad \textit{and} \quad \beta_n' = \sum_{\ell=0}^{n}\frac{q^{\ell^2}}{(q)_{n-\ell}}(q^n+q^{-\ell})\beta_\ell.$$

Remark 3.3. If (α_n, β_n) is a Bailey pair relative to 1, and we apply first Lemma 2.2 and then Lemma 3.2 to it, we obtain the Bailey pair $(\alpha_n^{(1)}, \beta_n^{(1)})$, where

$$\alpha_n^{(1)} = q^{2n^2-n}(1+q^{2n})\alpha_n, \quad \beta_n^{(1)} = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell}\frac{q^{\ell^2+m^2}}{(q)_{n-\ell}(q)_{\ell-m}}(q^n+q^{-\ell})\,\beta_m.$$

If we apply first Lemma 3.2 and then Lemma 2.2 to it, we obtain the Bailey pair $(\alpha_n^{(2)}, \beta_n^{(2)})$, where

$$\alpha_n^{(2)} = q^{2n^2-n}(1+q^{2n})\alpha_n, \quad \beta_n^{(2)} = \sum_{\ell=0}^{n}\sum_{m=0}^{\ell}\frac{q^{\ell^2+m^2}}{(q)_{n-\ell}(q)_{\ell-m}}(q^\ell+q^{-m})\beta_m.$$

Given that $\alpha_n^{(1)} = \alpha_n^{(2)}$ and that $(\alpha_n^{(1)}, \beta_n^{(1)})$ and $(\alpha_n^{(2)}, \beta_n^{(2)})$ are Bailey pairs, we know that $\beta_n^{(1)} = \beta_n^{(2)}$, even if it is less obvious from the expressions for $\beta_n^{(1)}$ and $\beta_n^{(2)}$ given above.

This remark plays a key role in proving the "Moreover, the j factors q^{-s_1} and $q^{-s_i}(1+q^{s_{i-1}+s_i})$, $2 \le i \le j$ may be replaced by any j-element subset of $\{q^{-s_1}\} \cup \{q^{-s_i}(1+q^{s_{i-1}+s_i}) : 2 \le i \le k-r\}$" part of the four main theorems.

Now we use Lemma 3.2 to prove two propositions.

Proposition 3.4. *If (α_n, β_n) is a Bailey pair relative to q, then, for all integers $j, r \ge 0$ and $k \ge 1$ such that $r + j \le k$, $(\alpha_n^{(k+1)}, \beta_n^{(k+1)})$ is a Bailey pair relative to 1, where*

$$\begin{aligned}\alpha_n^{(k+1)} &= (1-q)(1+q^{2n})^j q^{(k+1)n^2+(r+1-j)n}\left(\frac{\alpha_n}{1-q^{2n+1}} - \frac{q^{-2rn-1}\alpha_{n-1}}{1-q^{2n-1}}\right),\\ \beta_n^{(k+1)} &= \sum_{n\ge s_1\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_1^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}(q^n+q^{-s_1})(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{n-s_1}(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}.\end{aligned} \tag{3.3}$$

Moreover, $(q^n+q^{-s_1})(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})$ can be replaced by any product of j factors taken from $\{(q^n+q^{-s_1})\} \cup \{(q^{s_{i-1}}+q^{-s_i}) : 2 \le i \le k-r\}$.

Proof. Let (α_n, β_n) be a Bailey pair relative to q. First apply Lemma 2.2 to it $r+1$ times with $a = q$. This gives a Bailey pair $(\alpha_n^{(r+1)}, \beta_n^{(r+1)})$ relative to q, where

$$\alpha_n^{(r+1)} = q^{(r+1)n^2+(r+1)n}\alpha_n,$$

and

$$\beta_n^{(r+1)} = \beta_{s_{k-r}}^{(r+1)} = \sum_{s_{k-r}\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_{k-r+1}^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}}{(q)_{s_{k-r}-s_{k-r+1}}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}.$$

Next apply Lemma 2.7 with $a = q$ once. This yields a Bailey pair $(\tilde{\alpha}_n^{(r+1)}, \tilde{\beta}_n^{(r+1)})$ relative to 1, where

$$\tilde{\alpha}_n^{(r+1)} = (1-q)q^{(r+1)n^2+(r+1)n}\left(\frac{\alpha_n}{1-q^{2n+1}} - \frac{q^{-2rn-1}\alpha_{n-1}}{1-q^{2n-1}}\right),$$

and

$$\tilde{\beta}_n^{(r+1)} = \tilde{\beta}_{s_{k-r}}^{(r+1)} = \sum_{s_{k-r}\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_{k-r+1}^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}}{(q)_{s_{k-r}-s_{k-r+1}}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}.$$

Now apply Lemma 2.2 $k-r-j$ times with $a = 1$. This gives a Bailey pair $(\alpha_n^{(k-j+1)}, \beta_n^{(k-j+1)})$ relative to 1, where

$$\alpha_n^{(k-j+1)} = (1-q)q^{(k-j+1)n^2+(r+1)n}\left(\frac{\alpha_n}{1-q^{2n+1}} - \frac{q^{-2rn-1}\alpha_{n-1}}{1-q^{2n-1}}\right),$$

and

$$\beta_n^{(k-j+1)} = \beta_{s_j}^{(k-j+1)} = \sum_{s_j\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_{j+1}^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}}{(q)_{s_j-s_{j+1}}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}.$$

Finally, apply j times Lemma 3.2. This results in a Bailey pair $(\alpha_n^{(k+1)}, \beta_n^{(k+1)})$ relative to 1, where

$$\begin{aligned}\alpha_n^{(k+1)} &= (1-q)(1+q^{2n})^j q^{(k+1)n^2+(r-j+1)n}\left(\frac{\alpha_n}{1-q^{2n+1}} - \frac{q^{-2rn-1}\alpha_{n-1}}{1-q^{2n-1}}\right),\\ \beta_n^{(k+1)} &= \sum_{n\ge s_1\ge\cdots\ge s_{k+1}\ge 0} \frac{q^{s_1^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}(q^n+q^{-s_1})(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{n-s_1}(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}.\end{aligned}$$

By Remark 3.3, the $k-r-j$ applications of Lemma 2.2 and the j applications of Lemma 3.2 can be done in any order. Hence, the product $(q^n+q^{-s_1})(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})$ can be replaced by any product of j factors taken from $\{(q^n+q^{-s_1})\} \cup \{(q^{s_{i-1}}+q^{-s_i}) : 2 \le i \le k-r\}$. ∎

We can now take a limit and obtain the following.

Proposition 3.5. *Let $(\alpha_n,\ \beta_n)$ be a Bailey pair relative to q. Then, for all integers $j, r \geq 0$ and $k \geq 1$ such that $r + j \leq k$, we have*

$$\sum_{s_1\geq\cdots\geq s_{k+1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}$$
$$= \frac{1}{(q)_\infty}\sum_{\ell=0}^{\infty} q^{(k+1)\ell^2+(r-j+1)\ell}\frac{1-q}{1-q^{2\ell+1}}\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right)\alpha_\ell.$$

Moreover $q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})$ can be replaced by any product of j factors taken from $\{q^{-s_1}\}\cup\{(q^{s_{i-1}}+q^{-s_i}) : 2 \leq i \leq k-r\}$.

Proof. Letting $n \to \infty$ in (2.1) gives that if (α_n, β_n) is a Bailey pair relative to 1, then

$$\beta_\infty = \frac{1}{(q)_\infty^2}\sum_{\ell=0}^{\infty}\alpha_\ell. \tag{3.4}$$

Let (α_n, β_n) be a Bailey pair relative to q. Keeping the notation of Proposition 3.4, $(\alpha_n^{(k+1)},\ \beta_n^{(k+1)})$ is a Bailey pair relative to 1. Thus we combine (3.4) and (3.3) with $n \to \infty$ to obtain

$$\beta_\infty^{(k+1)} = \sum_{s_1\geq\cdots\geq s_{k+1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k+1}^2+s_{k-r+1}+\cdots+s_{k+1}}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_\infty(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}} \tag{3.5}$$
$$= \frac{1-q}{(q)_\infty^2}\sum_{\ell=0}^{\infty}(1+q^{2\ell})^j q^{(k+1)\ell^2+(r-j+1)\ell}\left(\frac{\alpha_\ell}{1-q^{2\ell+1}}-\frac{q^{-2r\ell-1}\alpha_{\ell-1}}{1-q^{2\ell-1}}\right),$$

where we recall that $\alpha_{-\ell} = 0$ for $\ell > 0$. Rearranging the last expression gives

$$\beta_\infty^{(k+1)} = \frac{1}{(q)_\infty^2}\sum_{\ell=0}^{\infty} q^{(k+1)\ell^2+(r-j+1)\ell}\frac{1-q}{1-q^{2\ell+1}}\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right)\alpha_\ell. \tag{3.6}$$

Equating (3.5) and (3.6) gives the desired result. ☐

3.2. **Proofs of Theorems 1.4, 1.6, 1.8, and 1.11.** The proofs of all four of our main theorems will use Proposition 3.5, but applied to different Bailey pairs (and some modifications in k and r). We start by proving Theorem 1.4.

Proof of Theorem 1.4. Apply Proposition 3.5 to the unit Bailey pair for $a = q$ (see Definition 2.4):

$$\alpha_n = (-1)^n q^{\binom{n}{2}}\frac{1-q^{2n+1}}{1-q}, \qquad \beta_n = \delta_{n,0}.$$

This yields

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_k}} \tag{3.7}$$
$$= \frac{1}{(q)_\infty}\sum_{\ell=0}^{\infty} q^{(k+\frac{3}{2})\ell^2+(r-j+\frac{1}{2})\ell}(-1)^\ell\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right).$$

Split the sum on the right-hand side into two parts. In the second sum, apply the change of variables $\ell \mapsto -\ell-1$. This transforms the expression into

$$\frac{1}{(q)_\infty}\sum_{\ell\in\mathbb{Z}} q^{(k+\frac{3}{2})\ell^2+(r-j+\frac{1}{2})\ell}(-1)^\ell(1+q^{2\ell})^j.$$

Now apply the binomial theorem to expand $(1+q^{2\ell})^j$:

$$\frac{1}{(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+3)\binom{\ell}{2}+(k+r-j+2s+2)\ell}.$$

Finally, apply the Jacobi triple product identity (2.4) with $q \mapsto q^{2k+3}$ and $z = q^{k+2+r-j+2s}$, yielding

$$\frac{1}{(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

Note that, in the first sum in (3.7), $q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})$ can be replaced by any product of j factors taken from $\{q^{-s_1}\}\cup\{(q^{s_{i-1}}+q^{-s_i}) : 2\le i\le k-r\}$. This completes the proof. □

We also prove Theorems 1.6, 1.8, and 1.11 in a similar way. We prove Theorem 1.6 by applying our procedure not to the unit Bailey pair, but to the Bailey pair (D'4) in Definition 2.12.

Proof of Theorem 1.6. Consider the Bailey pair (D'4) in (2.5) with $a=q$:

$$\alpha_n=(-1)^n q^{n^2}\frac{1-q^{2n+1}}{1-q},\qquad \beta_n=\frac{1}{(q^2;q^2)_n}, \tag{3.8}$$

and apply Proposition 3.5 with $r\to r-1, k\to k-1$ to it. (Note that, as (D'4) is obtained from the unit Bailey pair by one instance of (D4), the ranges for k,r,j are still valid.) This yields

$$\begin{aligned}
&\sum_{s_1\ge\cdots\ge s_k\ge 0}\frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}\\
&\quad=\frac{1}{(q)_\infty}\sum_{\ell=0}^{\infty}q^{(k+1)\ell^2+(r-j)\ell}(-1)^\ell\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right)\\
&\quad=\frac{1}{(q)_\infty}\sum_{\ell\in\mathbb{Z}}q^{(k+1)\ell^2+(r-j)\ell}(-1)^\ell(1+q^{2\ell})^j\\
&\quad=\frac{1}{(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+2)\binom{\ell}{2}+(k+r-j+2s+1)\ell}\\
&\quad=\frac{1}{(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\frac{(q^{k+1-r+j-2s},q^{k+2+r-j+2s},q^{2k+2};q^{2k+2})_\infty}{(q)_\infty},
\end{aligned}$$

where again the penultimate equality follows from the binomial theorem, and the last one from the Jacobi triple product (2.4). As before, in the first line, the product $q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})$ may be replaced by any product of j factors chosen from $\{q^{-s_1}\}\cup\{(q^{s_{i-1}}+q^{-s_i}) : 2\le i\le k-r\}$. □

Now we give a proof of Theorem 1.8 which comes from the Bailey pair (D'1) in Definition 2.13.

Proof of Theorem 1.8. Consider the Bailey pair (D'1) in (2.6) with $a=q$:

$$\alpha_n=(-1)^n q^{n^2-n}\frac{1-q^{4n+2}}{1-q^2},\qquad \beta_n=\frac{q^n}{(q^2;q^2)_n}, \tag{3.9}$$

and apply Proposition 3.5 with $r\to r-1, k\to k-1$ to it (again the ranges for r,j,k are valid). This gives

$$\begin{aligned}
&\sum_{s_1\ge\cdots\ge s_k\ge 0}\frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+2s_k}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}\\
&\quad=\frac{1}{(q)_\infty}\sum_{\ell=0}^{\infty}q^{(k+1)\ell^2+(r-j-1)\ell}(-1)^\ell\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right)\frac{1+q^{2\ell+1}}{1+q}\\
&\quad=\frac{1}{(1+q)(q)_\infty}\sum_{\ell\in\mathbb{Z}}q^{(k+1)\ell^2+(r-j-1)\ell}(-1)^\ell(1+q^{2\ell})^j(1+q^{2\ell+1})\\
&\quad=\frac{1}{(1+q)(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\sum_{\ell\in\mathbb{Z}}\left[(-1)^\ell q^{(2k+2)\binom{\ell}{2}+(k+r-j+2s)\ell}+q(-1)^\ell q^{(2k+2)\binom{\ell}{2}+(k+r-j+2s+2)\ell}\right]\\
&\quad=\frac{1}{(1+q)(q)_\infty}\sum_{s=0}^{j}\binom{j}{s}\left[(q^{k+2-r+j-2s},q^{k+r-j+2s},q^{2k+2};q^{2k+2})_\infty\right.\\
&\qquad\qquad\left.+q(q^{k-r+j-2s},q^{k+2+r-j+2s},q^{2k+2};q^{2k+2})_\infty\right].
\end{aligned}$$

As before, in the first line, the product $q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})$ may be replaced by any product of j factors chosen from $\{q^{-s_1}\}\cup\{(q^{s_{i-1}}+q^{-s_i}) : 2\le i\le k-r\}$. □

We conclude with the proof of Theorem 1.11.

