321-avoiding affine permutations, heaps, and periodic parallelogram polyominoes

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Abstract

We give exact formulas for the bivariate generating series of 321-avoiding affine permutations with respect to rank and Coxeter length. We use two different combinatorial approaches, both based on the theory of heaps of pieces.

Keywords: Affine permutations, heaps, polyominoes, generating functions.

1 Introduction

A permutation is said to be 321-avoiding if it avoids the pattern 321. In [5], Billey-Jockusch-Stanley showed that a permutation is 321-avoiding if and only if it is fully commutative (FC), which means that any two of its reduced decompositions are related by a series of transpositions of adjacent commuting generators. It is well-known that the number of 321-avoiding permutations in Coxeter group of type A_{n-1} is given by the nth Catalan number. An analogous result of Green [7] shows that fully commutative elements in the affine Coxeter group of type $\widetilde{A}_{n-1}$, once interpreted as infinite permutations, are exactly the 321-avoiding affine permutations (see Section 3). There is an infinite number of such affine permutations, nevertheless it makes sense to compute how many of them have a fixed Coxeter length. For any infinite family of Coxeter groups,

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it is interesting to compute the series

$$W(x,q) = \sum_n W_n^{FC}(q)x^n, \quad \text{where} \quad W_n^{FC}(q) = \sum_{w \in W_n^{FC}} q^{\ell(w)},$$

and W_n^{FC} denotes the set of FC elements in the Coxeter group of type W and rank n. In the case of the affine group of type $\tilde{A}_{n-1}$, Hanusa and Jones [8] showed that the series $W_n^{FC}(q)$ is a rational function in q, with ultimately periodic coefficients, and provided an expression for the generating series $W(x,q)$, although rather complicated.

The goal of this extended abstract is to provide some closed form expressions for the series $W(x,q)$ for Coxeter groups of type A and $\tilde{A}$, and for their subsets of involutions. In type A, we recover a result of Barcucci et al. [1]. We use two different approaches and provide two different combinatorial proofs, both based on the work of Viennot on heaps of pieces [10]. More precisely, the first one uses the theory of heaps of cycles, and the second one generalizes a technique introduced by Bousquet-Mélou and Viennot [6] to enumerate parallelogram polyominoes, and uses heaps of segments. Analytic proofs of such results can be found in [2].

We now state our results. For $n \geq 0$, we denote the q-Pochhammer symbol by

$$(x)_n := (1-x)(1-xq)\cdots(1-xq^{n-1}),$$

where q is the length variable. This is extended to $n = \infty$.

Theorem 1.1 *Let $A(x,q) \equiv A$ and $\tilde{A}(x,q) \equiv \tilde{A}$ be the generating functions of fully commutative elements of type A and $\tilde{A}$, respectively defined by*

$$A := \sum_{n \geq 0} A_n^{FC}(q)x^n \qquad \textit{and} \qquad \tilde{A} := \sum_{n \geq 1} \tilde{A}_{n-1}^{FC}(q)x^n.$$

Then

$$A = \frac{1}{1-xq}\frac{J(xq)}{J(x)} \qquad \textit{and} \qquad \tilde{A} = -x\frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1-q^n},$$

where $J(x)$ is the following series:

$$J(x) := \sum_{n \geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (xq)_n}. \tag{1}$$

The counterpart of this result for involutions reads as follows.

Theorem 1.2 *With the above notation, the generating functions of fully commutative involutions of type* A *and* $\tilde{A}$ *are respectively:*

$$\mathcal{A} = \frac{\mathcal{U}(xq) + xq\mathcal{U}(xq^2)}{\mathcal{U}(x) - x\mathcal{U}(xq)} \qquad \text{and} \qquad \tilde{\mathcal{A}} = -x\frac{\mathcal{U}'(x) - xq\mathcal{U}'(xq) - \mathcal{U}(xq)}{\mathcal{U}(x) - x\mathcal{U}(xq)},$$

with

$$\mathcal{U}(x) := \sum_{n\geq 0} \frac{(-x^2)^n q^{n(2n-1)}}{(q^2;q^2)_n}. \tag{2}$$

2 Heaps of pieces

Definition 2.1 Let $\mathcal{P}$ be a set (of basic pieces) with a symmetric and reflexive binary relation $\mathcal{C}$, called the *concurrency relation*. A *heap* is a triple $(H, \preceq, \epsilon)$, where $(H, \preceq)$ is a finite poset, and $\epsilon : H \to \mathcal{P}$ is a labeling map such that:

(i) if $x, y \in H$ and $\epsilon(x)\mathcal{C}\epsilon(y)$, then either $x \preceq y$ or $y \preceq x$;

(ii) the relation $\preceq$ is the transitive closure of the relation from (i).

The elements of H are called *pieces*. When $x \preceq y$ we will say that the piece y is above the piece x. Note that $\mathcal{P}$ is not necessarily finite. The set of all heaps with pieces in $\mathcal{P}$ and concurrency relation $\mathcal{C}$ is denoted by $\mathcal{H}(\mathcal{P}, \mathcal{C})$.