Proof of Theorem 1.11. Let $(\alpha_n,\ \beta_n)$ be a Bailey pair relative to q. Apply Proposition 3.5 with $r \to r-1, k \to k-1$ to the Bailey pair (α'_n, β'_n) obtained from Lemma 2.3 with a replaced by q:

$$\alpha'_n = \frac{(-1)^n(\rho)_n q^{\frac{n^2}{2}+3\frac{n}{2}}}{(q^2/\rho)_n\rho^n}\,\alpha_n, \quad \text{and} \quad \beta'_n = \sum_{\ell=0}^{n} \frac{(-1)^\ell(\rho)_\ell q^{\frac{\ell^2}{2}+3\frac{\ell}{2}}}{(q^2/\rho)_n(q)_{n-\ell}\rho^\ell}\,\beta_\ell.$$

This gives

$$\sum_{s_1\geq\cdots\geq s_k\geq s_{k+1}\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k-s_{k+1}}(q^2/\rho)_{s_k}}\,\frac{(-1)^{s_{k+1}}(\rho)_{s_{k+1}}q^{\frac{s_{k+1}^2}{2}+3\frac{s_{k+1}}{2}}}{\rho^{s_{k+1}}}\beta_{s_{k+1}}$$
$$= \frac{1}{(q)_\infty}\sum_{\ell=0}^{\infty} q^{k\ell^2+(r-j)\ell}\,\frac{1-q}{1-q^{2\ell+1}}\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right)\frac{(\rho)_\ell(-1)^\ell q^{\frac{\ell^2}{2}+3\frac{\ell}{2}}}{(q^2/\rho)_\ell\rho^\ell}\alpha_\ell,$$

with the usual ranges for r, j, k. Now take (α_n, β_n) to be the unit Bailey pair (2.2) with $a = q$:

$$\alpha_n = (-1)^n q^{\binom{n}{2}}\,\frac{1-q^{2n+1}}{1-q}, \quad \text{and} \quad \beta_n = \delta_{n,0}.$$

Inserting this in the above equation yields

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k}q^{-s_1}(q^{s_1}+q^{-s_2})\cdots(q^{s_{j-1}}+q^{-s_j})}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}(q^2/\rho)_{s_k}}$$
$$= \frac{1}{(q)_\infty}\sum_{\ell=0}^{\infty} q^{k\ell^2+(r-j)\ell}\left((1+q^{2\ell})^j-(1+q^{2\ell+2})^j q^{(k-r+1)(2\ell+1)-j}\right)\frac{(\rho)_\ell q^{\ell^2+\ell}}{(q^2/\rho)_\ell\rho^\ell}.$$

Now set $\rho = -q^{3/2}$, and then replace q by q^2. This gives

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{2(s_1^2+\cdots+s_k^2+s_{k-r+1}+\cdots+s_k)}q^{-2s_1}(q^{2s_1}+q^{-2s_2})\cdots(q^{2s_{j-1}}+q^{-2s_j})}{(q^2;q^2)_{s_1-s_2}\cdots(q^2;q^2)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}(-q;q^2)_{s_k}}$$
$$= \frac{1}{(q^2;q^2)_\infty}\sum_{\ell=0}^{\infty} q^{(2k+2)\ell^2+(2r-2j-1)\ell}\left((1+q^{4\ell})^j-(1+q^{4\ell+4})^j q^{2(k-r+1)(2\ell+1)-2j}\right)(-1)^\ell\,\frac{1+q^{2\ell+1}}{1+q}$$
$$= \frac{1}{(1+q)(q^2;q^2)_\infty}\sum_{\ell\in\mathbb{Z}} q^{(2k+2)\ell^2+(2r-2j-1)\ell}(1+q^{4\ell})^j(-1)^\ell(1+q^{2\ell+1})$$
$$= \frac{1}{(1+q)(q^2;q^2)_\infty}\sum_{s=0}^{j}\binom{j}{s}\sum_{\ell\in\mathbb{Z}}\left[(-1)^\ell q^{(4k+4)\binom{\ell}{2}+(2k+2r-2j+4s+1)\ell}+q(-1)^\ell q^{(4k+4)\binom{\ell}{2}+(2k+2r-2j+4s+3)\ell}\right]$$
$$= \frac{1}{(1+q)(q^2;q^2)_\infty}\sum_{s=0}^{j}\binom{j}{s}\Big[(q^{4k+4},q^{2k+2r-2j+4s+1},q^{2k-2r+2j-4s+3};q^{4k+4})_\infty$$
$$+\,q(q^{4k+4},q^{2k+2r-2j+4s+3},q^{2k-2r+2j-4s+1};q^{4k+4})_\infty\Big].$$

Multiplying both sides by $(-q;q^2)_\infty$ completes the proof. Here, the product $q^{-2s_1}(q^{2s_1}+q^{-2s_2})\cdots(q^{2s_{j-1}}+q^{-2s_j})$ may be replaced by any product of j factors chosen from $\{q^{-2s_1}\}\cup\{(q^{2s_{i-1}}+q^{-2s_i}) : 2\leq i\leq k-r\}$. ☐

4. Proofs of non-binomial extensions

4.1. **Proofs of Theorems 1.5, 1.7, and 1.9.** Recall that to get Theorem 2.11, one uses once the Bailey lattice after a few iterations of the Bailey lemma (actually the special instances given in Lemmas 2.2 and 2.6). The clue here is to use twice the Bailey lattice instead of once. More precisely, iterate $r+1$ times Lemma 2.2, then use Lemma 2.6. Next, iterate $k-r-j-1$ times Lemma 2.2 with a replaced by a/q and use once again Lemma 2.6. Finally, iterate $j-1$ times Lemma 2.2 with a replaced by a/q^2. (If $j=0$, the two last steps have to be omitted. If $r+j=k$, the second and third steps have to be omitted, and a needs to be replaced by aq in the remaining steps.) Letting $n\to\infty$ in the relation (2.1) for the resulting Bailey pair (α_n, β_n) relative to a/q^2 gives the following.

Proposition 4.1 (New consequence of the Bailey lattice)**.** *Let $(\alpha_n,\,\beta_n)$ be a Bailey pair relative to a. Then, for all integers $j,r\geq 0$ and $k\geq 1$ such that $r+j\leq k$, we have*

$$\sum_{s_1\geq\cdots\geq s_{k+1}\geq 0}\frac{a^{s_1+\cdots+s_{k+1}}q^{s_1^2+\cdots+s_{k+1}^2-2s_1-\cdots-2s_j-s_{j+1}-\cdots-s_{k-r}}}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}} \tag{4.1}$$
$$=\frac{1}{(aq)_\infty}\sum_{\ell\geq 0}a^{(k+1)\ell}q^{(k+1)\ell^2+(r-j-k)\ell}\frac{1}{1-aq^{2\ell}}\left(\frac{1-(aq^{2\ell-1})^{j+1}}{1-aq^{2\ell-1}}-a^{k+1-r}q^{(2k+2-2r)\ell-j}\frac{1-(aq^{2\ell+1})^{j+1}}{1-aq^{2\ell+1}}\right)\alpha_\ell.$$

We now prove Theorems 1.5, 1.7, and 1.9 using Proposition 4.1.

Proof of Theorem 1.5. We apply Proposition 4.1. Inserting the unit Bailey pair (2.2) in (4.1) gives

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{a^{s_1+\cdots+s_k}q^{s_1^2+\cdots+s_k^2-2s_1-\cdots-2s_j-s_{j+1}-\cdots-s_{k-r}}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}=\frac{1}{(a)_\infty}\sum_{\ell\geq 0}(-1)^\ell a^{(k+1)\ell}q^{(k+1)\ell^2+\binom{\ell}{2}+(r-j-k)\ell} \tag{4.2}$$
$$\times\frac{(a)_\ell}{(q)_\ell}\left(\frac{1-(aq^{2\ell-1})^{j+1}}{1-aq^{2\ell-1}}-a^{k+1-r}q^{(2k+2-2r)\ell-j}\frac{1-(aq^{2\ell+1})^{j+1}}{1-aq^{2\ell+1}}\right).$$

Now take $a=q$. Then the left-hand side corresponds to that of (1.9), while the right-hand side becomes

$$\frac{1}{(q)_\infty}\sum_{\ell\geq 0}(-1)^\ell q^{(k+\frac{3}{2})\ell^2+(r-j+\frac{1}{2})\ell}\left(\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}}-q^{(2k+2-2r)\ell+k+1-r-j}\frac{1-q^{(2\ell+2)(j+1)}}{1-q^{2\ell+2}}\right).$$

Split the sum into two parts, and apply the change of variables $\ell\mapsto-\ell-1$ in the second sum. This yields

$$\frac{1}{(q)_\infty}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(k+\frac{3}{2})\ell^2+(r-j+\frac{1}{2})\ell}\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}}.$$

Expanding the quotient yields

$$\frac{1}{(q)_\infty}\sum_{s=0}^{j}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+3)\binom{\ell}{2}}q^{(k+r-j+2s+2)\ell}.$$

Finally, applying the Jacobi triple product identity (2.4) with $q\mapsto q^{2k+3}$ and $z=q^{k+r-j+2s+2}$, we obtain the right-hand side of (1.9). ☐

We give a similar proof of Theorem 1.7, starting this time from the Bailey pair (D'4).

Proof of Theorem 1.7. We now apply Proposition 4.1 with $a=q$, $k\to k-1$ and $r\to r-1$ to the Bailey pair (3.8). Note that the ranges $r\geq 0$, $k\geq 1$ and $r+j\leq k$ are still valid, as the above Bailey pair is obtained by using one instance of the change of base [BIS00, (D4)] to the unit Bailey pair. This gives

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}=\frac{1}{(q)_\infty}\sum_{\ell\geq 0}(-1)^\ell q^{(k+1)\ell^2+(r-j)\ell}$$
$$\times\left(\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}}-q^{(2k+2-2r)\ell+k+1-r-j}\frac{1-q^{(2\ell+2)(j+1)}}{1-q^{2\ell+2}}\right).$$

The left-hand side is the one of (1.10). On the right-hand side, split the sum over ℓ into two sums, and shift $\ell\mapsto-\ell-1$ in the second one to get

$$\frac{1}{(q)_\infty}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(k+1)\ell^2+(r-j)\ell}\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}},$$

which by expanding the quotient yields

$$\frac{1}{(q)_\infty}\sum_{s=0}^{j}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+2)\binom{\ell}{2}}q^{(k+r+1-j+2s)\ell}.$$

We get the right-hand side of (1.9) by applying the Jacobi triple product identity (2.4) with $q\mapsto q^{2k+2}$ and $z=q^{k+r+1-j+2s}$. ☐

We finally prove Theorem 1.9, starting this time from the Bailey pair (D'1).

Proof of Theorem 1.9. We now apply Proposition 4.1 with $a = q$, $k \to k-1$ and $r \to r-1$ to the Bailey pair (3.9). Again the ranges $r \geq 0$, $k \geq 1$ and $r + j \leq k$ are still valid, and this gives

$$\sum_{s_1\geq\cdots\geq s_k\geq 0} \frac{q^{s_1^2+\cdots+s_k^2-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_{k-1}+2s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}} = \frac{1}{(q)_\infty}\sum_{\ell\geq 0}(-1)^\ell q^{(k+1)\ell^2+(r-j-1)\ell}\frac{1+q^{2\ell+1}}{1+q} \times\left(\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}} - q^{(2k+2-2r)\ell+k+1-r-j}\frac{1-q^{(2\ell+2)(j+1)}}{1-q^{2\ell+2}}\right).$$

The left-hand side becomes the one of (1.11). The right-hand side can be written as

$$\frac{1}{(1+q)(q)_\infty}\sum_{\ell\geq 0}(-1)^\ell q^{(k+1)\ell^2+(r-j-1)\ell}\left(\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}} - q^{(2k+4-2r)\ell+k+2-r-j}\frac{1-q^{(2\ell+2)(j+1)}}{1-q^{2\ell+2}}\right) + \frac{1}{(1+q)(q)_\infty}\sum_{\ell\geq 0}(-1)^\ell q^{(k+1)\ell^2+(r-j+1)\ell+1}\left(\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}} - q^{(2k-2r)\ell+k-r-j}\frac{1-q^{(2\ell+2)(j+1)}}{1-q^{2\ell+2}}\right).$$

Split each sum and shift $\ell \mapsto -\ell-1$ in the second and fourth parts. This gives

$$\frac{1}{(1+q)(q)_\infty}\left(\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(k+1)\ell^2+(r-j-1)\ell}\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}} + q\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(k+1)\ell^2+(r-j+1)\ell}\frac{1-q^{2\ell(j+1)}}{1-q^{2\ell}}\right).$$

Expanding the quotients gives

$$\frac{1}{(1+q)(q)_\infty}\left(\sum_{s=0}^{j}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+2)\binom{\ell}{2}}q^{(k+r-j+2s)\ell} + q\sum_{s=0}^{j}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+2)\binom{\ell}{2}}q^{(k+r-j+2s+2)\ell}\right).$$

Applying the Jacobi triple product identity (2.4) with $q \mapsto q^{2k+2}$ and $z = q^{k+r-j+2s}$ or $z = q^{k+r-j+2s+2}$ yields the right-hand side of (1.9). □

4.2. **Non-binomial extension of the Bresssoud–Göllnitz–Gordon identities.** Inspired by the methods in [DJK25], we see that an extension of Proposition 4.1 is needed, through the following process (the method is the same as for Proposition 4.1, except that at two appropriate steps we keep one finite parameter ρ) (therefore using Lemma 2.3 instead of Lemma 2.2).

First use Lemma 2.3 with $\rho = b$ and iterate r times Lemma 2.2. Then use Lemma 2.6. Next iterate $k-r-j-1$ times Lemma 2.2 with a replaced by a/q and use once again Lemma 2.6. Iterate $j-2$ times Lemma 2.2 with a replaced by a/q^2, and finally once Lemma 2.3 with a replaced by a/q^2 and $\rho = c$. As for Proposition 4.1, one has to examine what we should do when the number of steps described above is negative: If $j = 0$ and $r = k$, we do not use any lattice in the $k+1$ steps of the process. If $j = 0$ and $r < k$ or if $j = 1$, we only use one lattice. Note that for some extremal cases, one has to replace Lemma 2.3 in the first or last step by the use of the Bailey lattice in Theorem 2.5 in which $\sigma \to \infty$ while we keep ρ. Again, letting $n \to \infty$ in the relation (2.1) for the resulting Bailey pair (α_n, β_n) relative to a/q^2 gives the following.

Proposition 4.2 (Another new consequence of the Bailey lattice)**.** *If (α_n, β_n) is a Bailey pair relative to a, then for all integers $k \geq 1$ and $0 \leq r, j \leq k$ with $r + j \leq k$, we have*

(4.3)
$$\sum_{s_1\geq\cdots\geq s_{k+1}\geq 0}\frac{(-1)^{s_1+s_{k+1}}(b)_{s_1}(c)_{s_{k+1}}}{b^{s_1}c^{s_{k+1}}(aq/c)_{s_k}}\frac{a^{s_1+\cdots+s_{k+1}}q^{\frac{s_1^2}{2}+s_2^2+\cdots+s_k^2+\frac{s_{k+1}^2}{2}+\frac{s_1}{2}-2s_1-\cdots-2s_j-s_{j+1}-\cdots-s_{k-r}+\frac{s_{k+1}}{2}}}{(q)_{s_1-s_2}\cdots(q)_{s_k-s_{k+1}}}\beta_{s_{k+1}}$$
$$= \frac{(a/bq)_\infty}{(aq)_\infty}\sum_{\ell\geq 0}\frac{a^{(k+1)\ell}}{(bc)^\ell}q^{k\ell^2+(r+1-j-k)\ell}\frac{(b,c)_\ell}{(a/bq,aq/c)_\ell}\frac{1}{1-aq^{2\ell}}$$
$$\times\left(\frac{1+a^{j+1}q^{(2j+1)\ell-j-1}\frac{1-bq^\ell}{b-aq^{\ell-1}}}{1-aq^{2\ell-1}} + a^{k+1-r}q^{(2k+1-2r)\ell-j}\frac{1-bq^\ell}{b-aq^{\ell-1}}\frac{1+a^{j+1}q^{(2j+1)\ell+j}\frac{1-bq^{\ell+1}}{b-aq^\ell}}{1-aq^{2\ell+1}}\right)\alpha_\ell.$$

Applying (4.3) to the unit Bailey pair (2.2), we get

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{(-1)^{s_1}(b)_{s_1}}{b^{s_1}(aq/c)_{s_k}}\frac{a^{s_1+\cdots+s_k}q^{\frac{s_1^2}{2}+s_2^2+\cdots+s_k^2+\frac{s_1}{2}-2s_1-\cdots-2s_j-s_{j+1}-\cdots-s_{k-r}}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$
$$=\frac{(a/bq)_\infty}{(a)_\infty}\sum_{\ell\geq 0}(-1)^\ell\frac{a^{(k+1)\ell}}{(bc)^\ell}q^{(k+\frac12)\ell^2+(r-j-k+\frac12)\ell}\frac{(a,b,c)_\ell}{(q,a/bq,aq/c)_\ell}$$
$$\times\left(\frac{1+a^{j+1}q^{(2j+1)\ell-j-1}\frac{1-bq^\ell}{b-aq^{\ell-1}}}{1-aq^{2\ell-1}}+a^{k+1-r}q^{(2k+1-2r)\ell-j}\frac{1-bq^\ell}{b-aq^{\ell-1}}\frac{1+a^{j+1}q^{(2j+1)\ell+j}\frac{1-bq^{\ell+1}}{b-aq^\ell}}{1-aq^{2\ell+1}}\right).\tag{4.4}$$

As for (4.2), the appropriate choice for a to derive Stanton type formulas is $a=q$ (this is clear by inspecting the powers of a and q on the left-hand side of (4.4)). Note that the alternative classical choice is usually $a=1$, which creates problems of convergence here. When $a=q$, we get

$$\sum_{s_1\geq\cdots\geq s_k\geq 0}\frac{(-1)^{s_1}(b)_{s_1}}{b^{s_1}(q^2/c)_{s_k}}\frac{q^{\frac{s_1^2}{2}+s_2^2+\cdots+s_k^2+\frac{s_1}{2}-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}}$$
$$=\frac{(1/b)_\infty}{(q)_\infty}\sum_{\ell\geq 0}\frac{(-1)^\ell}{(bc)^\ell}q^{(k+\frac12)\ell^2+(r-j+\frac32)\ell}\frac{(b,c)_\ell}{(1/b,q^2/c)_\ell}$$
$$\times\left(\frac{1+q^{(2j+1)\ell}\frac{1-bq^\ell}{b-q^\ell}}{1-q^{2\ell}}+q^{(2k+1-2r)\ell+k+1-r-j}\frac{1-bq^\ell}{b-q^\ell}\frac{1+q^{(2j+1)(\ell+1)}\frac{1-bq^{\ell+1}}{b-q^{\ell+1}}}{1-q^{2\ell+2}}\right).\tag{4.5}$$

As mentioned in the introduction, m-versions of Bressoud's extensions [Bre80, (3.6)–(3.9)] of the Göllnitz–Gordon identities are derived in [DJK25]. The method uses a bilateral version of a simpler case of Proposition 4.2 (in which only one instance of the Bailey lattice is used instead of two). As a result, it is shown that, as for the cases of the Andrews–Gordon and Bressoud identities, all results come in pairs, arising from the two choices $a=1$ and $a=q$. Surprisingly, it is for instance noticed that [Bre80, (3.6)] (which extends one of the Göllnitz–Gordon identities) arises in pair with Bressoud's identity (1.5), while [Bre80, (3.7)] (which extends another of the Göllnitz–Gordon identities) arises in pair with Bressoud's identity (1.6). Similarly, [Bre80, (3.8)] (resp. [Bre80, (3.9)] arises in pair with a new formula expressed in [DJK25, Corollary 2.10] (resp. [DJK25, Corollary 2.12].