Let H be a heap. A piece is said to be *maximal* (resp. *minimal*) if it is not concurrent with any piece above (resp. below) it. We denote by $\max(H)$ (resp. $\min(H)$) the set of maximal (resp. minimal) pieces of H. A *pyramid* is a heap with exactly one maximal element. A *trivial heap* is either the empty heap, or a heap consisting of pieces which are pairwise unrelated, namely they are minimal and maximal. We denote with $\mathcal{T}(\mathcal{P}, \mathcal{C})$ the set of trivial heaps. We denote with $|H|$ the number of pieces in the heap H.

In this paper we will be mostly interested in a particular family of heaps, namely that of *heaps of segments*. The set $\mathcal{P}$ of pieces is the set of *segments* of the form $[a, b]$, where $a, b \in \mathbb{N}$, $a \leq b$. Two pieces p and p' of $\mathcal{P}$ are *concurrent* if their intersection is nonempty. There is a well-known operation of composition of heaps. Intuitively, given two heaps H_1 and H_2, the *composition* $H_1 * H_2$ is the heap that results by putting H_2 on top of H_1 (for a rigorous definition we refer to [10]). In Figure 1 two heaps of segments and their composition are depicted: H_2 is a heap of *monomers* and *dimers*, which means that all its pieces are either points, or segments of the form $[a, a+1]$.

The following fundamental result is due to Viennot [10], and it is usually called the Inversion Lemma. It allows to enumerate some families of heaps

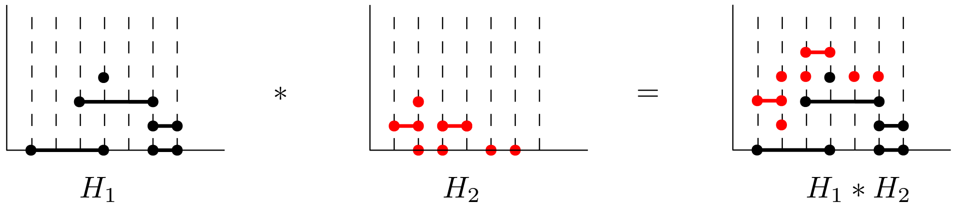


Fig. 1.

with respect to a certain valuation on their pieces. This means that each basic piece p of H has a weight $v(p)$, and that the *weight of* H, denoted by $v(H)$, is defined as the product of the weights of all the pieces of H. For us the weights will always be elements of some commutative ring with unity.

Lemma 2.2 (Inversion Lemma) *Let $\mathcal{M}$ be a subset of pieces of $\mathcal{P}$. Then the generating function for heaps with all maximal pieces in $\mathcal{M}$ is given by*

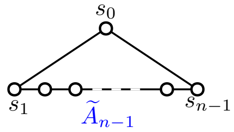
$$\sum_{\substack{H \in \mathcal{H}(\mathcal{P},\mathcal{C}) \\ \max(H) \subseteq \mathcal{M}}} v(H) = \frac{\displaystyle\sum_{T \in \mathcal{T}(\mathcal{P} \setminus \mathcal{M}, \mathcal{C})} (-1)^{|T|} v(T)}{\displaystyle\sum_{T \in \mathcal{T}(\mathcal{P},\mathcal{C})} (-1)^{|T|} v(T)},$$

where $\mathcal{T}(\mathcal{P},\mathcal{C})$ denotes the set of all trivial heaps with pieces from $\mathcal{P}$. In particular, the generating series for all heaps in $\mathcal{H}(\mathcal{P},\mathcal{C})$ is given by

$$\sum_{H \in \mathcal{H}(\mathcal{P},\mathcal{C})} v(H) = \frac{1}{\displaystyle\sum_{T \in \mathcal{T}(\mathcal{P},\mathcal{C})} (-1)^{|T|} v(T)}.$$

3 321-avoiding affine permutations

The Dynkin diagram of the affine Coxeter group of type $\tilde{A}_{n-1}$ is given by



There is a well-known combinatorial characterization of $\tilde{A}_{n-1}$ as a group of infinite permutations. More precisely, consider the set of bijective transformations $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that $\sigma(i+n) = \sigma(i) + n$ for all $i \in \mathbb{Z}$, as well as the normalization condition $\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i$. They form a group $\widetilde{S}_n$ under composition, called the group of affine permutations. It is well-known (see [9]) that $\widetilde{S}_n$ is isomorphic to the Coxeter system of type $\tilde{A}_{n-1}$, via the only morphism extending $s_i \mapsto ((i, i+1))$ for $i = 0, 1, \ldots, n-1$, where $((i, i+1))$ is the affine permutation exchanging $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$.