As can be seen in the cases of the Andrews–Gordon and Bressoud formulas, their extensions discovered by Stanton now embed both choices $a=1$ and $a=q$ in formulas involving two integral parameters j and r, instead of only one parameter as usual. As seen in the previous sections, these formulas can be derived from the Bailey lattice with only one choice $a=q$, but one needs to use twice the Bailey lattice instead of once.

Our goal is to do the same here for Bressoud's extensions of the Göllnitz–Gordon identities, by using Proposition 4.2. Nevertheless, as explained above, the only reasonable choice for a is q, resulting in formula (4.5).

The first specialisation used in [DJK25], namely $b\to\infty, c=-q$, applied to (4.5), yields Stanton's formula (1.10). The second specialisation in [DJK25] is $b\to\infty, c=-q^{1/2}$, which does not seem to yield interesting formulas. Alternatively, we are able to prove Theorem 1.12.

Proof of Theorem 1.12. Take $b\to\infty, c=-q^{3/2}$ in (4.5), replace q by q^2 and multiply both sides by $(-q;q^2)_\infty$. Then the left-hand side becomes the desired expression. On the right-hand side, we obtain

$$\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{\ell\geq 0}(-1)^\ell q^{(2k+2)\ell^2+(2r-2j-1)\ell}(1+q^{2\ell+1})\left(\frac{1-q^{4\ell(j+1)}}{1-q^{4\ell}}-\frac{1-q^{(2j+1)(\ell+1)}}{1-q^{2\ell+2}}q^{(2k+1-2r)\ell+k+1-r-j}\right).$$

Split the sum and apply the change of variables $\ell\mapsto -\ell-1$ in the second sum. This gives

$$\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+2)\ell^2+(2r-2j-1)\ell}(1+q^{2\ell+1})\frac{1-q^{4\ell(j+1)}}{1-q^{4\ell}}.$$

Expanding the quotient yields

$$\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{s=0}^{j}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(4k+4)\binom{\ell}{2}+(2k+2r-2j+4s+1)\ell}(1+q^{2\ell+1}).$$

Splitting the sum over ℓ into two sums and applying twice the Jacobi triple product identity (2.4) gives the desired result. □

Now we turn to the two last specialisations used in [DJK25] to extend [Bre80, (3.8)–(3.9)], namely $b = -q^{1/2}, c \to \infty$ and $b = -q^{1/2}, c = -q$, respectively. In view of (4.5), it seems hopeless to handle the right-hand side nicely with these specialisations. Alternatively, the choice $b = -1$ gives the next two results.

Theorem 4.3. *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. Then*

$$\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{\frac{s_1^2}{2}+s_2^2+\cdots+s_k^2+\frac{s_1}{2}-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}(-1)_{s_1}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q)_{s_k}} \tag{4.6}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\sum_{s=0}^{2j}\sum_{t=0}^{2r}(-1)^t(q^{2k+2}, q^{k+1-r-j+s+t}, q^{k+1+r+j-s-t}; q^{2k+2})_\infty.$$

Proof. Take $b = -1, c \to \infty$ in (4.5). We get the desired left-hand side, while the right-hand side becomes

$$\frac{(-1)_\infty}{(q)_\infty}\sum_{\ell \geq 0}(-1)^\ell q^{(k+1)\ell^2+(r-j+1)\ell}\left(\frac{1-q^{(2j+1)\ell}}{1-q^{2\ell}} - \frac{1-q^{(2j+1)(\ell+1)}}{1-q^{2\ell+2}}q^{(2k+1-2r)\ell+k+1-r-j}\right).$$

Split the sum into two parts, and in the second one, apply the change of variables $\ell \mapsto -\ell-1$. This transforms the sum into

$$\frac{(-1)_\infty}{(q)_\infty}\sum_{\ell \in \mathbb{Z}}(-1)^\ell q^{(k+1)\ell^2+(r-j+1)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^{2\ell}}.$$

Now apply the substitution $\ell \mapsto -\ell$ to this expression, obtaining

$$\frac{(-1)_\infty}{(q)_\infty}\sum_{\ell \in \mathbb{Z}}(-1)^\ell q^{(k+1)\ell^2+(r-j+1)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^{2\ell}}q^{-(2r+1)\ell}.$$

Adding the two above expressions and dividing by 2 gives

$$\frac{(-q)_\infty}{(q)_\infty}\sum_{\ell \in \mathbb{Z}}(-1)^\ell q^{(2k+2)\binom{\ell}{2}+(k+1-r-j)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^\ell}\frac{1+q^{(2r+1)\ell}}{1+q^\ell}.$$

Expanding the quotients and applying the Jacobi triple product identitiy (2.4) completes the proof. □

For $r = 0$, Formula (4.6) is [DJK25, Corollary 2.10] with $(r, i) \to (k+1, j)$. When $k = 1$, the three possible cases $(j, r) = (0, 0)$, $(1, 0)$ and $(0, 1)$ are equivalent to Slater's identities [Sl52, (8), (12), (13)].

Theorem 4.4. *Let $j, r \geq 0$ and $k \geq 1$ be integers such that $j + r \leq k$. If $r \geq 1$, then we have*

$$(1+q^{1/2})\sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{\frac{s_1^2}{2}+s_2^2+\cdots+s_k^2+\frac{s_1}{2}-s_1-\cdots-s_j+s_{k-r+1}+\cdots+s_k}(-1)_{s_1}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q,-q^{1/2})_{s_k}} \tag{4.7}$$
$$= \frac{(-q)_\infty}{(q)_\infty}\left(\sum_{s=0}^{2j}\sum_{t=0}^{2r-2}(-1)^t(q^{2k+1}, q^{k+3/2-r-j+s+t}, q^{k-1/2+r+j-s-t}; q^{2k+1})_\infty\right.$$
$$\left. + q^{1/2}\sum_{s=0}^{2j}\sum_{t=0}^{2r}(-1)^t(q^{2k+1}, q^{k+1/2-r-j+s+t}, q^{k+1/2+r+j-s-t}; q^{2k+1})_\infty\right),$$

and if $r = 0$, then we get [DJK25, Corollary 2.12] *with* $(r, i) \to (k+1, j)$.

Proof. Take $b = -1, c = -q^{3/2}$ in (4.5) and multiply both sides by $1 + q^{1/2}$, then we get the desired left-hand side, while the right-hand side is

$$\frac{(-1)_\infty}{(q)_\infty}\sum_{\ell \geq 0}(-1)^\ell q^{(k+1/2)\ell^2+(r-j)\ell}(1+q^{\ell+1/2})\left(\frac{1-q^{(2j+1)\ell}}{1-q^{2\ell}} - \frac{1-q^{(2j+1)(\ell+1)}}{1-q^{2\ell+2}}q^{(2k+1-2r)\ell+k+1-r-j}\right).$$

Split the sum into two parts, and in the second sum apply the substitution $\ell \mapsto -\ell-1$. This yields

$$\frac{(-1)_\infty}{(q)_\infty}\left(\sum_{\ell \in \mathbb{Z}}(-1)^\ell q^{(k+1/2)\ell^2+(r-j)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^{2\ell}} + q^{1/2}\sum_{\ell \in \mathbb{Z}}(-1)^\ell q^{(k+1/2)\ell^2+(r-j+1)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^{2\ell}}\right).$$

Now replace $\ell \mapsto -\ell$ in each sum, add to the original expressions, and divide by 2. We obtain

$$\frac{(-q)_\infty}{(q)_\infty}\Bigg(\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+1)\binom{\ell}{2}+(k-r-j+3/2)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^\ell}\frac{1+q^{(2r-1)\ell}}{1+q^\ell}$$
$$+q^{1/2}\sum_{\ell\in\mathbb{Z}}(-1)^\ell q^{(2k+1)\binom{\ell}{2}+(k-r-j+1/2)\ell}\frac{1-q^{(2j+1)\ell}}{1-q^\ell}\frac{1+q^{(2r+1)\ell}}{1+q^\ell}\Bigg).$$

If $r=0$, the two sums are the same, so one can extract a factor $1+q^{1/2}$. Expanding both quotients and applying the Jacobi triple product identity (2.4) yields [DJK25, Corollary 2.12] with $(r,i)\to(k+1,j)$. If $r\geq 1$, expanding the four quotients and applying (2.4) twice gives the result. □

5. Insertion map for multipartitions revisited

In this section we revisit the particle motion bijection, which was first used in [War97], and generalised recently in [DJK24], to make it more systematic and easier to apply. Warnaar's approach had a simpler combinatorial description but was less general, while the approach of [DJK24] was more general and gave rise to useful explicit formulas but was less intuitive to understand. Our goal here is to keep the best of both worlds by reformulating the approach of [DJK24] in the style of Warnaar, which leads both to a simple combinatorial description and explicit formulas.

In the next sections, we will use this bijection to give partition-theoretic interpretations of Stanton's non-binomial identities and prove Theorems 1.5, 1.7 and 1.9 by combining it with the Andrews–Gordon and Bressoud identities.

Recall from the introduction that a *partition* $\lambda=(\lambda_1,\dots,\lambda_\ell)$ is a weakly decreasing finite sequence of non-negative integers, that is, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_\ell\geq 0$. Each non-negative integer λ_i is called a part of λ and $\ell(\lambda)=\ell$ is called the length of λ. A *frequency sequence* $(f_i)_{i\geq 0}$ is a sequence of non-negative integers. Given a partition, its frequency sequence is defined by setting f_i to be the number of parts of size i for all $i\geq 0$. This yields a one-to-one correspondence between frequency sequences and partitions. The *size* and *length* of a frequency sequence $(f_i)_{i\geq 0}$ are defined as $|f|:=\sum_{i\geq 0} i f_i$ and $\ell(f):=\sum_{i\geq 0} f_i$, respectively, which coincide with the size and length of the corresponding partition.

From now on, we use the notation of a multipartition, which simply refers to a finite sequence of partitions. For an integer $k\geq 1$, a *k-multipartition* $\boldsymbol{\lambda}=(\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(k)})$ is a tuple of partitions $\lambda^{(i)}$, and each of them may be empty. The *size* $|\boldsymbol{\lambda}|$ of $\boldsymbol{\lambda}$ is defined by $\sum_{i=1}^k|\lambda^{(i)}|$, and the *length* $\ell(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ is defined by $\sum_{i=1}^k \ell(\lambda^{(i)})$.

A *frame sequence* is a frequency sequence $(f_i)_{i\geq 0}$ such that $f_{2i}\geq f_{2i+2}$ and $f_{2i+1}=0$ for all $i\geq 0$. We can associate a frame sequence to each multipartition as follows. For a k-multipartition $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(k)})$, let $s_1,\dots,s_k$ be integers with $s_1\geq\cdots\geq s_k\geq 0$ such that the length of $\lambda^{(i)}$ is s_i-s_{i+1} for all i, where we set $s_{k+1}:=0$. Then, the frame sequence $\mathrm{fs}(\boldsymbol{\lambda})$ corresponding to $\boldsymbol{\lambda}$ is defined as $(f_0,f_1,f_2,\dots)$, where the entries $f_0,f_2,f_4,\dots$ are, in this order, s_k copies of k, $(s_{k-1}-s_k)$ copies of $k-1$, …, and (s_1-s_2) copies of 1. That is,

$$\mathrm{fs}(\boldsymbol{\lambda}):=(\underbrace{k,0,\dots,k,0}_{s_k\text{ pairs}},\dots,\underbrace{i,0,\dots,i,0}_{s_i-s_{i+1}\text{ pairs}},\dots,\underbrace{1,0,\dots,1,0}_{s_1-s_2\text{ pairs}},0,\dots).$$

The following two sets are used throughout the paper.

Definition 5.1. For a non-negative integer k, let $\mathcal{P}_k$ denote the set of all pairs $(\boldsymbol{\lambda},\mathrm{fs}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda}$ is a k-multipartition and $\mathrm{fs}(\boldsymbol{\lambda})$ is the frame sequence corresponding to $\boldsymbol{\lambda}$. Let $\mathcal{A}_k$ denote the set of all frequency sequences $(f_i)_{i\geq 0}$ such that $f_i+f_{i+1}\leq k$ for all $i\geq 0$.

In this section, we introduce an *insertion map* Λ for multipartitions, which defines a size-preserving bijection between the sets $\mathcal{P}_k$ and $\mathcal{A}_k$, where the size of an element $(\boldsymbol{\lambda},\mathrm{fs}(\boldsymbol{\lambda}))\in\mathcal{P}_k$ is defined as $|(\boldsymbol{\lambda},\mathrm{fs}(\boldsymbol{\lambda}))|:=|\boldsymbol{\lambda}|+|\mathrm{fs}(\boldsymbol{\lambda})|$. Although $\mathrm{fs}(\boldsymbol{\lambda})$ is uniquely determined by $\boldsymbol{\lambda}$, we regard Λ as a map from $\mathcal{P}_k$ to $\mathcal{A}_k$, rather than directly from the set of multipartitions $\boldsymbol{\lambda}$, in order to emphasize that it preserves the size.

Particle motion. Let $f=(f_0,f_1,\dots)$ be a frequency sequence. Suppose that u is a non-negative integer such that there exists $h\geq 1$ with $f_u+f_{u+1}=h$ and $f_i+f_{i+1}\leq h$ for all $i\geq u$. We now describe the procedure for applying m particle motions in f, starting from the pair (f_u,f_{u+1}). Consider the pair (f_u,f_{u+1}). If the local condition $f_{u+1}+f_{u+2}<h$ is satisfied, we perform the following (single) *particle motion*:

$$(f_u,f_{u+1})\mapsto(f_u-1,f_{u+1}+1).$$

As long as the local condition remains satisfied for the current pair (f_u,f_{u+1}), we continue to apply this particle motion repeatedly at the current pair. Once the local condition is no longer satisfied, that is, if $f_{u+1}+f_{u+2}=h$, then we increment u by 1, shift our focus to the next pair (f_{u+1},f_{u+2}), and repeat the same procedure. This process continues until exactly m particle motions are performed.

The resulting sequence remains a frequency sequence; that is, all entries are non-negative integers. This follows from the fact that the single particle motion at (f_u, f_{u+1}) can only be performed when $f_u \geq 1$.

Definition 5.2. Let $f = (f_0, f_1, \dots)$ be a frequency sequence and m be a non-negative integer. Suppose that u is a non-negative integer such that there exists $h \geq 1$ with $f_u + f_{u+1} = h$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$. We define $\mathsf{pm}_u^{(m)}(f)$ to be the resulting frequency sequence by applying m particle motions starting from (f_u, f_{u+1}) in f. Moreover, if $\mathsf{pm}_u^{(m)}(f) = (\overline{f}_0, \overline{f}_1, \dots)$ and the final focus is on the pair $(\overline{f}_v, \overline{f}_{v+1})$, then we say that the pair (f_u, f_{u+1}) moves to $(\overline{f}_v, \overline{f}_{v+1})$.

Note that shifting the focus is part of the procedure for locating the next applicable pair and does not affect the frequency sequence. The single particle motion $(f_u, f_{u+1}) \mapsto (f_u - 1, f_{u+1} + 1)$ implies both $f_u + f_{u+1} \leq h$ and $f_{u+1} + f_{u+2} \leq h$. Therefore, by construction of the particle motion, the frequency sequence $\overline{f} = \mathsf{pm}_u^{(m)}(f)$ satisfies $\overline{f}_i + \overline{f}_{i+1} \leq h$ for all $i \geq u$. Note also that if the pair (f_u, f_{u+1}) moves to $(\overline{f}_v, \overline{f}_{v+1})$, then $\overline{f}_v + \overline{f}_{v+1} = f_u + f_{u+1} = h$.

Example 5.3. A frequency sequence $(f_i)_{i \geq 0}$ is represented by placing boxes above the x-axis. Starting from the 0th column, the number of boxes in the ith column corresponds to f_i. The current focus is indicated by shading the corresponding boxes in gray and marking the corresponding position on the x-axis with a bold line.

Let $f = (4, 0, 2, 0, 3, 1, 0, 0, \dots)$ be a frequency sequence. By Figure 1, we obtain $\overline{f} = \mathsf{pm}_0^{(9)}(f)$ where

$$\overline{f} = (2, 0, 3, 1, 0, 3, 1, 0, \dots).$$

Here, $(f_0, f_1) = (4, 0)$ moves to $(\overline{f}_5, \overline{f}_6) = (3, 1)$.

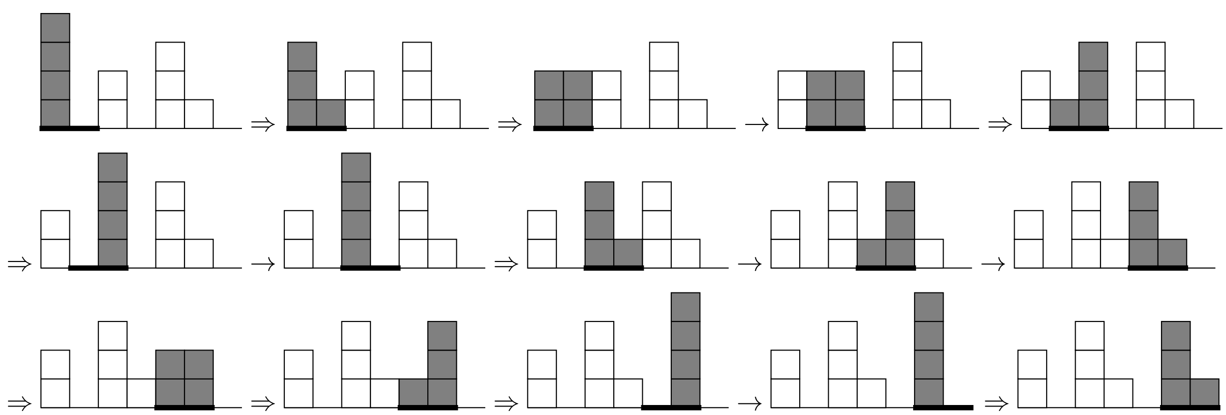

FIGURE 1. Illustration of applying 9 particle motions starting from (f_0, f_1) in the frequency sequence $f = (4, 0, 2, 0, 3, 1, 0, 0, \cdots)$. The symbol $\Rightarrow$ indicates a single particle motion, and $\rightarrow$ indicates a focus shift.

In this paper, we only apply the particle motions to pairs (f_u, f_{u+1}) of the form $(h, 0)$ with a positive integer h that occurs in the frame sequence. Therefore, from now on, we assume that the starting pair (f_u, f_{u+1}) is of the form $(h, 0)$ with a positive integer h.

In [DJK24], the authors describe the frequency sequence obtained by iteratively applying the particle motion. As a result, the resulting frequency sequence can be computed directly in a single step, without performing each individual particle motion. In the next proposition, we show that applying m particle motions as described above gives exactly the same frequency sequence.

Proposition 5.4. *Let $f = (f_0, f_1, \dots)$ be a frequency sequence. Suppose that u is a non-negative integer such that there exists $h \geq 1$ with $(f_u, f_{u+1}) = (h, 0)$ and $f_i + f_{i+1} \leq h$ for all $i \geq u$. For a non-negative integer m, let $\overline{f} = (\overline{f}_0, \overline{f}_1, \dots)$ be the frequency sequence obtained by applying m particle motions in f, starting from the pair (f_u, f_{u+1}). To determine the position where the pair (f_u, f_{u+1}) moves, define*

$$v := \min \left\{ t \geq u + 2 : \sum_{i=u+2}^{t} (h - (f_{i-1} + f_i)) \geq m \right\}. \tag{5.1}$$

Then (f_u, f_{u+1}) *moves to* $(\overline{f}_{v-2}, \overline{f}_{v-1})$. *The frequency sequence* $\overline{f} = \mathsf{pm}_u^{(m)}(f)$ *is given explicitly by:*

$$\overline{f}_i = \begin{cases} f_i & \text{if } 0 \le i < u, \\ f_{i+2} & \text{if } u \le i < v-2, \\ f_v + \sum_{j=u+2}^{v}(h - (f_{j-1} + f_j)) - m & \text{if } i = v-2, \\ f_{v-1} + m - \sum_{j=u+2}^{v-1}(h - (f_{j-1} + f_j)) & \text{if } i = v-1, \\ f_i & \text{if } i \ge v. \end{cases} \tag{5.2}$$

Proof. Let $(f_u, f_{u+1}) = (h, 0)$ move to $(\overline{f}_{v-2}, \overline{f}_{v-1})$ for some $v-2 \ge u$. We use the following facts: (1) $\overline{f}_{v-2} + \overline{f}_{v-1} = h$; (2) the entries originally between f_{u+2} and f_{v-1} are shifted two steps to the left; and (3) all other entries remain unchanged. Facts (1) and (3) are straightforward. Fact (2) requires a brief explanation. The focus moves from (f_u, f_{u+1}) to (f_{u+1}, f_{u+2}) only when $f_u + f_{u+1} = f_{u+1} + f_{u+2}$, that is, when $f_u = f_{u+2}$. From this point on, the value of the uth entry remains equal to f_{u+2}. Therefore, we may regard the original value of f_{u+2} as being shifted two steps to the left.

Using the facts above, we have

$$\begin{aligned} |\overline{f}| &= \sum_{i=0}^{u-1} i f_i + \sum_{i=u}^{v-3} i f_{i+2} + (v-2)\overline{f}_{v-2} + (v-1)\overline{f}_{v-1} + \sum_{i \ge v} i f_i \\ &= \sum_{i \ge 0} i f_i - 2\sum_{i=u}^{v-3} f_{i+2} - u f_u - (u+1) f_{u+1} + (v-2)\overline{f}_{v-2} + (v-1)\overline{f}_{v-1} \\ &= |f| - 2\sum_{i=u+2}^{v-1} f_i - uh + (v-2)\overline{f}_{v-2} + (v-1)\overline{f}_{v-1}. \end{aligned}$$

Since $\overline{f}$ is obtained from f via m particle motions, its size is $|\overline{f}| = |f| + m$. Therefore, we obtain

$$(v-2)\overline{f}_{v-2} + (v-1)\overline{f}_{v-1} = m + uh + \sum_{i=u+2}^{v-1} f_i.$$

Using $\overline{f}_{v-2} + \overline{f}_{v-1} = h$, we can express the left-hand side as either $(v-2)h + \overline{f}_{v-1}$ or $(v-1)h - \overline{f}_{v-2}$. These two expressions determine $\overline{f}_{v-2}$ and $\overline{f}_{v-1}$, as given in (5.2). One can also check the formula (5.1). ☐

Example 5.5. Let $f = (4, 0, 2, 0, 3, 1, 0, 0, \dots)$, $u = 0$ and $m = 9$. Then $f_0 + f_1 = 4 = h$. We have $v = 7$, since

$$\sum_{i=2}^{6}(h - (f_{i-1} + f_i)) = 8 < 9, \quad \text{and} \quad \sum_{i=2}^{7}(h - (f_{i-1} + f_i)) = 12 \ge 9.$$

By the explicit formula (5.2),

$$\overline{f}_i = \begin{cases} f_{i+2} & \text{if } 0 \le i < 5, \\ 0 + 12 - 9 = 3 & \text{if } i = 5, \\ 0 + 9 - 8 = 1 & \text{if } i = 6, \\ 0 & \text{if } i \ge 7. \end{cases}$$

Hence, we obtain

$$\mathsf{pm}_0^{(9)}((4, 0, 2, 0, 3, 1, 0, 0, \dots)) = (2, 0, 3, 1, 0, 3, 1, 0, \dots),$$

which coincides with the result in Example 5.3.

Now we reformulate the explicit map Λ defined in [DJK24] in terms of particle motions.

The map Λ. We now construct the map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$. For a k-multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$, define a sequence $(s_1, \dots, s_k)$ of non-negative integers with $s_1 \ge \cdots \ge s_k \ge 0$ such that, for all i, $\lambda^{(i)}$ has a length $s_i - s_{i+1}$ where we set $s_{k+1} := 0$. Recall that the frame sequence $\mathrm{fs}(\boldsymbol{\lambda})$ consists of $s_i - s_{i+1}$ pairs equal to $(i, 0)$ for $i = 1, \dots, k$. The map Λ produces the frequency sequence from $\mathrm{fs}(\boldsymbol{\lambda})$ by applying the particle motion starting from $(1, 0)$ with step sizes given by the parts of $\lambda^{(1)}$, from $(2, 0)$ with step sizes given by the parts of $\lambda^{(2)}$, …, and from $(k, 0)$ with step sizes given by the parts of $\lambda^{(k)}$. We give a formal definition of this map.

Let $\lambda = (\lambda_0, \dots, \lambda_{s_1 - 1})$ be the sequence of non-negative integers defined by

$$(\lambda_{s_1-1}, \lambda_{s_1-2}, \dots, \lambda_1, \lambda_0) = (\lambda_1^{(1)}, \dots, \lambda_{s_1-s_2}^{(1)}, \dots, \lambda_1^{(k)}, \dots, \lambda_{s_k}^{(k)}). \tag{5.3}$$

That is, let $(\lambda_{s_i-1}, \lambda_{s_i-2} \dots, \lambda_{s_{i+1}}) = (\lambda^{(i)}_1, \dots, \lambda^{(i)}_{s_i-s_{i+1}})$ for each $i = 1, \dots, k$. Since $\boldsymbol{\lambda}$ can be obtained directly from $\mathrm{fs}(\boldsymbol{\lambda})$ and λ, we regard the pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ and $(\lambda, \mathrm{fs}(\boldsymbol{\lambda}))$ as essentially the same.

The map Λ associates to a pair $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda}$ is a k-multipartition and $\mathrm{fs}(\boldsymbol{\lambda})$ is the frequency sequence corresponding to $\boldsymbol{\lambda}$, the frequency sequence

$$\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) = \left(\mathsf{pm}_0^{(\lambda_0)} \circ \mathsf{pm}_2^{(\lambda_1)} \circ \dots \circ \mathsf{pm}_{2(s_1-2)}^{(\lambda_{s_1-2})} \circ \mathsf{pm}_{2(s_1-1)}^{(\lambda_{s_1-1})}\right)(\mathrm{fs}(\boldsymbol{\lambda}))\,.$$

This map Λ can also be described, as was done in [DJK24], by using a sequence of intermediate frequency sequences, constructed recursively by

$$\mathrm{fs}(\boldsymbol{\lambda}) =: \theta^{(s_1)}, \theta^{(s_1-1)}, \dots, \theta^{(1)}, \theta^{(0)} := \Lambda(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda}), \tag{5.4}$$

where each $\theta^{(i)}$ is obtained from $\theta^{(i+1)}$ by

$$\theta^{(i)} = \mathsf{pm}_{2i}^{(\lambda_i)}\left(\theta^{(i+1)}\right) \quad \text{for } i = s_1 - 1, \dots, 1, 0. \tag{5.5}$$

We index the parts $\lambda_{s_1-1}, \dots, \lambda_0$ and the recursive steps $\theta^{(s_1)}, \dots, \theta^{(0)}$ in reverse order to ensure consistency with the starting positions $2(s_1 - 1), \dots, 2, 0$ of the particle motions. Note that if $(\theta^{(i+1)}_{2i}, \theta^{(i+1)}_{2i+1}) = (h, 0)$, then λ_i is an entry of the partition $\lambda^{(h)}$.

Example 5.6. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}) = ((3, 1), \emptyset, (6, 6, 5, 3), (19, 0))$ be a 4-multipartition. Then the corresponding frame sequence is $\mathrm{fs}(\boldsymbol{\lambda}) = (4, 0, 4, 0, 3, 0, 3, 0, 3, 0, 3, 0, 1, 0, 1, 0, \dots) = \theta^{(8)}$. See Figure 2 for the process of applying the map Λ to $(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$. Hence, we have

$$\begin{aligned}\Lambda(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda}) &= \left(\mathsf{pm}_0^{(0)} \circ \mathsf{pm}_2^{(19)} \circ \mathsf{pm}_4^{(3)} \circ \mathsf{pm}_6^{(5)} \circ \mathsf{pm}_8^{(6)} \circ \mathsf{pm}_{10}^{(6)} \circ \mathsf{pm}_{12}^{(1)} \circ \mathsf{pm}_{14}^{(3)}\right)(\mathrm{fs}(\boldsymbol{\lambda}))) \\ &= (4, 0, 0, 3, 0, 1, 2, 1, 1, 2, 1, 2, 0, 3, 1, 0, 0, 1, 0, \dots).\end{aligned}$$

The size of the frequency sequence $\Lambda(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$ is equal to

$$|\,\mathrm{fs}(\boldsymbol{\lambda})| + |\lambda^{(1)}| + |\lambda^{(2)}| + |\lambda^{(3)}| + |\lambda^{(4)}| = 118 + 4 + 0 + 20 + 19 = 171.$$

By construction, $\Lambda(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$ is a frequency sequence satisfying $f_i + f_{i+1} \le k$ for all $i \ge 0$. Hence the map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ is well-defined. In addition, the map Λ is in fact a bijection from $\mathcal{P}_k$ to $\mathcal{A}_k$. The following property proved in [DJK24] plays an important role in constructing the inverse map of Λ.

Proposition 5.7 (reformulation of Proposition 3.15 of [DJK24])**.** *Let $\boldsymbol{\lambda}$ be a k-multipartition with $\ell(\boldsymbol{\lambda}) = s$. Let $\theta^{(s)}, \dots, \theta^{(1)}, \theta^{(0)}$ be the sequence of frequency sequences in the construction* (5.4) *of $\Lambda(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$, that is,*

$$\theta^{(i)} = \mathsf{pm}_{2i}^{(\lambda_i)}(\theta^{(i+1)}) \quad \textit{and} \quad (\theta^{(i+1)}_{2i}, \theta^{(i+1)}_{2i+1}) = (h, 0),$$

for some $1 \le h \le k$. Assume that $(\theta^{(i+1)}_{2i}, \theta^{(i+1)}_{2i+1})$ moves to $(\theta^{(i)}_v, \theta^{(i)}_{v+1})$ via the map $\mathsf{pm}_{2i}^{(\lambda_i)}$. Then, the integer h is the largest value of $\theta^{(i)}_u + \theta^{(i)}_{u+1}$ for all $u \ge 2i$, and v is the smallest integer u such that $u \ge 2i$ and $\theta^{(i)}_u + \theta^{(i)}_{u+1} = h$.