An affine permutation σ is 321-avoiding if there are no $i < j < k$ in $\mathbb{Z}$ satisfying $\sigma(i) > \sigma(j) > \sigma(k)$. Green showed in [7] that an affine permutation is fully commutative if and only if it is 321-avoiding. We have the following characterization [7],[4].

Proposition 3.1 *An element w of type $\tilde{A}_{n-1}$ is* 321*-avoiding if and only if, in any reduced decomposition of w, the occurrences of s_i and s_{i+1} alternate for all $i \in \{0, \ldots, n-1\}$, where we set $s_n = s_0$.*

Thanks to this characterization, there is a convenient way to represent FC elements by particular diagrams, called FC-heaps in [4]. They are actually heaps in the sense of Definition 2.1, but we do not need this characterization in this note, so we prefer to call them just *alternating diagrams*. Without entering into the details of the formal definition, we show how to construct an alternating diagram through an example. Consider $\sigma = [6, -3, -1, 8] \in \tilde{A}_3$. Note that in complete notation it looks like $\sigma = \ldots \mid 2, -7, -5, 4 \mid 6, -3, -1, 8 \mid 10, 1, 3, 12 \mid \ldots$. In Figure 2, it is depicted how to obtain the alternating diagram of σ (right) starting from its line diagram (left).

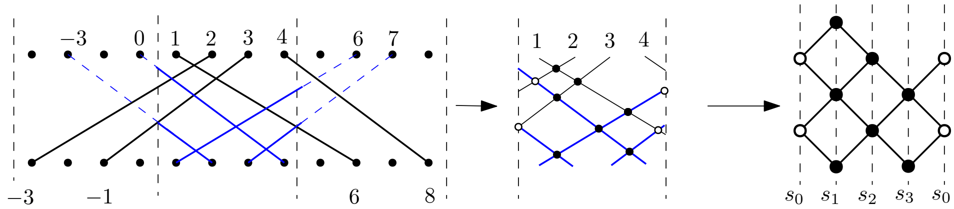


Fig. 2. The element $\sigma = [6, -3, -1, 8] = s_1 s_3 s_0 s_2 s_1 s_3 s_0 s_2 s_1$.

The alternating diagram of a 321-avoiding affine permutation $\sigma \in \tilde{A}_{n-1}$ has several important features. For example, the alternating diagram of σ has in column i as many points as the number of occurrences of the generator s_i in any reduced expression of σ. The last column is a copy of the first one and records the occurrences of s_0. The points between two consecutive columns i and $i+1$ alternate exactly as the corresponding occurrences of s_i and s_{i+1} do in the reduced decomposition of σ (from bottom to top).

Alternating diagrams can be encoded by lattice paths as follows. Given an alternating diagram H of type $\tilde{A}_{n-1}$, denote by $|H_{s_i}|$ the number of points in column i. For $i = 0, \ldots, n-1$, draw a step from $P_i = (i, |H_{s_i}|)$ to $P_{i+1} =$

$(i+1, |H_{s_{i+1}}|)$ as follows. If $|H_{s_i}| = |H_{s_{i+1}}| > 0$, we label the corresponding step by L (*resp.* R) if the lowest element of the chain $H_{\{s_i, s_{i+1}\}}$ has label s_{i+1} (*resp.* s_i). If $|H_{s_{i+1}}| = |H_{s_i}| = 0$, we label the ith step by L. Following [4] we define $\varphi'(H)$ to be the walk $(P_0, P_1, \ldots, P_n)$ with its labels; here we set $s_n = s_0$. This forms a path $\varphi'(H)$ of length n, with both P_0 and P_n at height $|H_{s_0}|$. If $w \in \tilde{A}^{FC}_{n-1}$, we set $\varphi'(w) := \varphi'(H(w))$, where $H(w)$ is the alternating diagram associated to w.

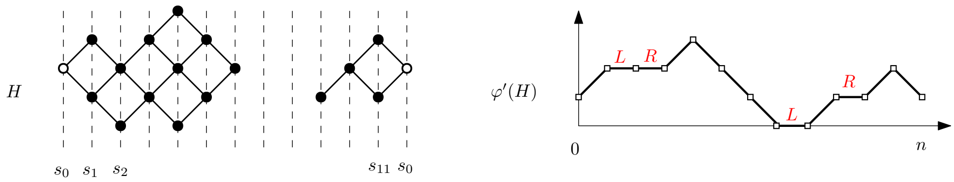


Fig. 3. The path $\varphi'(H)$ associated to the heap H.

Define $\mathcal{O}_n$ as the set of paths having n steps in the set $\{(1,1), (1,-1), (1,0)\}$ such that all horizontal steps $(1,0)$ are labeled either by L or by R, and whose starting and ending points are at the same height. A walk is said to satisfy condition $(*)$ if all horizontal steps of the form $(i,0) \to (i+1,0)$ (i.e. at height 0) have label L. The subset of walks satisfying condition $(*)$ is denoted $\mathcal{O}^*_n$. Define also $h'(P) = \sum_{i=0}^{n-1} h(P_i)$, which corresponds to the area under the walk P, (here $h(P_i)$ denotes the height of the point P_i). Finally, denote by $\mathcal{E}_n \subseteq \mathcal{O}^*_n$ the set of walks with all vertices at the same positive height, and all n steps with the same label (either L or R). For a proof of the following theorem we refer to [4].