We briefly describe the construction of the inverse map Γ of Λ. Given a frequency sequence, find the leftmost pair of adjacent entries whose sum is maximal. Apply reverse particle motions to move this pair to the first and second positions, until the second position becomes zero. Record the number of reverse particle motions applied during this step. Then, excluding the first and second entries, repeat the same procedure: find the next leftmost maximal pair among the remaining entries, move it to the third and fourth position using reverse particle motions, until the fourth position becomes zero. Again, record how many reverse particle motions are applied. Continue this process until all entries of the remaining sequence are zero. The resulting sequence represents the frame sequence $\mathrm{fs}(\boldsymbol{\mu})$, and the recorded numbers form the sequence $(\mu_{s-1}, \dots, \mu_0)$, from which the multipartition $\boldsymbol{\mu}$ can be recovered.

Based on the above description, we now formulate the construction of Γ more precisely.

Reverse particle motions. We define the reverse step of the particle motions to construct the map Γ. Let $f = (f_0, f_1, \dots)$ be a frequency sequence and u be a non-negative integer such that $f_{u-1} = 0$, where we set $f_{-1} = 0$. We now describe the procedure for applying *reverse particle motions in f ending at u*.

Let h be the largest value of $f_i + f_{i+1}$ for all $i \ge u$. Choose the smallest index $v \ge u$ such that $f_v + f_{v+1} = h$. Consider the pair (f_v, f_{v+1}). If the reverse local condition $f_{v-1} + f_v < h$ is satisfied, we perform the following (single) *reverse particle motion*:

$$(f_v, f_{v+1}) \mapsto (f_v + 1, f_{v+1} - 1).$$

As long as the reverse local condition remains satisfied at the current pair, we continue applying this reverse particle motion repeatedly at the same pair. Once the condition is no longer satisfied, we decrement v by 1, shift our focus to the previous

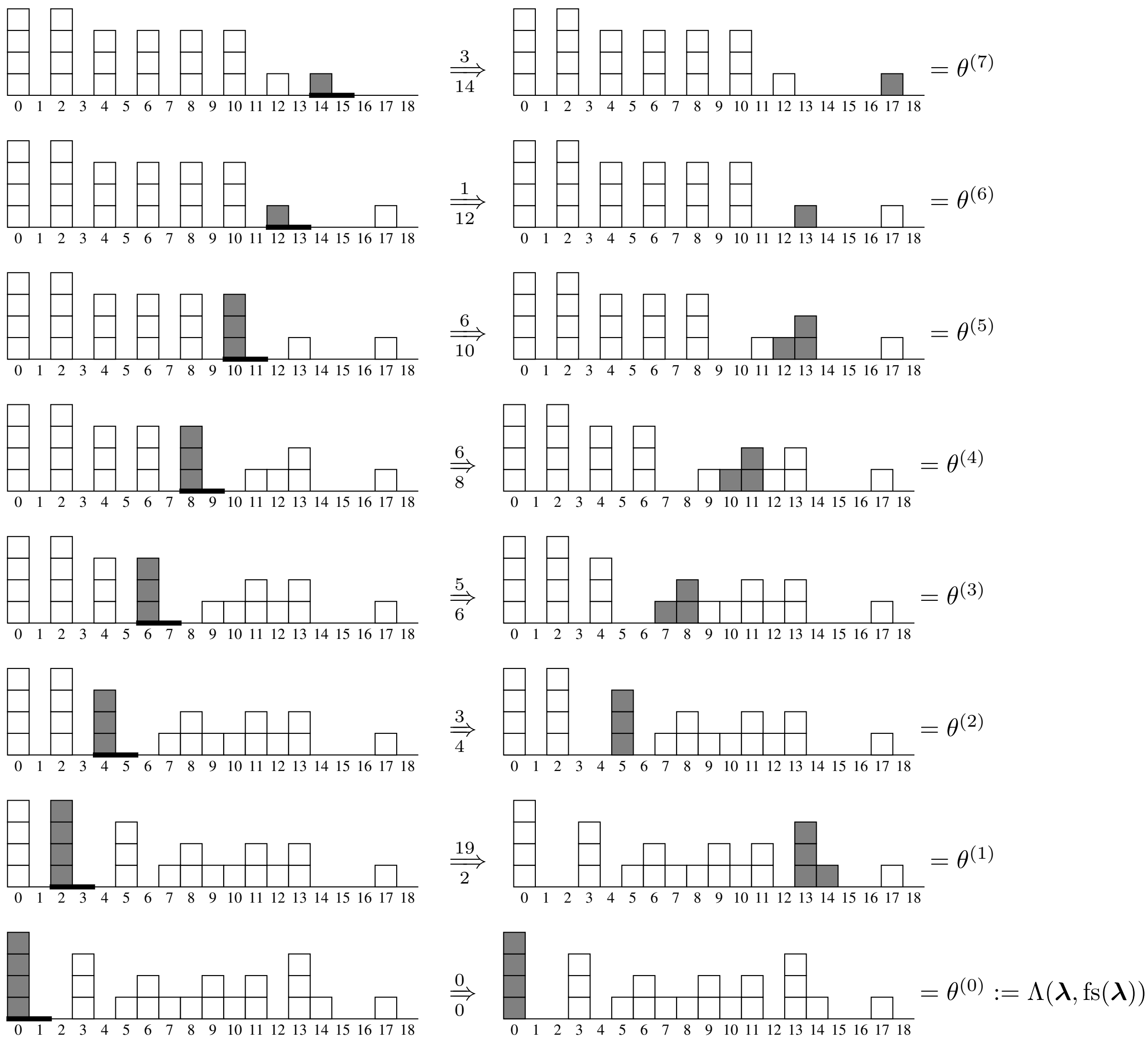

FIGURE 2. The process of applying the map Λ to $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ where $\boldsymbol{\lambda} = ((3,1), \emptyset, (6,6,5,3), (19,0))$. Here, the notation $f \overset{m}{\underset{u}{\Longrightarrow}} \overline{f}$ indicates that $\overline{f} = \mathsf{pm}_u^{(m)}(f)$.

pair (f_{v-1}, f_v), and resume the same procedure. This process continues until the focus reaches the pair (f_u, f_{u+1}) and the condition $f_{u+1} = 0$ is satisfied. Record the resulting frequency sequence and the total number of reverse particle motions applied during the process.

Definition 5.8. Let $f = (f_0, f_1, \dots)$ be a frequency sequence and u be a non-negative integer such that $f_{u-1} = 0$. We define $\mathsf{rpm}_u(f)$ to be the frequency sequence obtained by applying reverse particle motions in f ending at u, and $\mathsf{rstep}_u(f)$ to be the total number of reverse particle motions applied, as described above.

Note that in general, the reverse particle motion is not the inverse of the particle motion. Suppose that, for a given frequency sequence f and non-negative integer u, the pair $(f_u, f_{u+1}) = (h, 0)$ moves to $(\overline{f}_v, \overline{f}_{v+1})$ by applying particle motions multiple times in f starting from (f_u, f_{u+1}). To recover f via the reverse particle motion of $\overline{f}$ ending at u, the index $v \geq u$ must be the smallest index such that $f_v + f_{v+1} = h$. For instance, in Example 5.3, we have $(2,0,3,1,0,3,1,0,\cdots) = \mathsf{pm}_0^{(9)}((4,0,2,0,3,1,0,\cdots))$, but

$$\mathsf{rpm}_0((2,0,3,1,0,3,1,0,\cdots)) = (4,0,2,0,0,3,1,0,\cdots) \neq (4,0,2,0,3,1,0,\cdots).$$

In this case, the reverse particle motion is not the inverse of the particle motion. On the other hand, in Example 5.6, we have $\mathsf{rpm}_{2i}(\theta^{(i)}) = \theta^{(i+1)}$ for all $i = 0, \dots, 7$. Thus, in these eight cases, the reverse particle motion is indeed the inverse of the particle motion. In fact, by Proposition 5.7, all particle motions appearing in Λ have reverse particle motion as their inverse map.

Similar to the particle motions, the authors in [DJK24] gave explicit formulas for $\mathsf{rpm}_u(f)$ and $\mathsf{rstep}_u(f)$.

Proposition 5.9. *Let f be a frequency sequence, and let u be a non-negative integer such that $f_{u-1} = 0$, where $f_{-1} := 0$. Set*

$$h = \max\{f_i + f_{i+1} : i \geq u\}, \quad \text{and} \quad v = \min\{i \geq u+2 : f_{i-2} + f_{i-1} = h\}.$$

Then the frequency sequence $\mathsf{rpm}_u(f) = (\overline{f}_0, \overline{f}_1, \dots)$ and the non-negative integer $\mathsf{rstep}_u(f)$ are given explicitly by:

$$\overline{f}_i = \begin{cases} f_i & \text{if } 0 \leq i < u, \\ f_{i-2} & \text{if } u+2 \leq i < v, \\ f_i & \text{if } i \geq v, \end{cases} \quad \text{and} \quad (\overline{f}_u, \overline{f}_{u+1}) = (h, 0), \tag{5.6}$$

and

$$\mathsf{rstep}_u(f) = h - f_u + \sum_{i=u}^{v-3} (h - (f_i + f_{i+1})). \tag{5.7}$$

Proof. The proof uses a similar idea as in the proof of Proposition 5.4. By applying reverse particle motion, the intermediate entries originally between f_u and f_{v-3} are shifted two steps to the right. From this fact, (5.6) follows. The formula (5.7) then follows directly from $|\mathsf{rpm}_u(f)| + \mathsf{rstep}_u(f) = |f|$. □

The map Γ. We now construct the map $\Gamma : \mathcal{A}_k \to \mathcal{P}_k$. Let f be a frequency sequence such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$. The image of f by the map Γ is defined recursively as follows: Let $\eta^{(0)} = f$. Construct $\eta^{(i+1)}$ and μ_i for $i = 0, 1, \dots$ recursively by

$$\eta^{(i+1)} = \mathsf{rpm}_{2i}(\eta^{(i)}), \quad \text{and} \quad \mu_i = \mathsf{rstep}_{2i}(\eta^{(i)}). \tag{5.8}$$

Since f has finitely many nonzero entries, the smallest integer s such that $\eta_i^{(s)} = 0$ for all $i \geq 2s$ is well-defined. The sequence $\eta^{(s)}$ is a frame sequence, and from this together with $(\mu_0, \dots, \mu_{s-1})$, one can immediately obtain the k-multipartition $\boldsymbol{\mu}$. Therefore, we define $\Gamma(f) := (\boldsymbol{\mu}, \mathrm{fs}(\boldsymbol{\mu}))$. As an example, let $f = (4,0,0,3,0,1,2,1,1,2,1,2,0,3,1,0,0,1,0,\cdots) \in \mathcal{A}_4$. Then $\Gamma(f)$ is obtained by reversing the procedure described in Example 5.6.

We conclude this section with the main result: the map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ is a bijection and its inverse is given by Γ. This was proved in [DJK24], but the proof there was quite tedious. With the new formulation of this paper, the proof is only a few lines.

Theorem 5.10 ([DJK24])**.** *The map $\Lambda : \mathcal{P}_k \to \mathcal{A}_k$ is a size-preserving bijection, with inverse map Γ. More precisely, let $\boldsymbol{\lambda}$ be a multipartition with $\ell(\boldsymbol{\lambda}) = s$, and let $(\theta^{(s)}, \dots, \theta^{(0)})$ be the sequence of frequency sequences obtained in the process (5.4) of applying Λ to $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$. Let $(\eta^{(0)}, \dots, \eta^{(t)})$ be the sequence of frequency sequences obtained in the process (5.8) of applying Γ to $\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$. Then we have $s = t$, and $\theta^{(i)} = \eta^{(i)}$ for each $i = 0, \dots, s$.*

Proof. Recall that the sequence $(\theta^{(s)}, \dots, \theta^{(0)})$ of frequency sequences is defined by

$$\mathrm{fs}(\boldsymbol{\lambda}) =: \theta^{(s)}, \theta^{(s-1)}, \dots, \theta^{(1)}, \theta^{(0)} := \Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})),$$

where

$$\theta^{(i)} = \mathsf{pm}_{2i}^{(\lambda_i)}(\theta^{(i+1)}) \quad \text{for } i = s-1, \dots, 1, 0.$$

Assume that $(\theta_{2i}^{(i+1)}, \theta_{2i+1}^{(i+1)})$ moves to $(\theta_v^{(i)}, \theta_{v+1}^{(i)})$ via the map $\mathsf{pm}_{2i}^{(\lambda_i)}$. Then, by Proposition 5.7, v is the smallest index $u \geq 2i$ such that $\theta_u^{(i)} + \theta_{u+1}^{(i)} = \max\{\theta_u^{(i)} + \theta_{u+1}^{(i)} : u \geq 2i\}$. By the construction of Γ, it follows that

$$\mathsf{rpm}_{2i}(\theta^{(i)}) = \theta^{(i+1)}, \quad \text{and} \quad \mathsf{rstep}_{2i}(\theta^{(i)}) = \lambda_i.$$

The converse follows directly from the construction. Therefore, for each $i = 0, \dots, s-1$, we have

$$\theta^{(i)} = \mathsf{pm}_{2i}^{(\lambda_i)}(\theta^{(i+1)}) \iff \left(\mathsf{rpm}_{2i}(\theta^{(i)}), \mathsf{rstep}_{2i}(\theta^{(i)})\right) = \left(\theta^{(i+1)}, \lambda_i\right).$$

Since every step in the construction of Λ is invertible, the map Λ is a bijection, with the inverse map Γ. Moreover, the following holds:

$$|\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))| = |\,\mathrm{fs}(\boldsymbol{\lambda})| + \sum_{i=0}^{s-1} \lambda_i = |\,\mathrm{fs}(\boldsymbol{\lambda})| + |\boldsymbol{\lambda}| = |(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))|.$$

Hence, Λ is size-preserving, which completes the proof. □

6. A combinatorial proof of Theorem 1.5

Our strategy is as follows. We first define a subset $\mathcal{X}_{j,r,k} \subseteq \mathcal{P}_k$ whose generating function corresponds to the sum side of Theorem 1.5 (Proposition 6.2). On the other hand, we define a subset $\mathcal{Y}_{j,r,k} \subseteq \mathcal{A}_k$ whose generating function corresponds to the product side of Theorem 1.5 (Proposition 6.5), using the Andrews–Gordon identities. However, the set $\mathcal{Y}_{j,r,k}$ is an artifact introduced solely to match the desired generating function. To connect $\mathcal{X}_{j,r,k}$ and $\mathcal{Y}_{j,r,k}$, we define a new subset $\mathcal{Z}_{j,r,k} \subseteq \mathcal{A}_k$, and show that there exists a size-preserving bijection between $\mathcal{Y}_{j,r,k}$ and $\mathcal{Z}_{j,r,k} \subseteq \mathcal{A}_k$ (Proposition 6.7). We then prove that the map Λ gives a bijection between $\mathcal{X}_{j,r,k}$ and $\mathcal{Z}_{j,r,k}$ (Proposition 6.8). Combining these results gives a partition interpretation of the identity, and this also provides a combinatorial proof of Theorem 1.5.

6.1. **The sum side of Theorem 1.5.**

Definition 6.1. Let j, r, and k be non-negative integers with $j + r \leq k$. Define $\mathcal{X}_{j,r,k}$ to be the set of all pairs $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ is a k-multipartition and $\mathrm{fs}(\boldsymbol{\lambda})$ is the frame sequence corresponding to $\boldsymbol{\lambda}$, subject to the condition that each part of $\lambda^{(m)}$ is at least $m - j + \max\{m - (k - r), 0\}$ for each $m = 1, \dots, k$.