Theorem 3.2 *The map $\varphi' : \tilde{A}^{FC}_{n-1} \to \mathcal{O}^*_n \setminus \mathcal{E}_n$ is a bijection such that $\ell(w) = h'(\varphi'(w))$.*

The alternating diagram of an involution in $\tilde{A}^{FC}_{n-1}$ is horizontally symmetric (see [3]). This means that two consecutive nonempty columns always differ by exactly one point. Hence the restriction of the map φ' to the subset of involutions gives paths in $\mathcal{O}^*_n$ having horizontal steps only at height 0.

It is clear that FC elements of type A_{n-1} are in bijection with FC elements of type $\tilde{A}_{n-1}$ whose reduced expressions have no occurrence of s_0. Therefore, by Theorem 3.2, FC elements of type A_{n-1} are in bijection with lattice paths with starting and ending points at height 0 and satisfying condition $(*)$. Once again, the subset of involutions will be in bijection with the subset of such paths having horizontal steps only at height 0.

4 Solutions via lattice paths and heaps of monomers and dimers

In this section we give proofs of Theorems 1.1 and 1.2, based on the Inversion Lemma applied to heaps of monomer and dimers.

4.1 Heaps of cycles

Starting with the path encoding defined in the previous section, we use a specialization of a bijection due to Viennot [10] in the general theory of heaps of cycles, to encode our lattice paths, and thus affine FC permutations, by heaps of monomers and dimers.

Due to space limitation, we only illustrate such a bijection with an example. First consider a path in $\mathcal{O}_n^*$, where the last point has been marked, and rotate it clockwise by 90 degrees. Then transform each vertical step into a monomer (with its label), and form a set of dimers by matching pairs of down-up or up-down steps, precisely by attaching the top one to the first below it that starts and ends in the same columns. Finally, drop the resulting dimers and monomers from top to bottom. An example is shown in Figure 4.

The result of such transformations is a heap of monomer and dimers, where monomers might have two labels L and R, except at abscissa 0, where they have only label L (this is the translation on heaps of the $(*)$-condition). Moreover, such heaps have a unique maximal marked element. Hence they are marked pyramids. By Viennot's theory, this map is a bijection.

To summarize, there is a bijection between the set of affine FC permutations and pyramids of monomers and dimers marked in their maximal piece, where the monomers have two labels L or R, except in column 0, where they have label L. Let us denote this set by $\dot{\Pi}$. When we restrict this map to affine FC involutions the image is the set of marked pyramids of monomers and dimers, where the monomers can only occur at column 0, with label L.

Once again, when we consider FC permutations (resp. their subset of involutions), the image is the set of half-pyramids (resp. the maximal piece is [0,1]) of monomers and dimers, where the monomers have two labels L or R, except in column 0, where they have label L (resp. the monomers can occur only at column 0, with label L).

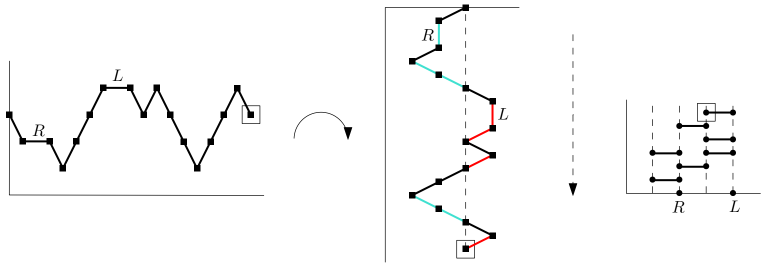


Fig. 4. The bijection between paths in $\mathcal{O}^*$ and marked pyramids.

4.2 Fully commutative elements in type A

We will prove the first result of Theorem 1.1, under the following equivalent form:

$$A = \frac{j(xq)}{j(x)}, \quad \text{where} \quad j(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}} (xq^{n+1})_\infty}{(q)_n} = (xq)_\infty J(x). \tag{3}$$

It will be convenient to introduce two more series H and h, closely related to J and j:

$$H(x) := \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}}}{(q)_n (x)_n} \qquad \text{and} \qquad h(x) := (x)_\infty H(x) = \sum_{n\geq 0} \frac{(-x)^n q^{\binom{n}{2}} (xq^n)_\infty}{(q)_n}. \tag{4}$$

It is not difficult to give a combinatorial proof of the following lemma.

Lemma 4.1 *The series $j(x)$ given by (3) is the signed generating function of trivial heaps of monomers and dimers satisfying (*). The series $h(x)$ defined by (4) is the signed generating function of trivial heaps of monomers and dimers.*

Let us now consider FC elements of type A_n. As explained after Theorem 3.2, they can be encoded by paths with $n+1$ steps satisfying Condition $(*)$. The length of the FC element is equal to the total height of the associated path. Moreover, as discussed in Section 4.1, these paths are themselves in bijection with half-pyramids of monomers and dimers. The monomers have a label L or R, except those at abscissa 0 that are all labeled by L. By giving a weight xq^i per monomer located at abscissa i, and a weight x^2q^{2i+1} per dimer at abscissa $[i, i+1]$, we have that an FC element w in A_n is bijectively encoded by a half-pyramid of weight $x^n q^{\ell(w)}$. The above bijection between FC elements and heaps of monomers and dimers, combined with the Inversion

Lemma, gives:

$$xA = \frac{h_0(x)}{j(x)} - 1, \tag{5}$$

where $j(x)$ is defined as in Lemma 4.1, and $h_0(x)$ is the signed generating function of trivial heaps that have no monomer or dimer at abscissa 0. Since such heaps are obtained by translating one step to the right any trivial heap, we have $h_0(x) = h(xq)$. One concludes the proof of (3) (left) by combining (5) and the identity $j(x) + xj(xq) = h(xq)$, which can be derived directly by the definitions of these series.

Let us now restrict our attention to involutions. They correspond to half-pyramids in which the only monomers lie at abscissa 0 (and then come in only one colour). By the Inversion Lemma,

$$x\mathcal{A} = \frac{\mathfrak{h}(xq)}{\mathfrak{j}(x)} - 1$$

where $\mathfrak{j}(x)$ is the signed generating function of trivial heaps having no monomer at positive abscissa, and $\mathfrak{h}(x)$ counts trivial heaps of dimers. It is easy to see that

$$\mathfrak{h}(x) = \mathcal{U}(x) \qquad \text{and} \qquad \mathfrak{j}(x) = \mathcal{U}(x) - x\mathcal{U}(xq), \tag{6}$$

where $\mathcal{U}(x)$ is defined by (2). Plugging these values of $\mathfrak{h}$ and $\mathfrak{j}$ in the expression of $x\mathcal{A}$, we obtain the first identity of Theorem 1.2.

4.3 Fully commutative elements in type $\tilde{A}$

Recall that $\dot{\Pi}$ denotes the set of pyramids whose monomers at column 0 have label L. Let us define $\dot{\Pi}(x) := \sum_{H\in\dot{\Pi}} v(H)$, where as before the weight of a monomer located at abscissa i is xq^i, and the weight of a dimer at abscissa $[i, i+1]$ is x^2q^{2i+1}. By the arguments of the previous section and Theorem 3.2 we obtain

$$\tilde{A} = \dot{\Pi}(x) - 2\sum_{n\geq 1} \frac{x^n q^n}{1-q^n}. \tag{7}$$

Consider a heap satisfying $(*)$, with one piece marked. By the Inversion Lemma, the associated generating function is $x\left(\frac{1}{j(x)}\right)' = -x\frac{j'(x)}{j(x)^2}$, where derivatives are taken with respect to x. By pushing the marked piece downwards, this marked heap factors into a marked pyramid and an arbitrary heap, both satisfying $(*)$ (Figure 5). This gives

$$-x\frac{j'(x)}{j(x)^2} = \frac{\dot{\Pi}(x)}{j(x)},$$

so that the generating function of marked pyramids is $\dot{\Pi}(x) = -x\frac{j'(x)}{j(x)}$. Returning to (7), this gives the second result of Theorem 1.1 since $j(x) = (xq)_\infty J(x)$.

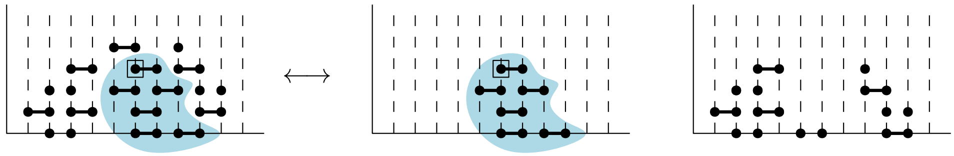


Fig. 5. Factorisation of a marked heap.

We now restrict our attention to FC affine involutions. Again, this means that the only monomers lie at abscissa 0. Note that this automatically rules out pyramids consisting of monomers lying at positive abscissa. Using the same argument as before, we obtain

$$\tilde{\mathcal{A}} = -x\frac{\mathsf{j}'(x)}{\mathsf{j}(x)}.$$

Since $\mathsf{j}(x) = \mathcal{U}(x) - x\mathcal{U}(xq)$, this gives the second result of Theorem 1.2.

5 Periodic parallelogram polyominoes and heaps of segments

Here we introduce and enumerate periodic parallelogram polyominoes using heaps of segments, which extends the theory in [6] for usual parallelogram polyominoes. We then apply this to the q-enumeration of FC elements in affine type $\widetilde{A}$, therefore exhibiting an alternative proof of our enumerative results in this type.