The size of $((f_i)_{i\geq 0}, \boldsymbol{\lambda}) \in \mathcal{X}_{j,r,k}$ is defined by $\sum_{i\geq 0} i f_i + \sum_{i=1}^{k} |\lambda^{(i)}|$. We give a combinatorial model for the left-hand side of the equation in Theorem 1.5. We use the following facts. A simple calculation (as shown in [DJK24, (2.15)]) shows that the weight of $\mathrm{fs}(s_1, \dots, s_k)$ is

$$|\,\mathrm{fs}(s_1, \dots, s_k)| = s_1^2 + \cdots + s_k^2 - (s_1 + \cdots + s_k). \tag{6.1}$$

Let $P_{\ell,m}(n)$ be the number of partitions of n of length ℓ into parts at least d. Then

$$\sum_{n\geq 0} P_{\ell,d}(n) q^n = \frac{q^{d\ell}}{(q)_\ell}. \tag{6.2}$$

Proposition 6.2. *For non-negative integers j, r, and k with $j + r \leq k$, we have*

$$\sum_{(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) \in \mathcal{X}_{j,r,k}} q^{|(\mathrm{fs}(\boldsymbol{\lambda}), \boldsymbol{\lambda})|} = \sum_{s_1 \geq \cdots \geq s_k \geq 0} \frac{q^{s_1^2 + \cdots + s_k^2 - (s_1 + \cdots + s_j) + (s_{k-r+1} + \cdots + s_k)}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-1} - s_k} (q)_{s_k}}.$$

Proof. For non-negative integers $s_1, \dots, s_k$ with $s_1 \geq \cdots \geq s_k \geq 0$, define $X_{j,r}(s_1, \dots, s_k)$ to be the set of k-multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ such that, for each m,

- the length of $\lambda^{(m)}$ is $s_m - s_{m+1}$, and
- each part of $\lambda^{(m)}$ is at least $m - j + \max\{m - (k - r), 0\}$; that is,

$$\lambda_1^{(m)} \geq \cdots \geq \lambda_{s_m - s_{m+1}}^{(m)} \geq m - j + \max\{m - (k - r), 0\}.$$

Let $\mathrm{fs}(s_1, \dots, s_k)$ be the frequency sequence corresponding to k-multipartitions in $X_{j,r}(s_1, \dots, s_k)$. We can express $\mathcal{X}_{j,r,k}$ as

$$\mathcal{X}_{j,r,k} = \bigsqcup_{s_1 \geq \cdots \geq s_k \geq 0} \{\mathrm{fs}(s_1, \dots, s_k)\} \times X_{j,r}(s_1, \dots, s_k).$$

By (6.2), we have

$$\begin{aligned} \sum_{\boldsymbol{\lambda} \in X_{j,r}(s_1, \dots, s_k)} q^{|\lambda|} &= \prod_{m=1}^{j} \frac{1}{(q)_{s_m - s_{m+1}}} \prod_{m=j+1}^{k-r} \frac{q^{(m-j)(s_m - s_{m+1})}}{(q)_{s_m - s_{m+1}}} \prod_{m=k-r+1}^{k-1} \frac{q^{(2m-k+r-j)(s_m - s_{m+1})}}{(q)_{s_m - s_{m+1}}} \frac{q^{(k+r-j)s_k}}{(q)_{s_k}} \\ &= \frac{q^{(s_{j+1} + \cdots + s_{k-r}) + 2(s_{k-r+1} + \cdots + s_k)}}{(q)_{s_1 - s_2} \cdots (q)_{s_{k-1} - s_k} (q)_{s_k}}. \end{aligned}$$

This, together with (6.1), completes the proof. $\square$

6.2. **The product side of Theorem 1.5.**

Definition 6.3. For non-negative integers j, r, and k with $j + r \leq k$, we define $\mathcal{Y}_{j,r,k}$ to be the set of all frequency sequences $(f_i)_{i\geq 0}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$, subject to the condition that

$$f_0 \in \{\ell + \max\{\ell - (j - r), 0\} : 0 \leq \ell \leq j\}.$$

We also define $Y_{s,k}$ to be the set of all frequency sequences $(f_i)_{i\geq 0}$ such that $f_i + f_{i+1} \leq k$ for all i, and $f_0 = s$.

Using Theorem 1.1, we immediately have the following.

Lemma 6.4 (The product side of the Andrews–Gordon identities)**.** *For non-negative integers s and k with $0 \le s \le k$, we have*

$$\sum_{f \in Y_{s,k}} q^{|f|} = \frac{(q^{2k+3}, q^{k+1-s}, q^{k+2+s}; q^{2k+3})_\infty}{(q)_\infty}.$$

We give a combinatorial model for the right-hand side of the equation in Theorem 1.5.

Proposition 6.5. *For non-negative integers j, r, and k with $j + r \le k$, we have*

$$\sum_{f \in \mathcal{Y}_{j,r,k}} q^{|f|} = \sum_{s=0}^{j} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}.$$

Proof. If $j \le r$, then the values $\ell + \max\{\ell - (j-r), 0\} = r - j + 2\ell$ for $0 \le \ell \le j$, are between 0 and k. The proof follows immediately from Lemma 6.4 and

$$\{\ell + \max\{\ell - (j-r), 0\} : 0 \le \ell \le j\} = \{r - j + 2s : 0 \le s \le j\}.$$

Now suppose $j > r$. Then the set

$$\{\ell + \max\{\ell - (j-r), 0\} : 0 \le \ell \le j\} = \{0, 1, \dots, j-r, j-r+2, \dots, j+r\}$$

can be expressed as the disjoint union of the two sets

$$\{j+r, j+r-2, j+r-4, \cdots\} \sqcup \{j-r-1, j-r-3, j-r-5, \cdots\}.$$

The first set is given by

$$\{r - j + 2s : s = \lfloor (j-r)/2 \rfloor, \lfloor (j-r)/2 \rfloor + 1, \dots, j\},$$

and the second set is

$$\{j - r - 1 - 2s : s = 0, 1, \dots, \lfloor (j-r)/2 \rfloor - 1\}.$$

We divide $\mathcal{Y}_{j,r,k}$ into two parts depending on whether f_0 belongs to the first set or the second set. By Lemma 6.4, the corresponding generating functions for these two parts are, respectively,

$$\sum_{s=\lfloor (j-r)/2 \rfloor}^{j} \frac{(q^{2k+3}, q^{k+1-(r-j+2s)}, q^{k+2+(r-j+2s)}; q^{2k+3})_\infty}{(q)_\infty}, \tag{6.3}$$

and

$$\sum_{s=0}^{\lfloor (j-r)/2 \rfloor - 1} \frac{(q^{2k+3}, q^{k+1-(j-r-1-2s)}, q^{k+2+(j-r-1-2s)}; q^{2k+3})_\infty}{(q)_\infty} = \sum_{s=0}^{\lfloor (j-r)/2 \rfloor - 1} \frac{(q^{2k+3}, q^{k+1-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+3})_\infty}{(q)_\infty}, \tag{6.4}$$

where the equality in (6.4) follows from switching the first two terms in the Pochhammer symbol in the numerator. Adding (6.3) and (6.4) completes the proof. ☐

6.3. **A combinatorial proof of Theorem 1.5.**

Definition 6.6. For non-negative integers j, r, and k with $j + r \le k$, we define $\mathcal{Z}_{j,r,k}$ to be the set of all frequency sequences $(f_i)_{i \ge 0}$ such that $f_i + f_{i+1} \le k$ for all $i \ge 0$, subject to the condition that

$$f_0 \le j - \max\{f_0 + f_1 - (k - r), 0\}.$$

Note that this condition is equivalent to

$$f_0 \le j \quad \text{and} \quad 2f_0 + f_1 \le k - r + j.$$

Proposition 6.7. *There exists a size-preserving bijection from $\mathcal{Y}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$.*

Proof. The additional conditions on $\mathcal{Y}_{j,r,k}$ and $\mathcal{Z}_{j,r,k}$ are given respectively by

$$f_0 \in \{\ell + \max\{\ell - (j-r), 0\} : 0 \le \ell \le j\}, \quad \text{and} \quad f_0 \le j - \max\{f_0 + f_1 - (k-r), 0\}.$$

We construct a bijection between $\mathcal{Y}_{j,r,k}$ and $\mathcal{Z}_{j,r,k}$ by modifying only the value of f_0.

Suppose that $j \ge r$. Then the condition on $\mathcal{Y}_{j,r,k}$ is $f_0 \in \{0, 1, \dots, j-r, j-r+2, \dots, j+r\}$. Define a map ϕ on $\mathcal{Y}_{j,r,k}$ by

$$(f_0, f_1, f_2, \dots) \mapsto (f_0', f_1, f_2, \dots),$$

where

$$f_0' = \begin{cases} f_0 & \text{if } f_0 \le j-r, \\ j-r+\ell & \text{if } f_0 = j-r+2\ell \text{ for some } \ell \in \{1,\dots,r\}. \end{cases}$$

We first claim that $\phi(\mathcal{Y}_{j,r,k}) \subseteq \mathcal{Z}_{j,r,k}$. Since all entries except the first one remain unchanged, $f_i + f_{i+1} \le k$ for $i \ge 1$. For $i = 0$, we have $f_0' + f_1 \le f_0 + f_1 \le k$. It remains to show that $f_0' \le j - \max\{f_0' + f_1 - (k-r), 0\}$. We verify this inequality case by case. If $f_0' + f_1 \le k - r$, then $\max\{f_0' + f_1 - (k-r), 0\} = 0$. By construction of ϕ, we have $f_0' \le j$, hence, $f_0' \le j - \max\{f_0' + f_1 - (k-r), 0\}$ holds. Suppose $f_0' + f_1 > k - r$. If $f_0 \le j - r$, then $f_0' = f_0$ and we have

$$\begin{aligned} f_0' + \max\{f_0' + f_1 - (k-r), 0\} &= 2f_0 + f_1 - (k-r) \\ &= f_0 + (f_0 + f_1) - (k-r) \\ &\le (j-r) + k - (k-r) = j. \end{aligned}$$

If $f_0 > j - r$, then $f_0 = j - r + 2\ell$ and $f_0' = j - r + \ell$ for some $\ell \in \{1,\dots,r\}$. We have

$$\begin{aligned} f_0' + \max\{f_0' + f_1 - (k-r), 0\} &= 2f_0' + f_1 - (k-r) \\ &= 2(j-r+\ell) + f_1 - (k-r) \\ &= f_0 + (j-r) + f_1 - (k-r) \\ &= (f_0 + f_1) - k + j \\ &\le k - k + j = j. \end{aligned}$$

Hence, the claim follows.

We now construct the inverse map of ϕ. Define a map π on $\mathcal{Z}_{j,r,k}$ by

$$(g_0, g_1, g_2, \dots) \mapsto (g_0', g_1, g_2, \dots),$$

where

$$g_0' = \begin{cases} g_0 & \text{if } g_0 \le j-r, \\ j-r+2\ell & \text{if } g_0 = j-r+\ell \text{ for some } \ell \in \{1,\dots,r\}. \end{cases}$$

Similarly, we claim that $\pi(\mathcal{Z}_{j,r,k}) \subseteq \mathcal{Y}_{j,r,k}$. Since $(g_0, g_1, \dots) \in \mathcal{Z}_{j,r,k}$, we have $g_i + g_{i+1} \le k$ for $i \ge 1$. If $g_0 \le j - r$, then

$$g_0' + g_1 = g_0 + g_1 \le k.$$

If $g_0 = j - r + \ell$, then $g_0' = j - r + 2\ell$ and

$$g_0' + g_1 = (j - r + 2\ell) + g_1 = 2g_0 - (j-r) + g_1 \le k,$$

where the last equality follows from the condition of $(g_0, g_1, \dots) \in \mathcal{Z}_{j,r,k}$ that $2g_0 + g_1 \ge k - r + j$. By the construction of π, we have

$$g_0' \in \{\ell + \max\{\ell - (j-r), 0\} : 0 \le \ell \le j\}.$$

Hence, the claim follows. Since the map $\pi : \mathcal{Z}_{j,r,k} \to \mathcal{Y}_{j,r,k}$ is the inverse of ϕ, it follows that ϕ is a bijection from $\mathcal{Y}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$. Moreover, since this map changes only the 0th entry, it does not affect the size of the partition; hence, it is size-preserving.

The case $j < r$ is proved in a similar way. Suppose that $j < r$. Then $f_0 \in \{r-j, r-j+2, \dots, r+j\}$. Define a map ϕ on $\mathcal{Y}_{j,r,k}$ by

$$(f_0, f_1, f_2, \dots) \mapsto (f_0', f_1, f_2, \dots),$$

where $f_0' = \ell$ for $f_0 = r - j + 2\ell$ with $\ell \in \{0,\dots,j\}$. Then, we have $\phi(\mathcal{Y}_{j,r,k}) \subseteq \mathcal{Z}_{j,r,k}$, since

$$f_0' + f_1 = \ell + f_1 \le 2\ell + (r-j) + f_1 = f_0 + f_1 \le k,$$

and

$$\begin{aligned} f_0' + \max\{f_0' + f_1 - (k-r), 0\} &\le 2f_0' + f_1 - (k-r) \\ &= 2\ell + f_1 - (k-r) \\ &= f_0 - (r-j) + f_1 - (k-r) \\ &= (f_0 + f_1) - k + j \\ &\le k - k + j = j. \end{aligned}$$

The inverse map π on $\mathcal{Z}_{j,r,k}$ is defined by

$$(g_0, g_1, g_2, \dots) \mapsto (g_0', g_1, g_2, \dots),$$

where $g_0' = r - j + 2\ell$ for $g_0 = \ell$ with $\ell \in \{0, \dots, j\}$. Then we have $\pi(\mathcal{Z}_{j,r,k}) \subseteq \mathcal{Y}_{j,r,k}$ since

$$g_0' \in \{r - j, r - j + 2, \dots, r + j\},$$

and

$$g_0' + g_1 = r - j + 2\ell + g_1 = r - j + 2g_0 + g_1 \leq k,$$

where the last equality follows from the condition of $(g_0, g_1, \dots) \in \mathcal{Z}_{j,r,k}$ that

$$j \geq g_0 + \max\{g_0 + g_1 - (k - r), 0\} \geq 2g_0 + g_1 - (k - r).$$

The map ϕ is a bijection from $\mathcal{Y}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$ in the case $j < r$ as well. □

Proposition 6.8. *The map Λ is a size-preserving bijection from $\mathcal{X}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$.*

It is shown in Theorem 5.10 that the map Λ is size-preserving and has an inverse map Γ. To complete the proof, it suffices to show that $\Lambda(\mathcal{X}_{j,r,k}) \subseteq \mathcal{Z}_{j,r,k}$ and $\Gamma(\mathcal{Z}_{j,r,k}) \subseteq \mathcal{X}_{j,r,k}$, which we prove in Lemmas 6.9 and 6.10.

Lemma 6.9. *Let $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda})) \in \mathcal{X}_{j,r,k}$ with $\ell(\boldsymbol{\lambda}) = s$. Suppose that $(\theta^{(s)}, \dots, \theta^{(0)})$ denotes the sequence of frequency sequences obtained recursively from $(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ via the map Λ, as in (5.4). Then, for all $i \in \{s, \dots, 0\}$, we have*

$$\theta_{2i}^{(i)} \leq j - \max\{\theta_{2i}^{(i)} + \theta_{2i+1}^{(i)} - (k - r), 0\}. \tag{6.5}$$

Moreover, $\Lambda(\mathcal{X}_{j,r,k}) \subseteq \mathcal{Z}_{j,r,k}$.

Proof. The proof follows a similar approach to that of [DJK24, Proposition 4.1], using backward induction on $i \in \{s, \dots, 0\}$. The base case $i = s$ holds trivially, since $\theta^{(s)} = \mathrm{fs}(\boldsymbol{\lambda})$ and the both entries at positions $2s$ and $2s + 1$ in the sequence $\mathrm{fs}(\boldsymbol{\lambda})$ are zero. Assume that $\theta_{2i+2}^{(i+1)} \leq j - \max\{\theta_{2i+2}^{(i+1)} + \theta_{2i+3}^{(i+1)} - (k - r), 0\}$ holds. Recall (5.5) that

$$\theta^{(i)} = \mathsf{pm}_{2i}^{(\lambda_i)}\left(\theta^{(i+1)}\right).$$

Suppose that $(\theta_{2i}^{(i+1)}, \theta_{2i+1}^{(i+1)}) = (h, 0)$ for some $h \geq 1$. Then λ_i is a part of the partition $\lambda^{(h)}$. By the condition of $\mathcal{X}_{j,r,k}$, we have $\lambda_i \geq h - j + \max\{h - (k - r), 0\}$. It suffices to prove that $\theta_{2i}^{(i)} \leq j - \max\{\theta_{2i}^{(i)} + \theta_{2i+1}^{(i)} - (k - r), 0\}$, which is equivalent to showing that $\theta_{2i}^{(i)} \leq j$ and $2\theta_{2i}^{(i)} + \theta_{2i+1}^{(i)} \leq k - r + j$.

Let v be the value defined in (5.1), so that $(\theta_{2i}^{(i+1)}, \theta_{2i+1}^{(i+1)})$ moves to $(\theta_{v-2}^{(i)}, \theta_{v-1}^{(i)})$. We use the explicit formula (5.2) for $\theta^{(i)}$ to determine $\theta_{2i}^{(i)}$ and $\theta_{2i+1}^{(i)}$.

- If $v > 2i+3$, then $(\theta_{2i}^{(i)}, \theta_{2i+1}^{(i)}) = (\theta_{2i+2}^{(i+1)}, \theta_{2i+3}^{(i+1)})$, and the claim follows immediately from the induction hypothesis.
- If $v = 2i + 3$, then

$$\begin{aligned}\theta_{2i}^{(i)} &= \theta_{2i+2}^{(i+1)},\\ \theta_{2i+1}^{(i)} &= \theta_{2i+3}^{(i+1)} + (h - \theta_{2i+1}^{(i+1)} - \theta_{2i+2}^{(i+1)}) + (h - \theta_{2i+2}^{(i+1)} - \theta_{2i+3}^{(i+1)}) - \lambda_i\\ &= 2h - \lambda_i - \theta_{2i+1}^{(i+1)} - 2\theta_{2i+2}^{(i+1)}.\end{aligned}$$

  We have $\theta_{2i}^{(i)} = \theta_{2i+2}^{(i+1)} \leq j$, by the induction hypothesis. Using two facts that $\theta_{2i+1}^{(i+1)} = 0$ and $\lambda_i \geq h - j + \max\{h - (k - r), 0\}$, we have

$$\begin{aligned}2\theta_{2i}^{(i)} + \theta_{2i+1}^{(i)} &= 2\theta_{2i+2}^{(i+1)} + (2h - \lambda_i - \theta_{2i+1}^{(i+1)} - 2\theta_{2i+2}^{(i+1)})\\ &= 2h - \lambda_i\\ &\leq 2h - (h - j + \max\{h - (k - r), 0\})\\ &\leq k - r + j.\end{aligned}$$

- If $v = 2i + 2$, then

$$\begin{aligned}\theta_{2i}^{(i)} &= \theta_{2i+2}^{(i+1)} + (h - \theta_{2i+1}^{(i+1)} - \theta_{2i+2}^{(i+1)}) - \lambda_i = h - \lambda_i,\\ \theta_{2i+1}^{(i)} &= \theta_{2i+1}^{(i+1)} + \lambda_i = \lambda_i.\end{aligned}$$

  Similarly, we have

$$\theta_{2i}^{(i)} = h - \lambda_i \leq h - (h - j + \max\{h - (k - r), 0\}) \leq j,$$

  and

$$2\theta_{2i}^{(i)} + \theta_{2i+1}^{(i)} = 2h - \lambda_i \leq k - r + j.$$

Therefore, in all cases, we have $\theta^{(i)}_{2i} \le j$ and $2\theta^{(i)}_{2i} + \theta^{(i)}_{2i+1} \le k - r + j$. This completes the proof by induction. Moreover, by (6.5) with $i = 0$, we obtain $\Lambda(\mathcal{X}_{j,r,k}) \subseteq \mathcal{Z}_{j,r,k}$.

$\square$

Lemma 6.10. *Let $f \in \mathcal{Z}_{j,r,k}$. Suppose that $(\eta^{(0)}, \dots, \eta^{(s)})$ denotes the sequence of frequency sequences obtained recursively from f via the map Γ, as in (5.8). Then, for all $i \in \{0, \dots, s\}$, we have*

$$\eta^{(i)}_{2i} \le j - \max\{\eta^{(i)}_{2i} + \eta^{(i)}_{2i+1} - (k - r), 0\}. \tag{6.6}$$

Moreover, $\Gamma(\mathcal{Z}_{j,r,k}) \subseteq \mathcal{X}_{j,r,k}$.

Proof. The proof follows a similar approach to [DJK24, Proposition 4.3 and Corollary 4.4], using induction on $i \in \{0, \dots, s\}$. The base case $i = 0$ holds clearly from $\eta^{(0)} = f \in \mathcal{Z}_{j,r,k}$. Assume that $\eta^{(i)}_{2i} \le j - \max\{\eta^{(i)}_{2i} + \eta^{(i)}_{2i+1} - (k - r), 0\}$ holds. To obtain $\eta^{(i+1)}$ from $\eta^{(i)}$, set

$$h = \max\{\eta^{(i)}_j + \eta^{(i)}_{j+1} : j \ge 2i\}, \quad \text{and} \quad v = \min\{j \ge 2i + 2 : \eta^{(i)}_{j-2} + \eta^{(i)}_{j-1} = h\},$$

and recall from (5.8) that $\eta^{(i+1)} = \mathsf{rpm}_{2i}(\eta^{(i)})$ and $\mu_i = \mathsf{rstep}_{2i}(\eta^{(i)})$. We first prove that $\eta^{(i+1)}_{2i+2} \le j - \max\{\eta^{(i+1)}_{2i+2} + \eta^{(i+1)}_{2i+3} - (k - r), 0\}$, which is equivalent to showing that $\eta^{(i+1)}_{2i+2} \le j$ and $2\eta^{(i+1)}_{2i+2} + \eta^{(i+1)}_{2i+3} \le k - r + j$. We use the explicit formula (5.6) to compute $\eta^{(i+1)}_{2i+2}$ and $\eta^{(i+1)}_{2i+3}$.

- If $v > 2i+3$, then $(\eta^{(i+1)}_{2i+2}, \eta^{(i+1)}_{2i+3}) = (\eta^{(i)}_{2i}, \eta^{(i)}_{2i+1})$, and the claim follows immediately from the induction hypothesis.
- If $v = 2i + 3$, then $(\eta^{(i+1)}_{2i+2}, \eta^{(i+1)}_{2i+3}) = (\eta^{(i)}_{2i}, \eta^{(i)}_{2i+3})$. The inequality $\eta^{(i)}_{2i+2} + \eta^{(i)}_{2i+3} \le h = \eta^{(i)}_{2i+1} + \eta^{(i)}_{2i+2}$ implies $\eta^{(i)}_{2i+3} \le \eta^{(i)}_{2i+1}$. Using this together with the induction hypothesis, we have
$$\eta^{(i+1)}_{2i+2} = \eta^{(i)}_{2i} \le j,$$
$$2\eta^{(i+1)}_{2i+2} + \eta^{(i+1)}_{2i+3} \le 2\eta^{(i)}_{2i} + \eta^{(i)}_{2i+1} \le k - r + j.$$
- If $v = 2i + 2$, then$(\eta^{(i+1)}_{2i+2}, \eta^{(i+1)}_{2i+3}) = (\eta^{(i)}_{2i+2}, \eta^{(i)}_{2i+3})$. Using inequalities $\eta^{(i)}_{2i+1} + \eta^{(i)}_{2i+2} \le h = \eta^{(i)}_{2i} + \eta^{(i)}_{2i+1}$ and $\eta^{(i)}_{2i+2} + \eta^{(i)}_{2i+3} \le h = \eta^{(i)}_{2i} + \eta^{(i)}_{2i+1}$ together with the induction hypothesis, we have
$$\eta^{(i+1)}_{2i+2} = \eta^{(i)}_{2i+2} \le \eta^{(i)}_{2i} \le j,$$
$$2\eta^{(i+1)}_{2i+2} + \eta^{(i+1)}_{2i+3} = 2\eta^{(i)}_{2i+2} + \eta^{(i)}_{2i+3} = (\eta^{(i)}_{2i+2} + \eta^{(i)}_{2i+3}) + \eta^{(i)}_{2i+2} \le 2\eta^{(i)}_{2i} + \eta^{(i)}_{2i+1} \le k - r + j.$$

Let $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(k)})$ be the k-partition obtained from f via the map Γ. It remains to prove that $(\mathrm{fs}(\boldsymbol{\mu}), \boldsymbol{\mu}) \in \mathcal{X}_{j,r,k}$. Since μ_i is a part of the partition $\mu^{(h)}$, it suffices to show that $\mu_i = \mathsf{rstep}_{2i}(\eta^{(i)}) \ge h - j + \max\{h - (k - r), 0\}$. We use the formula (5.7) to compute μ_i, and consider two cases: when $v \ge 2i + 3$ and when $v = 2i + 2$. If $v \ge 2i + 3$, then

$$\mu_i = \mathsf{rstep}_{2i}(\eta^{(i)}) = h - \eta^{(i)}_{2i} + \sum_{j=2i}^{v-3} (h - (\eta^{(i)}_j + \eta^{(i)}_{j+1})) \ge 2h - 2\eta^{(i)}_{2i} - \eta^{(i)}_{2i+1}.$$

Using $h \ge \eta^{(i)}_{2i} + \eta^{(i)}_{2i+1}$, and the induction hypotheses $\eta^{(i)}_{2i} \le j$ and $2\eta^{(i)}_{2i} + \eta^{(i)}_{2i+1} \le k - r + j$, we obtain the following two inequalities:

$$\mu_i \ge 2h - 2\eta^{(i)}_{2i} - \eta^{(i)}_{2i+1} \ge h - \eta^{(i)}_{2i} \ge h - j, \quad \text{and}$$
$$\mu_i \ge 2h - 2\eta^{(i)}_{2i} - \eta^{(i)}_{2i+1} \ge 2h - (k - r + j) = h - j + (h - (k - r)).$$

Combining the two inequalities above, we obtain $\mu_i \ge h - j + \max\{h - (k - r), 0\}$. If $v = 2i + 2$, then $\mu_i = h - \eta^{(i)}_{2i}$ and $h = \eta^{(i)}_{2i} + \eta^{(i)}_{2i+1}$. Hence, by (6.6), we have

$$\begin{aligned} \mu_i &= h - \eta^{(i)}_{2i} \\ &\ge h - j + \max\{\eta^{(i)}_{2i} + \eta^{(i)}_{2i+1} - (k - r), 0\} \\ &= h - j + \max\{h - (k - r), 0\}, \end{aligned}$$

which completes the proof. $\square$

7. A combinatorial proof of Theorems 1.7 and 1.9

The combinatorial proof of Theorem 1.7 follows the same approach as that of Theorem 1.5, but with Bressoud's identity in place of Andrews–Gordon's identity as key ingredient. The sum and product sides are obtained as the generating functions of new sets $\mathcal{X}'_{j,r,k}$ and $\mathcal{Z}'_{j,r,k}$, respectively. We then define a new set $\mathcal{Y}'_{j,r,k}$, and give bijections between $\mathcal{X}'_{j,r,k}$ and $\mathcal{Y}'_{j,r,k}$, and between $\mathcal{Y}'_{j,r,k}$ and $\mathcal{Z}'_{j,r,k}$. However, the case of the Kurşungöz type identities is somewhat different. One can define $\widetilde{\mathcal{X}}'_{j,r,k}$, $\widetilde{\mathcal{Y}}'_{j,r,k}$, and $\widetilde{\mathcal{Z}}'_{j,r,k}$ each satisfying the opposite parity condition, as analogues of $\mathcal{X}'_{j,r,k}$, $\mathcal{Y}'_{j,r,k}$, and $\mathcal{Z}'_{j,r,k}$. While the sum side arises as the generating function of $\widetilde{\mathcal{X}}'_{j,r,k}$, the product side does not correspond to the generating function of $\widetilde{\mathcal{Z}}'_{j,r,k}$. Unlike in the previous cases, $\widetilde{\mathcal{Z}}'_{j,r,k}$ is not in bijection with $\widetilde{\mathcal{Y}}'_{j,r,k}$. However, by using its relation to $\mathcal{Z}'_{j,r,k}$, we can determine the generating function for $\widetilde{\mathcal{Z}}'_{j,r,k}$, which in turn yields a new identity of the Kurşungöz type, namely Theorem 1.9.

Definition 7.1. Let j,r, and k be non-negative integers with $j+r\le k$. Define $\mathcal{X}'_{j,r,k}$ (resp. $\widetilde{\mathcal{X}}'_{j,r,k}$) to be the set of all pairs $(\boldsymbol{\lambda},\mathrm{fs}(\boldsymbol{\lambda}))$, where $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(k)})$ is a k-multipartition and $\mathrm{fs}(\boldsymbol{\lambda})$ is the frame sequence corresponding to $\boldsymbol{\lambda}$, subject to the conditions that

- each part of the partition $\lambda^{(m)}$ is at least $m-j+\max\{m-(k-r),0\}$ for each $m=1,\dots,k$, and
- each part of the last partition $\lambda^{(k)}$ has the same parity as $k+r-j$ (resp. $k+r-j+1$).

Proposition 7.2. *Let j,r, and k be non-negative integers with $j+r\le k$. Then we have*

$$\sum_{(\mu,\lambda)\in\mathcal{X}'_{j,r,k}} q^{|(\mu,\lambda)|}=\sum_{s_1\ge\cdots\ge s_k\ge 0}\frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{k-r+1}+\cdots+s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}},\tag{7.1}$$

$$\sum_{(\mu,\lambda)\in\widetilde{\mathcal{X}}'_{j,r,k}} q^{|(\mu,\lambda)|}=\sum_{s_1\ge\cdots\ge s_k\ge 0}\frac{q^{s_1^2+\cdots+s_k^2-(s_1+\cdots+s_j)+(s_{k-r+1}+\cdots+s_{k-1}+2s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.\tag{7.2}$$

Proof. We use the following identity. Let $P_{\ell,m,s}(n)$ be the number of partitions λ of n of length ℓ into parts at least m, such that each part of λ has the same parity as s. Then

$$\sum_{n\ge0}P_{\ell,m,m}(n)q^n=\frac{q^{m\ell}}{(q^2;q^2)_\ell},\quad\text{and}\quad \sum_{n\ge0}P_{\ell,m,m+1}(n)q^n=\frac{q^{(m+1)\ell}}{(q^2;q^2)_\ell}.\tag{7.3}$$

The proofs are the same as that of Proposition 6.2, except for the last term in the generating functions. By (6.2) and (7.3), we have

$$\begin{aligned}\sum_{\lambda\in X'_{j,r}(s_1,\dots,s_k)}q^{|\lambda|}&=\prod_{m=1}^{j}\frac{1}{(q)_{s_m-s_{m+1}}}\prod_{m=j+1}^{k-r}\frac{q^{(m-j)(s_m-s_{m+1})}}{(q)_{s_m-s_{m+1}}}\prod_{m=k-r+1}^{k-1}\frac{q^{(2m-k+r-j)(s_m-s_{m+1})}}{(q)_{s_m-s_{m+1}}}\frac{q^{(k+r-j)s_k}}{(q^2;q^2)_{s_k}}\\&=\frac{q^{(s_{j+1}+\cdots+s_{k-r})+2(s_{k-r+1}+\cdots+s_k)}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}},\end{aligned}$$

and

$$\begin{aligned}\sum_{\lambda\in \widetilde{X}'_{j,r}(s_1,\dots,s_k)}q^{|\lambda|}&=\prod_{m=1}^{j}\frac{1}{(q)_{s_m-s_{m+1}}}\prod_{m=j+1}^{k-r}\frac{q^{(m-j)(s_m-s_{m+1})}}{(q)_{s_m-s_{m+1}}}\prod_{m=k-r+1}^{k-1}\frac{q^{(2m-k+r-j)(s_m-s_{m+1})}}{(q)_{s_m-s_{m+1}}}\frac{q^{(k+r-j+1)s_k}}{(q^2;q^2)_{s_k}}\\&=\frac{q^{(s_{j+1}+\cdots+s_{k-r})+2(s_{k-r+1}+\cdots+s_{k-1})+3s_k}}{(q)_{s_1-s_2}\cdots(q)_{s_{k-1}-s_k}(q^2;q^2)_{s_k}}.\end{aligned}$$

This, together with (6.1), completes the proof. □

Definition 7.3. For non-negative integers j,r, and k with $j+r\le k$, we define $\mathcal{Y}'_{j,r,k}$ (resp. $\widetilde{\mathcal{Y}}'_{j,r,k}$) to be the set of all frequency sequences $(f_i)_{i\ge0}$ such that $f_i+f_{i+1}\le k$ for all $i\ge0$, subject to the conditions that

- $f_0\in\{\ell+\max\{\ell-(j-r),0\}:0\le\ell\le j\}$, and
- if $f_u+f_{u+1}=k$, then $uf_u+(u+1)f_{u+1}$ has the same parity as $k+r-j$ (resp. $k+r-j+1$).