5.1 Some special heaps of segments

In this section we let $\mathcal{H}$ be the set of heaps where the pieces are segments $[a, b]$ and $1 \leq a \leq b$ are integers. Let $\mathcal{S}$ be the set of finite sequences of pairs of positive integers (a_i, b_i), $1 \leq i \leq n$, satisfying $a_i \leq b_i$ for all i and $a_i \leq b_{i-1}$ for $i > 1$, i.e.

$$a_1 \leq b_1 \geq a_2 \leq b_2 \geq \cdots \leq b_{n-1} \geq a_n \leq b_n.$$

Following [6, Section III], to a sequence $(a_i, b_i)_{1\leq i\leq n} \in \mathcal{S}$, we associate an element of $\mathcal{H}$ by stacking the segments $[a_n, b_n]$, $[a_{n-1}, b_{n-1}]$, ..., finishing by $[a_1, b_1]$. More precisely, for $P \in \mathcal{S}$ defined as above, we consider

$$f(P) := E_n * \cdots * E_1,$$

where $E_i := [a_i, b_i]$, for $i = 1, \ldots, n$, and the composition $*$ is defined as in Section 2. An illustration is given in Figure 6 (right).

Lemma 5.1 *The map f is a bijection between $\mathcal{S}$ and $\mathcal{H}$. Moreover $[a_n, b_n]$ (resp. $[a_1, b_1]$) is the leftmost minimal (resp. rightmost maximal) segment in $f(P)$.*

In [6, Lemma 3.3(i)], it is already shown that $[a_n, b_n]$ is the leftmost minimal segment in the image by f of the previous sequence. Then the inverse image of any heap of segments is uniquely determined by successively removing the leftmost minimal segments in a heap of $\mathcal{H}$ and naming the first one $[a_n, b_n]$, the second one $[a_{n-1}, b_{n-1}]$, and so on until the last one, which is denoted $[a_1, b_1]$.

A *parallelogram polyomino* is a polyomino such that the intersection with every line perpendicular to the main diagonal is a segment. In [6], the set of parallelogram polyominoes is encoded by the subset of sequences $(a_i, b_i)_{1 \le i \le n} \in \mathcal{S}$ with the restriction $a_1 = 1$. Here b_i is the length of the ith column and a_i is the number of common rows between the $(i-1)$th and ith columns. In [6, Proposition 3.4(i)], it is shown that f induces a bijection between parallelogram polyominoes and *half-pyramids*, the heaps in $\mathcal{H}$ having a unique maximal segment of the form $[1, b]$. This is now a consequence of Lemma 5.1.

So the elements of $\mathcal{S}$ can be encoded by parallelogram polyominoes which are marked on their first column at a height a, where a is an integer between 1 and the height of the first column.

Definition 5.2 [Periodic parallelogram polyominoes] A *periodic parallelogram polyomino* of width $n \ge 1$ is a parallelogram polyomino marked on its first column in a, where a is an integer between 1 and the height of the last column. These naturally correspond to sequences $(a_i, b_i)_{1 \le i \le n} \in \mathcal{S}$ such that $a_1 \le b_n$.

An example of such a polyomino is represented in Figure 6, together with its induced image under f (here the dashed columns highlight the periodic structure). Now, consider the following condition on a nonempty heap $H \in \mathcal{H}$.

Definition 5.3 [Condition (I)] If $[a, b]$ is the rightmost maximal segment of H and $[a', b']$ is its leftmost minimal segment, then $a \le b'$.

Equivalently, condition (I) means that there is no minimal segment of H which occurs strictly left of a maximal one. Let $\mathcal{H}_I$ be the set of heaps satisfying (I). We obtain immediately

Proposition 5.4 *The bijection f induces a bijection between the set of periodic parallelogram polyominoes and $\mathcal{H}_I$.*

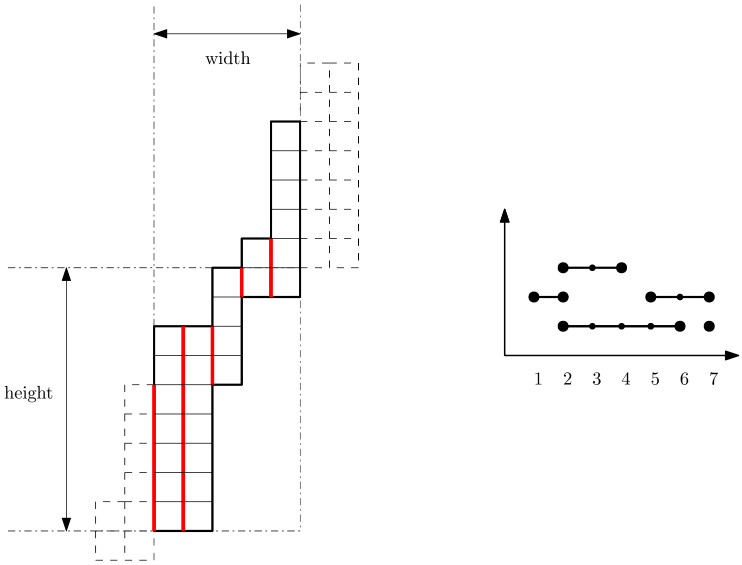


Fig. 6. A periodic parallelogram polyomino P of width $n = 5$, height 9, and its image $f(P)$, with $(a_1, \ldots, a_n) = (5, 7, 2, 1, 2)$ and $(b_1, \ldots, b_n) = (7, 7, 4, 2, 6)$.

Finally, in the correspondence of Proposition 5.4, the width of the periodic parallelogram polyomino P is equal to the number of segments in $f(P)$, the height of P to the sum of the lengths of the segments minus the number of segments in $f(P)$ (here $[a, b]$ has length $b - a + 1$, which differs from [6]), and the area of P to the sum of the right abscissas of the segments in $f(P)$. Here the *height* of a periodic parallelogram polyomino is defined as the numbers of rows between the bottom of the first column and the bottom of the first repeated column.

5.2 Counting heaps in $\mathcal{H}_I$

Following [6], we want to count heaps $H \in \mathcal{H}_I$ with respect to the following weight (note that for practical reasons due to the above choice for the length of a segment, we adopt here a slightly different, though equivalent, definition of weight from the one in [6]):

$$v(H) := y^{|H|} x^{\ell(H)} q^{e(H)},$$

where $|H|$ is the number of segments in H, $\ell(H)$ is the sum of the lengths of all segments and $e(H)$ is the sum of all right endpoints of segments.

Denote by $\mathcal{T} \subset \mathcal{H}$ the set of trivial heaps. Given any $E \in \mathcal{H}$ and $T \in \mathcal{T}$, define

$$\phi(E,T) := (T * E, T).$$

The function ϕ is clearly injective. We wish to determine the image of $\mathcal{H}_I \times \mathcal{T}$ by ϕ, and to this purpose we need the following definitions for a heap $F \in \mathcal{H}_I \setminus \mathcal{T}$:

- $S_F = [a_F, b_F]$ is the rightmost segment in $\max(F) \setminus \min(F)$;
- Y_F is the set of segments $[a,b]$ in $\max(F) \cap \min(F)$ satisfying $a > b_F$;
- X_F is the set of segments $[a,b] \in \min(F)$ satisfying $b < a_F$;
- $U_1(F)$ is the union $X_F \cup Y_F$;
- if $F \setminus \min(F) \notin \mathcal{H}_I$ and there exists $S_0 = S_0(F) \in \min(F)$ such that $F \setminus (\min(F) \setminus \{S_0\}) \in \mathcal{H}_I$, then we define $U_2(F)$ as $\min(F) \setminus \{S_0\}$, otherwise we simply set $U_2(F) := \min(F)$.

Note that if $S_0(F) = [a_0, b_0]$, then there exists $[a,b]$ such that $a \le a_0 \le b < a_F \le b_0$. From this we deduce that S_0 is unique if it exists, so that $U_2(F)$ is well-defined.

Lemma 5.5 *The pair (F,U) belongs to $\phi(\mathcal{H}_I \times \mathcal{T})$ if and only if $U \subseteq \min(F)$ and one of the following cases occurs:*

(i) *$F \in \mathcal{T}$ and $|F \setminus U| = 1$,*

(ii) *$F \notin \mathcal{T}$, $F \setminus U_1(F) \in \mathcal{H}_I$ and $U_1(F) \subseteq U \subseteq U_2(F)$.*

Thus by Lemma 5.5 we obtain

$$\left(\sum_{T\in\mathcal{T}} (-1)^{|T|} v(T)\right)\left(\sum_{E\in\mathcal{H}_I} v(E)\right) = \sum_{(F,U)\in\phi(\mathcal{H}_I\times\mathcal{T})} v(F)(-1)^{|U|}$$

$$= \sum_{F\in\mathcal{T}} |F|(-1)^{|F|-1} v(F) + \sum_{F, F\setminus U_1(F)\in\mathcal{H}_I} v(F)\left(\sum_{U_1(F)\subseteq U\subseteq U_2(F)} (-1)^{|U|}\right). \quad (8)$$

Recall that the generating function $N \equiv N(x,y,q) := \sum_{T\in\mathcal{T}} (-1)^{|T|} v(T)$ was computed in [6]; its expression is

$$N(x,y,q) = \sum_{n\ge 0} \frac{(-xy)^n q^{\binom{n+1}{2}}}{(q)_n (xq)_n}.$$

It is easy to see that the first sum in (8) is equal to $-y\partial_y N$. It is clear that the terms in the second sum vanish when $U_1 \ne U_2$. It turns out that the remaining sum with $U_1 = U_2$ also vanishes, which can be proved by a complicated involution that we skip due to space constraints. Therefore we are left with the equality

$$N \times \sum_{E \in \mathcal{H}_I} v(E) = -y \partial_y N, \tag{9}$$

from which we obtain that the generating function for $\mathcal{H}_I$ is given by $-y \frac{\partial_y N}{N}$. Finally we get the following general result.