The following lemma was obtained in [DJK24] using Theorem 1.2.

Lemma 7.4 ([DJK24, Equation (2.12) and (2.13)]). *Let $Y'_{s,k}$ (resp. $\widetilde{Y}'_{s,k}$) be the set of all frequency sequences $(f_i)_{i\geq 0}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$, $f_0 = s$, and if $f_u + f_{u+1} = k$, then $uf_u + (u+1)f_{u+1}$ has the same parity as $k-s$ (resp. $k-s+1$). Then*

$$\sum_{\lambda \in Y'_{s,k}} q^{|\lambda|} = \frac{(q^{2k+2}, q^{k+1-s}, q^{k+1+s}; q^{2k+2})_\infty}{(q)_\infty}, \tag{7.4}$$

$$\sum_{\lambda \in \widetilde{Y}'_{s,k}} q^{|\lambda|} = \frac{(q^{2k+2}, q^{k-s}, q^{k+2+s}; q^{2k+2})_\infty}{(q)_\infty}. \tag{7.5}$$

Using this lemma, we deduce expressions for the generating functions of $\mathcal{Y}'_{j,r,k}$ and $\widetilde{\mathcal{Y}}'_{j,r,k}$ as sums of infinite products.

Proposition 7.5. *For non-negative integers j, r, and k with $j + r \leq k$, we have*

$$\sum_{\lambda \in \mathcal{Y}'_{j,r,k}} q^{|\lambda|} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}, \tag{7.6}$$

$$\sum_{\lambda \in \widetilde{\mathcal{Y}}'_{j,r,k}} q^{|\lambda|} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}. \tag{7.7}$$

Proof. We use a similar idea to the proof of Proposition 6.5, with the key difference that, in this case, the parity of f_0 needs to be taken into account. The proof of the second identity (7.7) is essentially the same as that of the first (7.6), so we prove only the first identity.

If $j \leq r$, then $\ell + \max\{\ell - (j-r), 0\} = r - j + 2\ell$ for $0 \leq \ell \leq j$. The values of f_0 are

$$\{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\} = \{r - j + 2s : 0 \leq s \leq j\},$$

and they all have same parity as $r-j$. Hence, the proof follows immediately from (7.4). Now suppose $j > r$. Then the set

$$\{\ell + \max\{\ell - (j-r), 0\} : 0 \leq \ell \leq j\} = \{0, 1, \dots, j-r, j-r+2, \dots, j+r\}$$

can be expressed as the disjoint union of the two sets

$$\{j+r, j+r-2, j+r-4, \cdots\} \sqcup \{j-r-1, j-r-3, j-r-5, \cdots\}.$$

Every element of the first set $\{r - j + 2s : s = \lfloor (j-r)/2 \rfloor, \lfloor (j-r)/2 \rfloor + 1, \dots, j\}$ has the same parity as $r-j$. On the other hand, every element of the second set $\{j - r - 1 - 2s : s = 0, 1, \dots, \lfloor (j-r)/2 \rfloor - 1\}$ has the same parity as $r-j+1$. We apply (7.4) to the first set and (7.5) to the second set. The corresponding generating functions for these two parts are, respectively,

$$\sum_{s=\lfloor (j-r)/2 \rfloor}^{j} \frac{(q^{2k+2}, q^{k+1-(r-j+2s)}, q^{k+1+(r-j+2s)}; q^{2k+2})_\infty}{(q)_\infty}, \tag{7.8}$$

and

$$\sum_{s=0}^{\lfloor (j-r)/2 \rfloor - 1} \frac{(q^{2k+2}, q^{k-(j-r-1-2s)}, q^{k+2+(j-r-1-2s)}; q^{2k+2})_\infty}{(q)_\infty} = \sum_{s=0}^{\lfloor (j-r)/2 \rfloor - 1} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}. \tag{7.9}$$

Therefore, adding (7.8) and (7.9) completes the proof. □

Definition 7.6. For non-negative integers j, r, and k with $j + r \leq k$, we define $\mathcal{Z}'_{j,r,k}$ (resp. $\widetilde{\mathcal{Z}}'_{j,r,k}$) to be the set of all frequency sequences $(f_i)_{i\geq 0}$ such that $f_i + f_{i+1} \leq k$ for all $i \geq 0$, subject to the conditions that

- $f_0 \leq j - \max\{f_0 + f_1 - (k-r), 0\}$, and
- if $f_u + f_{u+1} = k$, then $uf_u + (u+1)f_{u+1}$ has the same parity as $k + r - j$ (resp. $k + r - j + 1$).

Proposition 7.7. *The map Λ gives a bijection between $\mathcal{X}'_{j,r,k}$ (resp. $\widetilde{\mathcal{X}}'_{j,r,k}$) and $\mathcal{Z}'_{j,r,k}$ (resp. $\widetilde{\mathcal{Z}}'_{j,r,k}$).*

Proof. The proof is similar to the arguments in [DJK24, (4.2) and (4.3)]. Let $\theta^{(s)}, \dots, \theta^{(1)}, \theta^{(0)}$ be the sequence of frequency sequences in the construction of $\Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$. We have

$$\theta^{(i)} = \mathsf{pm}_{2i}^{(\lambda_i)}(\theta^{(i+1)}) \quad \text{and} \quad (\theta_{2i}^{(i+1)}, \theta_{2i+1}^{(i+1)}) = (h, 0),$$

for some $h \geq 1$, and the pair $(\theta_{2i}^{(i+1)}, \theta_{2i+1}^{(i+1)})$ moves to $(\theta_v^{(i)}, \theta_{v+1}^{(i)})$ for some $v \geq 2i$. We have the explicit formula (5.2) of $\theta^{(i)}$ from $\theta^{(i+1)}$ and λ_i, with the property that

$$|\theta^{(i)}| = |\theta^{(i+1)}| + \lambda_i.$$

An entry in the sequence that remains unchanged or is shifted two steps to the left does not affect the parity. Hence, we obtain

$$\begin{aligned} v\theta_v^{(i)} + (v+1)\theta_{v+1}^{(i)} &\equiv (2i) \cdot h + (2i+1) \cdot 0 + \lambda_i \pmod 2 \\ &\equiv \lambda_i \pmod 2. \end{aligned}$$

In Proposition 5.7, we showed that consecutive particle motions starting from pairs of the form $(h, 0)$ do not interfere with each other by verifying that $v + 2 \leq v_0$. Let $\lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_\ell^{(k)})$. Then, the last ℓ steps in the construction of Λ consist of ℓ particle motions starting from the pair $(k, 0)$. The pairs moved in these steps remain unchanged for the rest of the process, until the final frequency sequence $f = \Lambda(\boldsymbol{\lambda}, \mathrm{fs}(\boldsymbol{\lambda}))$ is obtained. From this, we immediately deduce the following: for any pair (f_i, f_{i+1}) satisfying $f_i + f_{i+1} = k$, there exists a part $\lambda_u^{(k)}$ of $\lambda^{(k)}$ such that $i \cdot f_i + (i+1) \cdot f_{i+1} \equiv \lambda_u^{(k)} \pmod 2$.

Conversely, the first ℓ steps in the construction of Γ satisfy the corresponding parity condition. More precisely, for any pair (f_i, f_{i+1}) satisfying $f_i + f_{i+1} = k$, there exists $\lambda_{u+1}^{(k)}$ for some $u = 0, \dots, \ell - 1$ such that $\mathsf{rstep}_{2u}(\eta^{(u)}) = \lambda_{u+1}^{(k)} \equiv i \cdot f_i + (i+1) \cdot f_{i+1} \pmod 2$.

It follows that the bijection $\Lambda : \mathcal{X}_{j,r,k} \to \mathcal{Y}_{j,r,k}$ naturally restricts to bijections $\Lambda : \mathcal{X}'_{j,r,k} \to \mathcal{Y}'_{j,r,k}$ and $\Lambda : \widetilde{\mathcal{X}}'_{j,r,k} \to \widetilde{\mathcal{Z}}'_{j,r,k}$. ∎

Proposition 7.8. *There exists a size-preserving bijection from $\mathcal{Y}'_{j,r,k}$ to $\mathcal{Z}'_{j,r,k}$.*

Proof. By Proposition 7.5, the right-hand side is the generating function for $\mathcal{Y}'_{j,r,k}$. Consider the bijection ϕ from $\mathcal{Y}_{j,r,k}$ to $\mathcal{Z}_{j,r,k}$ and its inverse map π, described in the proof of Proposition 6.7. We now prove that the bijection $\phi : \mathcal{Y}_{j,r,k} \to \mathcal{Z}_{j,r,k}$ restricts to a bijection from $\mathcal{Y}'_{j,r,k}$ to $\mathcal{Z}'_{j,r,k}$. In other words, the parity condition is preserved under the map ϕ.

Suppose $f = (f_i)_{i \geq 0} \in \mathcal{Y}'_{j,r,k}$. Recall that the map ϕ is defined by $(f_0, f_1, f_2, \dots) \mapsto (f'_0, f_1, f_2, \dots)$, where

$$f'_0 = \begin{cases} f_0 & \text{if } f_0 \leq j - r, \\ j - r + \ell & \text{if } f_0 = j - r + 2\ell \text{ for } \ell = 1, \dots, r, \end{cases}$$

if $j \geq r$, and $f'_0 = \ell$ if $f_0 = r - j + 2\ell$ for some $\ell \in \{0, \dots, j\}$ if $j < r$. We show that $\phi(f) \in \mathcal{Z}'_{j,r,k}$. Since ϕ modifies only the 0th entry, it suffices to show that if $f'_0 + f_1 = k$, then $0 \cdot f_0 + 1 \cdot f_1 = f_1 \equiv k - r + j \pmod 2$. By the definition of ϕ, we have $f_0 \geq f'_0$, and hence $f'_0 + f_1 \leq f_0 + f_1 \leq k$. Therefore, if $f'_0 + f_1 = k$, then f'_0 and f_0 are equal. Since $f \in \mathcal{Y}'_{j,r,k}$, we have $\phi(f) \in \mathcal{Z}'_{j,r,k}$, as required.

Suppose $g = (g_i)_{i \geq 0} \in \mathcal{Z}'_{j,r,k}$. Recall that the inverse map π is defined by $(g_0, g_1, g_2, \dots) \mapsto (g'_0, g_1, g_2, \dots)$, where

$$g'_0 = \begin{cases} g_0 & \text{if } g_0 \leq j - r, \\ j - r + 2\ell & \text{if } g_0 = j - r + \ell \text{ for } \ell = 1, \dots, r. \end{cases}$$

if $j \geq r$, and $g'_0 = r - j + 2\ell$ if $g_0 = \ell$ for some $\ell \in \{0, \dots, j\}$ if $j < r$. It suffices to show that if $g'_0 + g_1 = k$, then $g_1 \equiv k - r + j \pmod 2$. By construction, we have $g_0 \leq g'_0$. If $g_0 = g'_0$, then we have $\pi(g) \in \mathcal{Y}'_{j,r,k}$ by the assumption that $g \in \mathcal{Z}'_{j,r,k}$. Now suppose $g_0 < g'_0$. Then the pair (g_0, g'_0) is either of the form

$$(j - r + \ell, j - r + 2\ell) \quad \text{or} \quad (\ell, r - j + 2\ell),$$

for some ℓ, depending on whether $j \geq r$ or $j < r$, respectively. In both cases, we have $g'_0 \equiv j - r \pmod 2$. Hence, if $g'_0 + g_1 = k$, then $g_1 = k - g'_0 \equiv k + r - j \pmod 2$. Therefore, $\pi(g) \in \mathcal{Y}'_{j,r,k}$, as desired. ∎

The generating function for $\mathcal{X}'_{j,r,k}$ is given in Proposition 7.2. By Proposition 7.8, the generating function for $\mathcal{Z}'_{j,r,k}$ is given by

$$\sum_{\lambda \in \mathcal{Z}'_{j,r,k}} q^{|\lambda|} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+1-r+j-2s}, q^{k+1+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}.$$

The result then follows from Proposition 7.7. Therefore, this gives a combinatorial proof of Theorem 1.7.

Proposition 7.9. *For non-negative integers j, r, and k with $j + r \leq k$, we have*

$$(1+q)\sum_{\lambda\in\widetilde{\mathcal{Z}}'_{j,r,k}} q^{|\lambda|} = \sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k+2-r+j-2s}, q^{k+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty} + q\sum_{s=0}^{j} \frac{(q^{2k+2}, q^{k-r+j-2s}, q^{k+2+r-j+2s}; q^{2k+2})_\infty}{(q)_\infty}.$$

Proof. Observe that

- $\mathcal{Z}'_{j,r+1,k} \subseteq \widetilde{\mathcal{Z}}'_{j,r,k} \subseteq \mathcal{Z}'_{j,r-1,k}$, and
- The map $(f_0, f_1, f_2, \dots) \mapsto (f_0, f_1 - 1, f_2, \cdots)$ is a bijection from $\widetilde{\mathcal{Z}}'_{j,r,k} \setminus \mathcal{Z}'_{j,r+1,k}$ to $\mathcal{Z}'_{j,r-1,k} \setminus \widetilde{\mathcal{Z}}'_{j,r,k}$.

We have

$$\begin{aligned}(1+q)\sum_{\lambda\in\widetilde{\mathcal{Z}}'_{j,r,k}} q^{|\lambda|} &= \sum_{\lambda\in\widetilde{\mathcal{Z}}'_{j,r,k}} q^{|\lambda|} + \sum_{\lambda\in\widetilde{\mathcal{Z}}'_{j,r,k}\setminus\mathcal{Z}'_{j,r+1,k}} q^{|\lambda|+1} + q\sum_{\lambda\in\mathcal{Z}'_{j,r+1,k}} q^{|\lambda|} \\ &= \sum_{\lambda\in\widetilde{\mathcal{Z}}'_{j,r,k}} q^{|\lambda|} + \sum_{\lambda\in\mathcal{Z}'_{j,r-1,k}\setminus\widetilde{\mathcal{Z}}'_{j,r,k}} q^{|\lambda|} + q\sum_{\lambda\in\mathcal{Z}'_{j,r+1,k}} q^{|\lambda|} \\ &= \sum_{\lambda\in\mathcal{Z}'_{j,r-1,k}} q^{|\lambda|} + q\sum_{\lambda\in\mathcal{Z}'_{j,r+1,k}} q^{|\lambda|},\end{aligned}$$

which completes the proof. □

Together with the bijection Λ between $\widetilde{\mathcal{X}}'_{j,r,k}$ and $\widetilde{\mathcal{Z}}'_{j,r,k}$ in Proposition 7.7, and the generating functions for these sets given in Proposition 7.2 and Proposition 7.9, this yields a combinatorial proof of Theorem 1.9.

8. Final remarks

We conclude with a few final remarks.

(1) Stanton [Stan18] proved Theorem 1.5 using Theorem 1.4, and similarly showed that Theorem 1.6 implies Theorem 1.7. In contrast, our proofs rely on different Bailey pair constructions. From the perspective of Bailey pairs, it would be interesting to investigate whether Theorem 1.4 implies Theorem 1.5, or Theorem 1.6 implies Theorem 1.7, and similarly regarding Theorems 1.11 and 1.12.

(2) As mentioned in Problem 1.13, Stanton posed the challenge of finding a partition-theoretic interpretation not only for the non-binomial extension but also for the binomial extension. While we have treated only the non-binomial case, the question of a partition-theoretic interpretation for the binomial case remains open and intriguing.

(3) There are combinatorial interpretations for certain generalisations of the Göllnitz–Gordon identities, see for instance [Bre80] or [HZ23]. However, no study has applied the particle motion framework to these identities. Exploring such an approach may lead to new identities or provide combinatorial proofs of the Bresssoud–Göllnitz–Gordon identities (1.13), Theorem 1.11 and Theorem 1.12. We have not yet explored this direction in depth.

(4) One could also wonder if there could exist binomial versions of Theorems 4.3 and 4.4.

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