Proposition 5.6 *The generating function for periodic parallelogram polyominoes weighted with*

$$x^{\#rows+\#columns} y^{\#columns} q^{area}$$

is given by $-y \frac{\partial_y N}{N}$.

5.3 Back to FC heaps in affine type $\widetilde{A}$

Let P be a periodic parallelogram polyomino, associated with the sequence $(a_i, b_i)_{1 \leq i \leq n}$; we first remove the bottom box of each of its columns, thereby possibly obtaining columns of height zero. We then fill every box by a point, and rotate the polyomino by 45 degrees clockwise. Points are now distributed on $N = \sum_i (b_i - a_i + 1)$ vertical lines (repeated periodically), which we want to match with generators in the type $\widetilde{A}_{N-1}$. By picking an integer in the interval $[a_1, b_1]$, we can determine unambiguously which vertical line corresponds to the generator s_0, and obtain an alternating diagram of type $\widetilde{A}_{N-1}$.

This construction (due to Viennot) transforms a pair (P, i), where P is a periodic parallelogram polyomino and i is an integer between a_1 and b_1, in an alternating diagram. It is defined except when all columns of P have equal length, are aligned, and i is equal to the commun length of the columns. Moreover, it is bijective onto the disjoint union of alternating diagrams of $\widetilde{A}_{N-1}$ over all N.

From this we obtain an enumeration formula for $\widetilde{A}$ as for the periodic parallelogram polyominoes if we take into account a little shift due to the n boxes removed, a derivative with respect to x instead of y because of the integer $i \in [a_1, b_1]$, and a corrective term due to the forbidden polyominoes. More precisely we first get through Proposition 5.4:

$$\widetilde{A} = \sum_{H \in \mathcal{H}_I^*} x^{\ell(H)} q^{e(H) - |H|} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}, \tag{10}$$

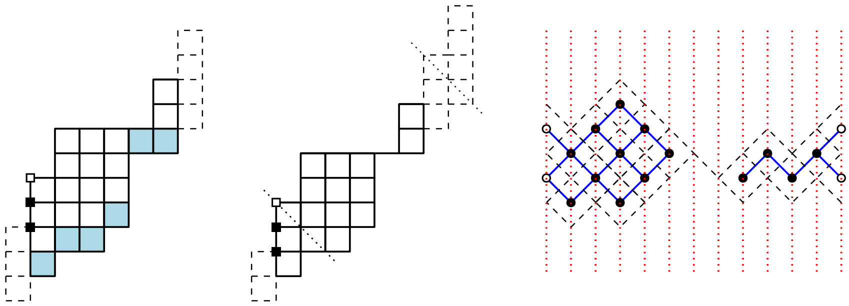


Fig. 7. From a periodic parallelogram polyomino to an alternating diagram.

where $\mathcal{H}_I^*$ is the set of heaps $H \in \mathcal{H}_I$ such that the rightmost maximal segment is marked on one of its points (recall that by Lemma 5.1 this segment is $[a_1, b_1]$). To handle the first sum in this expression, one needs to consider a marked version of Lemma 5.5.

Lemma 5.7 *We have:*

$$\sum_{H \in \mathcal{H}_I^*} y^{|H|} x^{\ell(H)} q^{e(H)} = -x \frac{\partial_x N}{N}.$$

To prove this result, we examine the image $\phi(H,T) = (F,T)$ where $H \in \mathcal{H}_I^*$ and $T \in \mathcal{T}$, as in Lemma 5.5, where the marked segment in H naturally becomes a marked segment in F. Each step in the argument of Section 5.2 is still valid by replacing $\mathcal{H}_I$ by $\mathcal{H}_I^*$, noting that, when F is trivial, any of its segments can be marked. When F is not trivial, we also obtain in this case a sum that vanishes. Therefore we get

$$\left(\sum_{T \in \mathcal{T}} (-1)^{|T|} v(T) \right) \left(\sum_{E \in \mathcal{H}_I^*} v(E) \right) = \sum_{F \in \mathcal{T}} \ell(F) (-1)^{|F|-1} v(F), \tag{11}$$

from which we derive the result. Summarizing, Lemma 5.7 and (10) together yield

$$\widetilde{A} = -x \frac{\partial_x N(x, 1/q, q)}{N(x, 1/q, q)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n} = -x \frac{J'(x)}{J(x)} - \sum_{n \geq 1} \frac{x^n q^n}{1 - q^n}.$$

Acknowledgements

We would like to thank Mireille Bousquet-Mélou for suggesting the problem to us and for many valuable suggestions given in the earlier stage of this paper.

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